$$\left[-\frac{1}{4\rho^3}\int_x^1 \lambda(t)u(t)e^{\rho(x-t)}dt - \frac{\sin\rho+\cos\rho}{4\rho^3}\int_0^x \lambda(t)u(t)e^{-\rho(1-x+t)}dt\right]$$
$$\cdot\left\{1-e^{-\rho}\sin\rho-e^{-\rho}\cos\rho\right\}^{-1}.$$

For t in $[x, 1]$, $x-t\leqq 0$; for t in $[0, x]$, $1-x+t\geqq 0$. Also, there exists a constant $c>0$ such that $1-e^{-\rho}\sin\rho-e^{-\rho}\cos\rho\geqq c$ for $\rho=\rho_n$. Call $M_n=\max_{[0,1]}|u_n(x)|$, $K=\int_0^1|\lambda(t)|dt$. Then

$$|u_n(x)|\leqq 2+\frac{5}{c}+\frac{17M_nK}{4c\rho_n^3},\qquad M_n\leqq 2+\frac{5}{c}+\frac{17M_nK}{4c\rho_n^3},$$
$$M_n\leqq\frac{2+5/c}{1-17K/(4c\rho_n^3)}.$$

Thus for n sufficiently great, $M_n\leqq 4+10/c$. The remaining n's form a finite set. Hence $u_n(x)$ is uniformly bounded in n. Thus

$$u_n(x)=\cos\rho_nx-\sin\rho_nx+e^{-\rho_n(1-x)}\sin\rho_n-e^{-\rho_nx}+r_n(x)/\rho_n^3,$$

and $\cos\rho_n=\phi(\rho_n)$, where $\lim_{n\to\infty}\phi(\rho_n)=0$. Thus $\rho_n=(n+1/2)\pi+\epsilon_n$, where $\lim_{n\to\infty}\epsilon_n=0$. It follows that

$$\begin{aligned}u_n(x)=&\cos(n+1/2)\pi x-\sin(n+1/2)\pi x\\&+(-1)^n\exp\{-(n+1/2)\pi(1-x)\}\\&-\exp\{-(n+1/2)\pi x\}+r_n(x)/n^3.\end{aligned}$$

The theorem for system V follows.

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AN ALMOST UNIVERSAL FORM

GORDON PALL

P. R. Halmos[1] obtained the 88 possible forms (a, b, c, d), $0<a\leqq b\leqq c\leqq d$, which represent all positive integers with one exception, and proved that property for all except for the form $h=(1, 2, 7, 13)$. A proof for h follows.

The forms $f=(1, 2, 7)$ and $g=(1, 1, 14)$ constitute the reduced forms of a genus.[2] Between them they represent all positive integers not of the form[3] $\Lambda=7^{2k+1}(7m+3, 5, 6)$. The identities

$$\begin{aligned}x^2+y^2+14z^2&=x^2+2((y+7z)/3)^2+7((y-2z)/3)^2\\&=y^2+2((x+7z)/3)^2+7((x-2z)/3)^2\end{aligned}$$

show that every number represented by g with either $y\equiv -z$ or $x\equiv -z$ (mod 3) is also represented by f. Hence every number $3n$ and $3n+1$ not of the form Λ is represented by f. For, $x\equiv y\equiv 0$, $z\not\equiv 0$, and $x, y\not\equiv 0$, $z\equiv 0$ (mod 3) both imply $g\equiv 2$. If $N=3n$ or $3n+1$ is of the form Λ, then $7|N$, so that $N-13\cdot 3^2\neq\Lambda$. Similarly, one of $3n+2-13$ and $3n+2-52$ is not of the form Λ; but neither of these is congruent to 2 (mod 3). These linear forms are positive if $n\geqq 39$; h represents all integers not less than 119. The only number less than 119 not represented in (1, 2, 7, 13) is found to be 5.

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[1] This Bulletin, vol. 44 (1938), pp. 141–144.
[2] See any table of positive ternaries.
[3] For example, see B. W. Jones, Transactions of this Society, vol. 33 (1931), pp. 111–124.