Vol. 102, No. 2 DUKE MATHEMATICAL JOURNAL © 2000

EXCEPTIONAL INTEGERS FOR GENERA OF INTEGRAL TERNARY POSITIVE DEFINITE QUADRATIC FORMS

RAINER SCHULZE-PILLOT

0. Introduction. In [2] it was shown that in a certain sense most integers represented by some form in the genus of a given integral ternary positive definite quadratic form are represented by all forms in this genus. More precisely, let (L,q) be a **Z**-lattice of rank 3 with integral positive definite quadratic form. Then all sufficiently large integers a that are represented primitively by some lattice in the spinor genus of (L,q) are represented by all lattices in that spinor genus (see corollary to Theorem 3 in [2]). The theorems of that article actually imply a slightly sharper characterization of the set of exceptional integers that are represented by some forms in the genus of L but not by all of them. Since there seems to be some interest to have available a characterization of this set that is as sharp as possible, I give such a description and some examples in this note. I also comment on the question of effectivity of the results and on results for primitive representations.

As in [2], a special role is played by the integers t in the square class of a primitive spinor exception, that is, in the square class of an integer represented primitively by some but not by all spinor genera in the genus of (L,q). The Fourier coefficients of the spinor generic theta series at integers tp^2 for primes p in certain arithmetic progressions do not grow for growing p. In view of the positivity of the Fourier coefficients of theta series, this implies that the Shimura lift with respect to such a square class of the difference of the theta series of lattices in the same spinor genus omits these primes in its Fourier expansion. One might therefore be tempted to speculate about a connection to CM-forms. By looking at an example, we see, however, that this is not the case; in general one has to expect that one is looking at the sum of a cusp form and of its quadratic twist.

Acknowledgement. I thank Peter Sarnak for stimulating discussions on the questions mentioned above.

1. Exceptional integers. Let (L,q) as above be a quadratic lattice of level N, that is, for the dual lattice $L^{\#}$, we have $q(L^{\#})\mathbf{Z} = N^{-1}\mathbf{Z}$. Let d denote the discriminant of (L,q). Let T denote the (finite) set of primes p for which (L,q) remains anisotropic over the p-adic completion $\mathbf{Q}_p$ (for $p \in T$, we have $p \mid N$) and write $\overline{q}(L)$ for the set of numbers represented by some lattice in the genus of (L,q) (or equivalently, locally everywhere by (L,q)), $\overline{q_r}(L)$ for the set of $t \in \overline{q}(L)$ that are divisible at

Received 5 March 1999.
1991 *Mathematics Subject Classification.* Primary 11E45; Secondary 11E12, 11E20, 11F27.

most to the rth power by the primes in T, and $\overline{q^*}(L)$ for the set of $t \in \overline{q}(L)$ that are represented primitively by some lattice in the genus of (L, q). For $t \in \overline{q_r}(L)$, there is an integer m (which is bounded) consisting only of primes in T such that $t/m^2 \in \overline{q^*}(L)$; all statements about representation of sufficiently large $t \in \overline{q_r}(L)$ are therefore immediately reduced to the corresponding statements for sufficiently large $t \in \overline{q^*}(L)$.

Recall the following facts about representation of numbers by spinor genera of ternary lattices (see [3], [8], [10], and [11]):

- There is a finite set of square classes $t_i\mathbf{Z}^2$ (the spinor exceptional square classes) such that numbers in $\overline{q}(L)$ outside these square classes are represented (primitively if they are in $\overline{q^*}(L)$) by all spinor genera in the genus of L (i.e., in each spinor genus, there is at least one lattice representing the number), and all spinor genera have the same measure of representation (or Darstellungsmaß) for such an integer. An integer t whose square-free part does not divide d is not in any of these exceptional square classes.
- For each of the spinor exceptional square classes, the set of spinor genera in the genus of (L, q) is divided into two half-genera containing equally many spinor genera such that all spinor genera in the same half-genus (primitively) represent the same numbers in that square class (and with equal representation measures). The numbers that are (primitively) represented only by one of the half-genera are called the (primitive) spinor exceptions of the genus; if t is a primitive spinor exception and m is an integer prime to the level N, then tm^2 is a primitive spinor exception too. The sets of (primitive) spinor exceptions have been explicitly determined in [10] and [4].
- For each t_i from above, let $E_i = \mathbf{Q}(\sqrt{-dt_i})$. If p is a prime that splits in $E_i/\mathbf{Q}$, then for all $t \in t_i\mathbf{Z}^2$, the integer tp^2 is a (primitive) spinor exception if and only if t is a (primitive) spinor exception and t and tp^2 are represented by the same spinor genera in the genus of (L, q).
- Let t_i, E_i be as above and let p be a prime that is inert in $E_i/\mathbf{Q}$. Let $t \in t_i\mathbf{Z}^2$ be a primitive spinor exception of the genus of (L, q) represented by the half-genus of $\mathrm{spn}(L)$. If $p \nmid N$, then tp^2 is primitively represented by the other half-genus not containing $\mathrm{spn}(L)$ (but not by the half-genus of L). In particular, the tm^2 with $(m, N) = 1$, for which at least one prime factor of m is inert in $E_i/\mathbf{Q}$, are primitive spinor exceptions but not spinor exceptions. If $p \mid N$, then either there is a ν_0 depending on N such that $tp^{2\nu}$ is not a primitive spinor exception for $\nu \geq \nu_0$ or the $tp^{2\nu}$ behave in the the same way as in the case $p \nmid N$.
- Let t_i, E_i be as above and let $t \in t_i\mathbf{Z}^2$, p be a prime. If p is ramified in $E_i/\mathbf{Q}$ and $p^{2\nu}$ divides t for $\nu \in \mathbf{N}$ large enough (depending on N), then t is neither a spinor exception nor a primitive spinor exception.

We proved in [2] for positive definite (L, q) that all sufficiently large integers that are primitively represented by the spinor genus of (L, q) are represented by all lattices in that spinor genus and gave an asymptotic formula for the number of representations. However, we made no statement about the representation behaviour of those integers that are primitive spinor exceptions but not spinor exceptions: If they are sufficiently large, they are represented by all lattices in the spinor genera representing them primitively and by at least one lattice in each of the spinor genera in the other half of the set of spinor genera in the genus.

The following theorem shows that most of these integers are also represented by all lattices in the genus with possibly (and usually) infinitely many exceptions.

THEOREM. *There is some constant c depending on N and r such that all $t \in \overline{q_r}(L)$ with $t \geq c$ are represented by all lattices in the genus of (L, q) unless one of the following conditions is satisfied:*

(i) *t is a spinor exceptional integer, in which case it is represented by all lattices in the half-genus representing t and by no lattice in the other half-genus;*

(ii) *t/p^2 is a spinor exceptional integer for some prime p that is inert in the quadratic extension $E/\mathbf{Q}$ associated to the square class of t, in which case t is represented by all lattices in the half-genus not representing t/p^2 and by precisely those lattices in the other half-genus that represent t/p^2 (hence by all of them if $t/p^2 \geq c$ holds).*

Proof. We assume without loss of generality that $t \in \overline{q^*}(L)$. By the results of [2], there is a constant c_0 depending on N such that all integers $t \geq c_0$ in $\overline{q_r}(L)$ that are primitively represented by some lattice in the spinor genus of (L, q) are represented by all lattices in the spinor genus of (L, q). Choosing $c \geq c_0$, we therefore have only to deal with t that are represented by the spinor genus of (L, q) but are not represented primitively by that spinor genus. Let $E/\mathbf{Q}$ be the quadratic extension associated to the square class of t. Let $\hat{t} \in \overline{q^*}(L) \cap t(\mathbf{Q}^\times)^2$ be such that all representations of $\hat{t}$ by the lattices in $\mathrm{gen}(L)$ are primitive; we call such a number a primitive element of $\overline{q^*}(L)$ and notice that such numbers are almost square-free in the following sense: There is some constant c_1 depending on N such that primitive elements of $\overline{q^*}(L)$ are divisible by m^2 only for $m \leq c_1$ (this is an easy consequence of the well-known representation properties of local lattices [9]). Hence there is a constant c_2 depending on N such that $\hat{t} \leq c_2$ holds for all primitive elements $\hat{t}$ of $\overline{q^*}(L)$ that are in one of the spinor exceptional square classes. We can choose $\hat{t}$ such that it divides our given t, write $t = \hat{t}m^2$, and decompose $m = m_a m_r m_i m_s$, where m_a is the largest divisor of m consisting only of primes p for which L_p is anisotropic, m_r is the largest divisor of m consisting only of primes ramified in $E/\mathbf{Q}$ and prime to m_a, and m_i is the largest divisor of m consisting only of primes inert in $E/\mathbf{Q}$ and prime to m_a. By assumption, the "anisotropic part" m_a of m is bounded, and we can restrict ourselves to the case $m_a = 1$; in particular, we can assume $t \in \overline{q^*}(L)$. Since, by assumption, t is not represented primitively by the spinor genus of L, it is a primitive spinor exception. The results of [4] quoted above imply then that m_r is bounded as well, and we can restrict to $m_r = 1$ (the restrictions made being justified by suitably enlargening c_0).

Let K be a lattice in the genus of L representing $\hat{t}$. The integer $t_1 := \hat{t}m_s^2$ is then a spinor exception that is primitively represented by some lattice K' in the half-genus of K; it is (primitively) represented by all classes in that genus if $t \geq c_0$ by the results of [2]. We are left with the case that $t_1 \leq c_0$ and $m_i \neq 1$ hold. We choose $c \geq c_0^2$. If m_i is composite, it has at least one proper divisor m_i' such that $t_1 m_i'^2 \geq c_0$ holds. Hence $t_1 m_i'^2$ is represented (primitively) by all classes in one half-genus H_1 in the genus of L, and for any prime $p \mid (m_i/m_i')$, we see that $t_1 m_i'^2 p^2$ is represented (primitively) by all classes in the other half-genus H_2 of the genus of L and represented (but not primitively) by all classes in H_1. Then, of course, t is represented by all classes in the genus of L as well, and our assertion is proved. □

Remark 1. An anologous result is true for representations with additional congruence conditions. The proof is the same as above.

Remark 2. The result we gave in [2] is not effective. It would be no principal problem to make the estimates on the error term in our asymptotic formula effective; in fact, for the estimates of Fourier coefficients of modular forms of half-integral weight greater than or equal to $5/2$, this was done in the *Diplom* thesis (or *Diplomarbeit*) of M. Bienert [1]. However, as already remarked in [2], the growth of the main term in our asymptotic formula is of the same order of magnitude as the growth of the class number of imaginary quadratic fields. The effective bound of Goldfeld [5], following from the work of Gross and Zagier, for the class number is much too weak for the present purpose. It is, however, well known that under the assumption of the generalized Riemann hypothesis, the class number bound $h(d) \gg d^{1/2-\epsilon}$ can be made effective. In fact, we have (from Siegel's proof of his class number estimate), for any $0 < \epsilon < 1$ for which the Dedekind zeta function $\zeta_D(s)$ of $\mathbf{Q}(\sqrt{-D})$ satisfies $\zeta_D(1-\epsilon/2) \leq 0$, the estimate

$$h(D) > \frac{wD^{1/2-\epsilon(1-2\epsilon)}}{4\pi e^{4\pi}}$$

(where w is the number of units of $\mathbf{Q}(\sqrt{-D})$. Under the assumption of the generalized Riemann hypothesis, our asymptotic formula (and the sharpening in the theorem above) therefore becomes effective too.

The author is frequently asked whether the results of [2] also hold for primitive representations. We take this occasion to state this fact in a corollary.

COROLLARY. *There is some constant c^* depending on N and r, such that all $t \in \overline{q_r}(L)$ with $t \geq c^*$ that are represented primitively by some lattice in the spinor genus of (L, q) are represented primitively by all lattices in the spinor genus of (L, q).*

The same is true for representations with additional congruence conditions. Moreover, the primitive representations of t satisfying these conditions by (L, q) are asymptotically equidistributed in the sense of [2, Theorem 3] on the ellipsoid surface

$$\{x \in L \otimes \mathbf{R} \mid q(x) = t\}.$$

Proof. Denote by $r(L, q, t) = \#\{x \in L \mid q(x) = t\}$ the number of representations of t by (L, q) and by $r^*(L, q, t)$ the number of primitive representations. By the

Möbius inversion formula, we have

$$r^*(L,q,t) = \sum_{d^2|t} \mu(d) r\Big(L,q,\frac{t}{d^2}\Big).$$

As usual, we denote by

$$r\big(\mathrm{spn}(L,q),t\big) := \frac{\sum_{\{K\}} r(K,q,t)/|O(K)|}{\sum_{\{K\}} 1/|O(K)|}$$

(where the summation goes over a set of representatives of the classes of lattices in the spinor genus of L and where $|O(K)|$ is the order of the group of units or isometries of (K,q)) the weighted mean of the representation numbers of t by the lattices in the spinor genus of L and analogously for the averaged primitive representation number $r^*(\mathrm{spn}(L,q),t)$. Obviously, we then have

$$r^*\big(\mathrm{spn}(L,q),t\big) = \sum_{d^2|t} \mu(d) r\Big(\mathrm{spn}(L,q),\frac{t}{d^2}\Big),$$

and hence

$$r^*(L,q,t) - r^*\big(\mathrm{spn}(L,q),t\big) = \sum_{d^2|t} \mu(d)\Big(r\Big(L,q,\frac{t}{d^2}\Big) - r\Big(\mathrm{spn}(L,q),\frac{t}{d^2}\Big)\Big).$$

The number of terms in the sum on the right-hand side is at most the number of divisors of t, hence $O(t^\epsilon)$ for all $\epsilon > 0$. Each summand $\mu(d)(r(L,q,t/d^2) - r(\mathrm{spn}(L,q),t/d^2))$ is estimated as a cusp form coefficient in the same way as in [2], and the main term $r^*(\mathrm{spn}(L,q),t)$ is at least of the order of magnitude of $t^{1/2-\epsilon}$ as in [2]. This proves the first assertion; the remainder of the corollary is proved in the same way as in [2]. □

2. Shimura correspondence. As usual, $\vartheta(K,z) = \sum_{x\in K} \exp(2\pi i q(x) z)$ is the theta series of the lattice K, and

$$\vartheta\big(\mathrm{spn}(L),z\big) := \frac{\sum_{\{K\}} \vartheta(K,z)/|O(K)|}{\sum_{\{K\}} 1/|O(K)|}$$

(where the summation goes over a set of representatives of the classes of lattices in the spinor genus of L and where $|O(K)|$ is the order of the group of units or isometries of (K,q)) is the theta series of the spinor genus of L. We proved in [11], [12] that the Shimura lift with respect to any t' of $\vartheta(L,z) - \vartheta(\mathrm{spn}(L),z)$ is cuspidal. Let us consider the following example taken from [7]: We look at the lattice K giving the quadratic form $4x^2 + 48y^2 + 49z^2 + 48yz + 4xz$ and the lattice K' giving the quadratic form $x^2 + 48y^2 + 144z^2$. The forms are in the same spinor genus, with another spinor genus in the same genus consisting of the lattice L with quadratic form $9x^2+16y^2+48z^2$ and the lattice L' with quadratic form $16x^2+25y^2+25z^2+14yz+16xz+16xy$.

The exceptional square class is just the set of integral squares, and the associated quadratic extension is $E=\mathbf{Q}(\sqrt{-3})$ by [10]. Write $f(z)=\vartheta(K',z)-\vartheta(K,z)$ and $g(z)=\vartheta(L,z)-\vartheta(L',z)$. By results from [12], f,g are "good" cusp forms, that is, forms whose Shimura lifting is cuspidal. The explicit calculation of $T(p^2)$ acting on theta series given in [12] also shows that $T(p^2)f$ is a scalar multiple $\lambda_p f$ of f for $p\equiv 1 \bmod 3$ and $\lambda_p g$ of g for $p\equiv -1 \bmod 3$, and vice versa with λ_p replaced by some μ_p for g. This follows from the fact that $T(p^2)f$ is in the first case, a cusp form in the one-dimensional space of cusp forms generated by $\vartheta(K,z)$, $\vartheta(K',z)$, and in the second case, a cusp form in the one-dimensional space of cusp forms generated by $\vartheta(L,z)$, $\vartheta(L',z)$. In fact, an explicit (computer-assisted) calculation of the theta series and their $T(p^2)$-images for the first few primes p shows that we have $T(p^2)f=\lambda_p f$, $T(p^2)g=\lambda_p g$ (respectively, $T(p^2)g=\lambda_p f$, $T(p^2)f=\lambda_p g$) with $\lambda_p=-2,0,-4,-2,4$ for $p=5,7,11,13,19$. But since for $p,p'\equiv 1 \bmod 3$ and $q,q'\equiv -1 \bmod 3$, we have $\lambda_p\mu_q=\lambda_q\mu_p$ and $\lambda_q\mu_{q'}=\mu_q\lambda_{q'}$, by the commutativity of the Hecke algebra, we see that $\lambda_p=\mu_p$ holds for all $p\neq 2,3$. The associated eigenforms $f+g$, $f-g$ and their Shimura lifts are thus finally seen to have Hecke eigenvalues λ_p, $\chi(p)\lambda_p$, where $\chi(p)=(-3/p)$ is the quadratic character associated to $E/\mathbf{Q}$, that is, they are quadratic twists of each other. In general, the situation may be slightly more complicated, but the apparent lacunarity of the Fourier coefficients of $\vartheta(\mathrm{spn}(L))-\vartheta(L)$ in the square class of a primitive spinor exception is also caused by adding up character twists of cusp forms for the quadratic character associated to the square class in question.

References

[1] M. Bienert, *Fourier-Koeffizienten von Modulformen halbganzen Gewichts vom Nebentyp*, Diplomarbeit, Universität zu Köln, 1996.

[2] W. Duke and R. Schulze-Pillot, *Representation of integers by positive ternary quadratic forms and equidistribution of lattice points on ellipsoids*, Invent. Math. **99** (1990), 49–57.

[3] A. Earnest, *Representation of spinor exceptional integers by ternary quadratic forms*, Nagoya Math. J. **93** (1984), 27–38.

[4] A. Earnest, J. S. Hsia, and D. Hung, *Primitive representations by spinor genera of ternary quadratic forms*, J. London Math. Soc. (2) **50** (1994), 222–230.

[5] D. Goldfeld, *Gauss's class number problem for imaginary quadratic fields*, Bull. Amer. Math. Soc. (N.S.) **13** (1985), 23–37.

[6] J. S. Hsia, *Representations by spinor genera*, Pacific J. Math. **63** (1976), 147–152.

[7] B. W. Jones and G. Pall, *Regular and semiregular positive definite ternary quadratic forms*, Acta Math. **70** (1940), 165–191.

[8] M. Kneser, *Darstellungsmaße indefiniter quadratischer Formen*, Math. Z. **77** (1961), 188–194.

[9] O. T. O'Meara, *Introduction to Quadratic Forms*, 2d ed., Grundlehren Math. Wiss. **117**, Springer-Verlag, New York, 1971.

[10] R. Schulze-Pillot, *Darstellung durch Spinorgeschlechter ternärer quadratischer Formen*, J. Number Theory **12** (1980), 529–540.

[11] ———, *Darstellungsmaße von Spinorgeschlechtern ternärer quadratischer Formen*, J. Reine Angew. Math. **352** (1984), 114–132.

[12] ———, *Thetareihen positiv definiter quadratischer Formen*, Invent. Math. **75** (1984), 283–299.

FACHBEREICH 9 MATHEMATIK, BAU 27, UNIVERSITÄT DES SAARLANDES, POSTFACH 15 11 50, D-66041 SAARBRÜCKEN, GERMANY; schulzep@math.uni-sb.de