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15 pages

A unified proof of regularity for 36 forms

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In the reference at the end of the paper there is a list of 913 positive definite ternary forms. All but 22 are regular; the stubborn 22 are plausible candidates. The list includes all regular forms.

Of the 913, 794 are alone in their genera and therefore are regular by default. There remain 119. For 97 = 119 - 22 there is a proof of regularity. Some of these proofs are scattered in the literature. The rest (due to Will Jagy and/or myself) are unpublished. The task of writing a coherent account suitable for publication has yet to be accomplished. In the meantime, this paper presents a single method that proves regularity for 36 forms.

Let f and g be forms. Suppose that we suspect that f dominates g (meaning that f represents every number that g represents). This question is at the heart of proving regularity. The trouble is that there is no known algorithm for checking domination; we seem to need a different trick every time.

Of course if there is a homothety from f to g then domination is immediate. (A homothety is a linear transformation that multiplies the form by a scalar.) But something weaker will do. Suppose, for a third form h, we happen to know that g and h represent exactly the same numbers and that there is a honothety from f to h. Then f dominates h and therefore dominates g. This simple minded idea actually works for 36 forms.

Homothety is testable algorithmically. Indeed, Will Jagy wrote a program for it. One feeds in 12 numbers: the coefficients of f and g. Out comes either the statement that there is no homothety or (a useful bonus) all homotheties.

Now for the details. At the end of the paper there is a hitherto unpublished table of the 119 forms at issue and their genus mates. We shall be referring to a form by its attached number. If in addition one of the letters A, B, C appears, this is one of the forms for which the method works. The letters stand for three infinite families.

A. It is known that the binary forms $x^2 + xy + y^2$ and $x^2 + 3y^2$ represent the same numbers. Therefore if a ternary form has a binary direct summand $a(x^2 + xy + y^2)$, this can be replaced by $a(x^2 + 3y^2)$.

B. The form (s, s, t, s, s, s) represents the same numbers as (s, 4s, t, 2s, 0, 2s).

C. The form (s, s, s, t, t, t) represents the same numbers as (s, 4s, 2s - t, 4s - 2t, 0, 2t). The proofs for B and C appear in Appendix I.

Remark. The second genus mate for form 37 can be switched to (4, 4, 4, -1, -1, -1) by replacing x by -x. A similar remark applies to form 58.

Appendix II records the 38 homotheties (there are 2 more than 36 because the forms 21 and 37 each have 2 genus mates). A more or less arbitrary choice was made of the homotheties offered by the computer. Here are the detail for form number 1: $f = (1, 1, 3, 0, 1, 0)$. The mate is $g = (1, 1, 4, 1, 1, 1)$ and its companion is $h = (1, 4, 4, 2, 0, 2)$. The homothety listed is $2z, x + y, y$ and indeed replacing x, y, z by $2z, x + y, y$ in f yields h.

Appendix I: Proofs for B and C.

B. The year is 2002, but there is still no concensus on a canonical form for positive definite ternary forms. Schiemann prefers (s, s, t, s, s, s) but Brandt and Intrau prefer $g = (s, s, t, s, 0, s)$. (To pass from the first to the second replace y by $-x + y$.) I shall use the second version. Sending y into $2y$ induces a homothety that sends g into $h = (s, 4s, t, 2s, 0, 2s)$. We must conversely show that any number $g(x, y, z)$ represented by g is also represented by h. This is immediate if y is even; so we assume that y is odd. Now replacing x by $-x - y$ is an automorphism of g; therefore the parity of x is at our disposal. Another automorphism of g is available: replacing y by $-x - y - z$. By appropriately choosing the parity of x we can thus make y even, as required.

C. In $g = (s, s, s, t, t, t)$ replace y by $y - z$ to get $g^* = (s, s, 2s - t, 2s - t, 0, t)$. Replace the new y by $2y$ to get $h = (s, 4s, 2s - t, 4s - 2t, 0, 2t)$. We have to show that a number $A = g(x, y, z)$ represented by g is also represented by h. Two of x, y, z must have the same parity, say y and z. In the corresponding variables for g^* the (new) y is even. This gives what is required.

Remark. It can be seen that B is a special case of C, but that is irrelevant here.

Notes: Will Jagy : WED 27 FEB 2008

Four families that appear different are really equivalent! Further, they are "quasi-decomposable" forms.

C: (S, S, S, T, T, T) is positive when $2S > T > -S$.

It represents the same numbers as
$(S, 2S-T, 3S-T, 0, 2(S-T), 0)$.
Notice this is a unary plus a binary.

B: $(A, A, B, A, A, A) \cong (B, B, B, 2B-A, 2B-A, 2B-A)$

Remark on page (2): $(S, S, S, -T, T, T) \cong (S, S, S, -T, -T, -T)$

Extra: $(3(S-T), S, S, -T, 2(S-T), 2(S-T)) \cong (S, S, S, -T, -T, -T)$

Recall $(S, 2S-T, 3S-T, 0, 2(S-T), 0)$ refers to
$G(x, y, z) = (2S-T)y^2 + (Sx^2 + (2S-2T)xz + (3S-T)z^2)$.

1. $2z$, ~~$x+y$~~ $x+y$, y
2. x, $y+z$, $-y+z$
3. $x+y+z$, $y-z$, $-y-z$
4. $x-z$, $2z$, y
5. $x-z$, $y+z$, $-y+z$
10. $x+y-z$, $2z$, y
12. $x-z$, y, $2z$
13. $x-z$, $y-z$, $2z$
15. $x+y-z$, y, $2z$
16. x, $y-z$, $2z$
18. $x-z$, $y-z$, $2z$
19. $x-z$, $2z$, y
20. $x+y-z$, y, $2z$

21a. $x-z$, y, $2z$

21b. $y+2z$, $x+y$, y

26. $x-z$, $y-z$, $2z$
28. x, $y-z$, $2z$
29. $x+z$, $-x+z$, y
30. $x-z$, $y-z$, $2z$
31. $x-y$, $2y$, z
32. $y+2z$, $x+y$, $-y$
33. $x-z$, $y-z$, $2z$
34. $x-y$, $y-z$, $y+z$
35. $x+y$, $-x+y$, z

~~23~~ ~~34~~

37a. $x-z$, $2z$, y

37b. $2x-y$, y, $y+z$ Switch 1 to -1

41. $x-z$, $y-z$, $2z$
42. $x+2y+z$, $-x+z$, $-y$
43. $x-z$, $-x-2y-z$, y
46. $x+y-z$, $2z$, y
47. $x-z$, $2z$, y
49. $x-y$, $y+z$, $y-z$
51. $x-y$, $2y$, z
52. x, $y-z$, $y+z$
55. $x+y-z$, y, $2z$
58. $2x-y$, $-y$, $y+z$
60. $x-z$, $y+z$, $-y+z$
61. $x+2y+z$, $-x+z$, $-y$

~~45~~

~~Because of 21, 37~~ 36

~~Need 38 homotheties~~

Table of the 119 regular forms and candidates

The 22 candidates are designated by an asterisk

The pairs 6 and 7; 8 and 9; 24 and 25 each constitute a genus

Odd forms

^		
1 ; 11 :	1 1 3 0 1 0	B
:	1 1 4 1 1 1	
^		
2 ; 15 :	1 2 2 1 0 0	A
:	1 1 5 0 0 1	
^		
3 ; 17 :	1 2 3 2 1 1	B
:	1 1 6 1 1 1	
^		
4 ; 21 :	1 2 3 0 0 1	A
:	1 1 7 0 0 1	
^		
5 ; 24 :	1 3 3 3 1 1	A
:	1 1 8 0 0 1	
^		
6 ; 27 :	1 1 7 0 1 0	
7 ; 27 :	1 2 4 1 0 1	
^		
8 ; 27 :	1 1 9 0 0 1	
9 ; 27 :	1 3 3 3 0 0	
^		
10 ; 32 :	1 3 3 1 0 1	B
:	1 1 11 1 1 1	
^		
11 ; 44 :	1 3 4 0 0 1	
:	1 1 15 1 1 1	
^		
12 ; 45 :	1 3 4 0 1 0	A
:	1 1 15 0 0 1	
^		
13 ; 48 :	1 3 5 3 1 0	A
:	1 1 16 0 0 1	
^		
14 ; 50 :	1 2 7 2 1 0	
:	1 1 17 1 1 1	
^		
15 ; 56 :	1 3 5 1 1 0	B
:	1 1 19 1 1 1	
^		
16 ; 63 :	1 3 6 3 0 0	A
:	1 1 21 0 0 1	
^		
17 ; 72 :	1 3 7 2 1 1	
:	3 3 3 1 2 3	

^			
18 ; 72 :		1 3 7 3 1 0	A
	:	1 1 24 0 0 1	
^			
19 ; 75 :		1 4 5 0 0 1	A
	:	1 5 5 5 0 0	
^			
20 ; 80 :		1 3 7 1 1 0	B
	:	1 1 27 1 1 1	
^			
21 ; 81 :		1 3 7 0 1 0	
	:	1 1 27 0 0 1	A
	:	3 3 4 3 3 3	B
^			
22 ; 108 :		1 1 36 0 0 1	
	:	3 3 4 0 0 3	
^			
23 ; 108 :		1 3 10 3 1 0	
	:	3 4 4 4 3 3	
^			
24 ; 108 :		1 4 7 0 1 0	
25 ; 108 :		1 5 7 5 1 1	
^			
26 ; 120 :		1 3 11 3 1 0	A
	:	1 1 40 0 0 1	
^			
27 ; 121 :		1 3 11 0 0 1	
	:	3 4 4 -3 2 2	
^			
28 ; 135 :		1 3 12 3 0 0	A
	:	1 1 45 0 0 1	
^			
29 ; 135 :		2 2 9 0 0 1	A
	:	3 3 5 0 0 3	
^			
30 ; 144 :		1 3 13 3 1 0	A
	:	1 1 48 0 0 1	
^			
31 ; 147 :		1 2 21 0 0 1	A
	:	1 7 7 7 0 0	
^			
32 ; 189 :		2 3 8 0 1 0	B
	:	3 3 8 3 3 3	
^			
33 ; 216 :		1 3 19 3 1 0	A
	:	1 1 72 0 0 1	

^		
34 ; 216 :	3 5 5 2 3 3	A
:	3 3 8 0 0 3	
^		
35 ; 225 :	2 2 15 0 0 1	A
:	3 5 5 5 0 0	
^		
36 ; 240 :	1 5 13 2 1 1	*
:	4 5 5 5 2 4	
^		
37 ; 243 :	1 7 9 0 0 1	
:	1 9 9 9 0 0	A
:	4 4 4 -1 1 1	C
^		
38 ; 243 :	2 3 11 3 1 0	
:	2 2 17 2 2 1	
:	3 5 5 2 0 3	
^		
39 ; 289 :	3 5 6 1 2 3	
:	3 6 6 -5 2 2	
^		
40 ; 297 :	1 6 13 3 1 0	*
:	4 4 6 -3 3 1	
^		
41 ; 360 :	1 3 31 3 1 0	A
:	1 1 120 0 0 1	
^		
42 ; 392 :	3 3 12 -2 2 1	C
:	5 5 5 3 3 3	
^		
43 ; 400 :	3 3 12 2 2 1	B
:	5 5 7 5 5 5	
^		
44 ; 405 :	2 5 11 2 2 1	*
:	5 5 6 3 3 5	
^		
45 ; 432 :	1 3 37 3 1 0	
:	1 1 144 0 0 1	
:	3 3 16 0 0 3	
:	3 7 7 5 3 3	
^		
46 ; 432 :	3 5 9 3 0 3	B
:	3 3 17 3 3 3	
^		
47 ; 441 :	3 6 7 0 0 3	A
:	3 7 7 7 0 0	

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^
48 ; 484 :   1 3 44 0 0 1
         :   5 5 5 -1 1 1
^
49 ; 600 :   5 7 7 6 5 5       A
         :   5 5 8 0 0 5
^
50 ; 648 :   1 7 25 5 1 1
         :   4 7 7 5 2 2
^
51 ; 675 :   1 4 45 0 0 1       A
         :   1 15 15 15 0 0
^
52 ; 675 :   5 6 6 3 0 0       A
         :   5 5 9 0 0 5
^
53 ; 720 :   3 5 15 3 3 3      *
         :   5 5 11 4 5 5
^
54 ; 972 :   1 7 36 0 0 1
         :   4 9 9 9 0 0
         :   7 7 7 5 5 5
^
55 ; 1080 :  3 9 11 3 3 0       B
          :  3 3 41 3 3 3
^
56 ; 1125 :  1 10 29 5 1 0     *
          :  4 9 11 9 4 3
^
57 ; 1125 :  2 7 22 -6 1 1     *
          :  7 7 8 2 7 4
^
58 ; 1323 :  2 8 21 0 0 1      C
          :  8 8 8 -5 5 5
^
59 ; 1620 :  5 8 11 -4 1 2     *
          :  5 5 23 4 5 5
^
60 ; 1800 :  5 11 11 7 5 5     A
          :  5 5 24 0 0 5
^
61 ; 2160 :  5 9 15 9 3 3
          :  5 5 29 1 2 5      *
          :  5 11 11 7 1 1
^
62 ; 2592 :  5 9 17 6 5 3      *
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             :  5 5 27 3 3 1
             :  9 9 11 3 3 9
^
63 ; 3375 :  2 15 32 15 1 0        *
             :  8 8 17 8 8 1
^
64 ; 4500 :  7  8 23  6  7  2      *
             :  7 13 13  1  3  3
^
65 ; 4536 :  5 9 27 0 3 3
             :  9  9 20  6  6  9   *
             :  9 11 17  8  9  9
^
66 ; 5400 :  7  7 28 -2  2  1      C
             :  13 13 13 11 11 11
^
67 ; 8232 :  5 13 33 -6  3  1      *
             :  13 13 17 10 11  9
^
68 ; 10125 :  9 11 29 -4  3  6     *
              :  9 11 35 10  0  9
^
69 ; 24696 :   11 15 39 -3  6  3   *
              :   11 23 29 13 11  5
^
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Please check discriminants 8640, 14400, 43200

~~13 August 1991~~

Even forms

70	; 16 :	1 1 16 0 0 0	
	:	2 2 5 2 2 0	
71	; 25 :	1 2 13 2 0 0	
	:	2 2 9 2 2 2	
72	; 64 :	1 2 32 0 0 0	
	:	2 4 9 4 0 0	
73	; 64 :	1 4 16 0 0 0	
	:	4 4 5 0 4 0	
74	; 64 :	1 5 13 2 0 0	
	:	4 5 5 4 0 4	
75	; 108 :	1 12 12 12 0 0	
	:	4 4 9 0 0 4	
76	; 108 :	1 3 36 0 0 0	
	:	3 4 9 0 0 0	
77	; 108 :	1 4 28 4 0 0	
	:	4 5 8 4 4 4	
78	; 144 :	1 4 36 0 0 0	
	:	4 4 9 0 0 0	
79	; 192 :	1 8 24 0 0 0	
	:	4 8 9 8 4 0	
80	; 224 :	3 6 14 4 2 2	*
	:	6 7 7 2 2 6	
81	; 256 :	1 16 16 0 0 0	
	:	4 9 9 2 4 4	

82 ~~13~~ ; 256 : 1 8 32 0 0 0
: 4 8 9 0 4 0

83 ~~14~~ ; 256 : 3 3 32 0 0 2
: 4 8 11 8 4 0

84 ~~15~~ ; 324 : 3 4 28 4 0 0
: 4 4 27 0 0 4
: 7 7 7 2 2 2

85 ~~16~~ ; 400 : 3 3 51 -2 2 2
: 8 8 11 8 8 8

86 ~~17~~ ; 432 : 1 12 36 0 0 0
: 4 9 12 0 0 0

87 ~~18~~ ; 448 : 5 8 12 0 4 0
: 5 5 20 4 4 2

88 ~~19~~ ; 512 : 1 8 64 0 0 0
: 4 8 17 0 4 0

89 ~~20~~ ; 576 : 3 8 24 0 0 0
: 8 11 11 10 8 8

90 ~~21~~ ; 768 : 1 16 48 0 0 0
: 4 16 17 16 4 0

91 ~~22~~ ; 960 : 5 8 24 0 0 0
: 8 13 13 6 8 8

92 ~~23~~ ; 1008 : 7 8 20 0 4 4 *
: 7 7 27 6 6 6

93 ~~24~~ ; 1024 : 3 11 32 0 0 2
: 11 11 12 -4 4 10

94 ~~25~~ ; 1280 : 7 12 16 0 0 4

: 7 7 28 -4 4 2

^

95 ~~26~~ ; 1296 : 5 8 36 0 0 4
: 8 9 20 0 8 0

^

96 ~~27~~ ; 1728 : 1 24 72 0 0 0
: 4 24 25 24 4 0

^

97 ~~28~~ ; 1728 : 1 48 48 48 0 0
: 9 16 16 16 0 0

^

98 ~~29~~ ; 1728 : 5 5 72 0 0 2
: 8 12 21 12 0 0

^

99 ~~30~~ ; 1728 : 8 9 24 0 0 0
: 8 17 17 10 8 8

^

100 ~~31~~ ; 1728 : 1 16 112 16 0 0
: 9 16 17 16 6 0

^

101 ~~32~~ ; 1728 : 4 13 37 2 4 4
: 13 13 16 -8 8 10

^

102 ~~33~~ ; 2112 : 7 15 23 -6 2 6 *
: 7 16 23 16 2 0

^

103 ~~34~~ ; 2304 : 3 16 48 0 0 0
: 12 16 19 16 12 0

^

104 ~~35~~ ; 2880 : 11 16 19 8 2 8 *
: 11 11 27 6 6 6

^

105 ~~36~~ ; 2880 : 8 15 24 0 0 0
: 8 23 23 22 8 8

^

106 ~~37~~ ; 3136 : 3 19 56 0 0 2
: 12 19 19 -18 4 4

^

```
107 ~~38~~ ; 4800 :   1 40 120 0 0 0
                  :   4 40 41 40 4 0

108 ~~39~~ ; 5184 :   3 16 112 16 0 0
                  :   16 16 27 0 0 16
                  :   19 19 19 -10 10 10

109 ~~40~~ ; 5184 :   7 15 55 -6 2 6
                  :   15 16 28 16 12 0

110 ~~41~~ ; 6336 :   5 20 68 -8 4 4
                  :   20 20 21 -12 12 8

111 ~~42~~ ; 6400 :   3 27 80 0 0 2
                  :   12 27 27 -26 4 4

112 ~~43~~ ; 6912 :   1 48 144 0 0 0
                  :   4 48 49 48 4 0
                  :   9 16 48 0 0 0
                  :   16 25 25 14 16 16

113 ~~44~~ ; 6912 :   9 17 48 0 0 6
                  :   17 17 32 -8 8 14

114 ~~45~~ ; 6912 :   5 20 77 20 2 4
                  :   20 20 21 12 12 8

115 ~~46~~ ; 8000 :   11 16 51 8 2 8
                  :   16 19 35 10 0 16

116 ~~47~~ ; 8640 :   13 24 28 0 4 0
                  :   13 13 52 4 4 2

117 ~~48~~ ; 14400 :  3 40 120 0 0 0
                   :  12 40 43 40 12 0

118 ~~49~~ ; 14400 :  7 23 92 12 4 2
                   :  23 28 28 -24 4 4
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      ^
119 ~~80~~ ; 43200 :    9 41 120   0   0   6
                   :   36 41  41 -38  12  12
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Reference

J K S paper