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Three theorems on ternary forms

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1. Introduction. The theory of ternary quadratic forms is full of challenges that have not yet been met. As yet there are not many theorems that apply to whole families of forms. In this paper I present three theorems of this kind.

2. Forms that represent the same integers. In [3, p. 174] Jones and Pall stated without proof that $x^2 + 3y^2 + 2yz + 4z^2$ and $x^2 + y^2 + 4z^2 + xy + xz$ represent the same integers. I now fit this in as the case $r = 4$ in an infinite set of statements. For notational simplicity I suppress the dependence of f and g on r.

Theorem 1. For any integer r the forms $f = x^2 + 3y^2 + 2yz + rz^2$ and $g = x^2 + y^2 + rz^2 + xy + xz$ represent the same integers.

Remarks. 1. We can allow r to be negative or 0, thereby acquiring some information on indefinite forms.

2. For $r = 9$, g is the first of the "near misses" listed in [4], that is, it seems to represent all positive odd integers with exactly one exception. (In retrospect it might have been better to call them candidates to be near misses.) Subsequently, in a search not yet finished, I found three more candidates. One was f with $r = 9$. So, although I still do not know whether these two forms actually represent all positive odd integers with one exception, at least I know that they will stand or fall together.

Here are the fifth and sixth candidates. The forms $2x^2 + 3y^2 + 3z^2 + 2xy + yz$ and $2x^2 + 3y^2 + 5z^2 + 2xy + 3yz$ both seem to represent all positive odd integers except 1. In fact the first of these forms seems even to represent all integers represented by its genus except the powers of 4 (including 1). Now I know there are 43 in all.

3. Theorem 1 leads one to wonder about other pairs that represent the same integers.

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They exist in an abundance that may make classification difficult. As a starter one gets many pairs by inserting $x^2 + 3y^2$ and $x^2 + xy + y^2$ into ternary forms. However, the problem looks more reasonable if restricted to diagonal forms (which might as well be primitive). In that case I know of only two examples: the pair 1, 1, 1, and 1, 2, 2; and the pair 1, 1, 2 and 1, 2, 4. (I am using the notation a, b, c for the form $ax^2+by^2+cz^2$, and shall assume $a \leqq b \leqq c$.) I have two partial results which I shall state without proof. Suppose that a, b, c and d, e, f represent the same integers. It is trivial that $a = d$. (i) Assume $b < e$. Then the forms are one of the two pairs listed above. (ii) Assume $b = e = 1$. Then the two forms are identical. I remark further that an inspection of the 102 primitive regular diagonal forms presented on pages 112-113 of [2] reveals that there are no other primitive diagonal pairs where both forms are regular.

Proof of Theorem 1. The method is standard and elementary. Suitable scalar multiples of f and g are diagonalized and a bridge is built between them by using still another form: $h = x^2 + 3y^2 + (3r - 1)z^2$. This calls for comparing three statements, where A is a given positive integer: (a) f represents A, (b) g represents A, (c) h represents 3A.

(a) $\rightarrow$ (c). From $f(x, y, z) = A$ we get

(1) $$3f = (3y + z)^2 + 3x^2 + (3r - 1)z^2 = 3A.$$

(b) $\rightarrow$ (c). From $g(x, y, z) = A$ we get

(2) $$12g = (3x + 2z)^2 + 3(x + 2y)^2 + (12r - 4)z^2 = 12A.$$

Now it is known that if $a^2 + 3b^2 = 4c$ then there exist d, e with $d^2 + 3e^2 = c$. By applying this to (2) and dividing by 4 we get that h represents 3A.

(c) $\rightarrow$ (a). We are given

(3) $$u^2 + 3v^2 + (3r - 1)w^2 = 3A$$

and our task is to find x, y, and z satisfying (1). From (3) we see that $u^2 - w^2$ is divisible by 3. By changing the sign of w, if necessary, we arrange that u - w is divisible by 3. Set $x = v$, $y = (u - w)/3$, $z = w$.

(c) $\rightarrow$ (b). Again we are given (3) and this time we need to reach (2). Again we arrange to have u - w divisible by 3. Set $x = 2(u - w)/3$ and $z = w$. Note that x is even and set $y = v - x/2$.

3. Two families of regular forms. Recall that a form is regular if it represents all integers represented by its genus.

Theorem 2. Suppose that $k = x^2 + 3y^2 + sz^2 + xz + 3yz$ and $m = x^2 + xy + y^2 + (4s - 4)z^2$ constitute a genus. Then k is regular.

Proof. Let A be an eligible integer (i. e., one represented by the genus). We must show that k represents A, and we can assume that m represents A (for otherwise we see at once that k represents A). Since the binary forms $x^2 + xy + y^2$ and $x^2 + 3y^2$ represent the same integers we can assume that $u^2 + 3v^2 + (4s - 4)w^2 = A$. Set $x = u - w$, $y = v - w$, $z = 2w$. Then $k(x, y, z) = A$.

Theorem 3. Suppose that $n = x^2 + 3y^2 + tz^2 + xz + yz$ and $p = x^2 + y^2 + (4t - 1)z^2 + xy + xz$ constitute a genus. Then n is regular. l.c. "z"

Proof. The plan is the same. With B eligible we may assume that p represents B and our task is to prove that n represents B. Now

$$12p = (3x + 2z)^2 + 3(x + 2y)^2 + (48t - 16)z^2.$$ l.c "tee"

So we have u, v, and w satisfying

(4) $$u^2 + 3v^2 + (48t - 16)w^2 = 12B.$$

We repeat the "$x^2 + 3y^2$" trick: since $u^2 + 3v^2$ is divisible by 4 there exist elements c and d with $u^2 + 3v^2 = 4(c^2 + 3d^2)$. After dividing (4) by 4 we have

$$c^2 + 3d^2 + (12t - 4)w^2 = 3B.$$

l.c. "del"

Next comes another repetition: $c^2 - w^2$ is divisible by 3 and we arrange that $c - w$ is divisible by 3. Set $x = d - w$, $y = (c - w)/3$, $z = 2w$. Then $n(x, y, z) = B$.

In [4] three forms were proved to be regular: numbers 13, 15, and 19 in the list at the end of that paper. These have been reprised here by the cases $s = 3$ of Theorem 2 and $t = 3, 5$ of Theorem 3, respectively. New regular forms are given by $s = 5, 7, 11, 13, 19, 31$ and $t = 7$, according to the Brandt-Intrau table [1]. Whether there are any more beyond the table seems doubtful. (Later information strongly suggests that there are none) (Still later: I know there are none)

References

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2. L. E. Dickson, Modern Elementary Theory of Numbers, Univ. of Chicago Press, 1939.
3. B. Jones and G. Pall, Regular and semi-regular positive ternary quadratic forms, Acta Math. 70 (1939), 165-191.
4. I. Kaplansky, Ternary positive quadratic forms that represent all odd positive integers, to appear in Acta Arithm.