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# A third genus of regular ternary forms

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In [4] I followed up a question raised by Hsia [2] and exhibited a second genus of positive definite ternary quadratic forms in which both forms are regular. I have now located a third. The genus in question is the first one of discriminant 108 in [1], consisting of

$$f = x^2 + 4y^2 + 7z^2 + xz, \qquad \left\{ \begin{matrix} 1 & 4 & 7 & 0 & 1 & 0 \\ 1 & 5 & 7 & 5 & 0 & 1 \end{matrix} \right\}$$

$$g = x^2 + 5y^2 + 7z^2 + xy + 5yz.$$

The integers not represented by the genus are as follows: those exactly divisible by 3, those of the form $4n + 2$, and those of the form $9^k(9n + 6)$. All others I call eligible. Any eligible integer can be written as $p^2 + q^2 + 3r^2$, since the numbers so representable are precisely those not of the form $9^k(3n + 6)$.

Before proceeding to the proofs that f and g both represent all eligible integers I present four lemmas that exhibit special properties of the binary form $x^2 + 3y^2$.

Lemma 1. If a and b are odd then $a^2 + 3b^2$ can be written $c^2 + 3d^2$ with c and d even.

Proof. We have

$$a^2 + 3b^2 = \left(\frac{a \pm 3b}{2}\right)^2 + 3\left(\frac{a \mp b}{2}\right)^2.$$

By the appropriate choice of sign we make $(a \mp b)/2$ even and then $(a \pm 3b)/2$ is also even.

Lemma 2. Suppose that $t = a^2 + 3b^2$ with a and b even and that $t/4$ is odd. Then t can be written $c^2 + 3d^2$ with c and d odd.

Remark. The hypothesis that $t/4$ is odd cannot be deleted: try $t = 16.$ *

Proof. Write $a = 2a^*$, $b = 2b^*$. Then $a^*$ and $b^*$ must have opposite parities. Take $c = a^* + 3b^*$, $d = a^* - b^*$.

Lemma 3. If a and b are both prime to 3 then $4(a^2 + 3b^2)$ can be written $c^2 + 27d^2$.

Proof. We have

$$4(a^2 + 3b^2) = (a \pm 3b)^2 + 3(a \mp b)^2.$$

The appropriate choice of sign will make $a \mp b$ divisible by 3.

Lemma 4. Suppose that a is prime to 3. Then $4(a^2 + 3b^2)$ can be written $c^2 + 3d^2$ with c and d both prime to 3.

Proof. It is not possible for $a + b$ and $a - b$ both to be divisible by 3. By changing the sign of b, if necessary, we make $a - b$ prime to 3. Of course $a + 3b$ is also prime to 3. Take $c = a + 3b$, $d = a - b$.

Proof that f is regular

Lemma 5. An integer A is represented by f if and only if 4A is represented by $u^2 + 16v^2 + 27w^2$.

Proof. We have $4f = (2x + z)^2 + 16y^2 + 27z^2$, so that the "only if" part is immediate. For the converse, assume $4A = u^2 + 16v^2 + 27w^2$. Then u and w must have the same parity. Set $z = w$, $y = v$, $x = (u - w)/2$.

We now assume that A is eligible and proceed to the proof that f represents A. The proof splits into several cases and subcases.

I. Assume that A is prime to 3 and expressible as $p^2 + q^2 + 3r^2$ with p, q, and r all odd. It is not possible for p and q both to be divisible by 3. Say p is prime to 3. Since q and r are

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* In fact, the condition is necessary

both odd we can, by Lemma 1, make a switch and assume that q and r are even. If r is divisible by 3 we are ready to multiply by 4 and cite Lemma 5. So assume that r is prime to 3. We multiply by 4 and apply Lemma 3 to $p^2 + 3r^2$ to reach our goal.

II. Assume that A is prime to 3 and representable as $p^2 + q^2 + 27r^2$. If p or q is even we are ready for multiplication by 4. So we amy assume that p and q are both odd. If r is even we have $A \equiv 2 \pmod 4$, contradicting the eligibility of A. Thus r is odd. We cite Case I.

III. Assume A is odd, prime to 3, and not representable as $p^2 + q^2 + 3r^2$ with p, q, and r odd. Then when we write $A = p^2 + q^2 + 3r^2$ two of p, q, and r must be even and the other odd.

Subcase (a). Suppose that r is the odd one. At least one of p and q is prime to 3; say p. If r is divisible by 3 we are ready for multiplication by 4; so we assume r prime to 3. We multiply by 4 and apply Lemma 3 to $p^2 + 3r^2$.

Subcase (b). Here r is even and p and q have opposite parities; say p is even and q is odd. If 3 divides r we cite Case II. So r can be assumed prime to 3.

Subsubcase $A \equiv 2 \pmod 3$. Here p and q are both prime to 3. We multiply by 4 and apply Lemma 3 to $q^2 + 3r^2$.

Subsubcase $A \equiv 1 \pmod 3$. Exactly one of p and q is divisible by 3. If that one is p we multiply by 4 and apply Lemma 3 to $q^2 + 3r^2$. So p can be assumed prime to 3. We have $p^2 + 3r^2 = 4[(p/2)^2 + 3(r/2)^2]$. Since p/2 and r/2 are both prime to 3 we can again use Lemma 3 to recast $p^2 + 3r^2$ in the form $c^2 + 27d^2$. That puts us in Case II.

At this point we have finished the case where A is odd and prime to 3. The rest of the argument is short.

IV. Assume A is prime to 3 and divisible by 4, $A = 4B$. B inherits the property of being expressible as $p^2 + q^2 + 3r^2$. We will be multiplying by 16 to go from B to 4A, so our only concern is to acquire a coefficient 27. If r is divisible by 3, we are done. If r is prime to 3 we note that p or q is prime to 3 and use Lemma 3.

V. Assume that A is divisible by 3 (and therefore by 9 by the eligibility of A). Write $A = 9C$. C inherits the property of being expressible as $p^2 + q^2 + 3r^2$. By Lemma 1 we can arrange that p (or q) is even. Multiplication of C by 36 to reach 4A completes the task.

Proof that g is regular

Lemma 6. An integer A is represented by g if and only if 4A is representable by $u^2 + 3v^2 + 16w^2$ with $v^2 \equiv w^2 \pmod 3$.

Proof. We make a change of basis in g, replacing z by z - y. After changing the -9yz that arises to 9yz we have the equivalent form $g^* = x^2 + 7y^2 + 7z^2 + xy + 9yz$. Now

$$4g^* = (2x + y)^2 + 3(3y + 2z)^2 + 16z^2.$$

When can we solve $2x + y = u$, $3y + 2z = v$, $z = w$ for x, y, and z? We need $v^2 \equiv w^2 \pmod 3$. After changing the sign of w (if necessary) this enables us to solve for y. Then to solve for x we need that y and u have the same parity. Now y and v have the same parity, so the final required condition is that u and v have the same parity. This is evident from $u^2 + 3v^2 + 16w^2 = 4A$.

For brevity we are going to say that v "agrees" with w if $v^2 \equiv w^2 \pmod 3$; in detail, this means that either v and w are both divisible by 3 or they are both prime to 3.

The following will come up several times: after writing $A = p^2 + q^2 + 3r^2$, success is achieved if we have that p (or q) is even and agrees with r; just mutiply by 4.

In Cases I and II, A is odd and prime to 3 and we write $A = p^2 + q^2 + 3r^2$, using Lemma 1 to assure that exactly one of p, q, r is odd.

I. The odd one is r (and hence p and q are even).

Subcase (a). p or q agrees with r. By the immediately preceding remark we are done.

Subcase (b). p and q both disagree with r. Note that this cannot happen if r is prime to 3 (for then p and q are divisible by 3, contradicting the assumption that A is prime to 3). So r is divisible by 3 and p and q are prime to 3. We use Lemma 4 to arrange that $4(q^2 + 3r^2) = v \;\; + 3w$ with w prime to 3. Multiply by 4.

II. The odd one is not r.

Subcase (a). $A \equiv 2 \pmod 3$. Then p and q re both prime to 3. Let us say that p is even and q odd. We finish this exactly as in Subcase (b) of I.

Subcase (b). $A \equiv 1 \pmod 3$ but $A \not\equiv 1 \pmod 8$. Again let us take p even and q odd. If p and r agree we are done. So assume that they disagree.

Subsubcase (i). p is divisible by 3 and r is prime to 3. Note that q is prime to 3, since $A \equiv 1 \pmod 3$. We use Lemma 3 to arrange $4(q^2 + 3r^2) = v^2 + 3w^2$ with w divisible by 3. Multiply by 4.

Subsubcase (ii). p is prime to 3 and r is divisible by 3. Recall that r is even. If $(p^2 + 3r^2)/4$ is even we have $A \equiv 1 \pmod 8$, contrary to our assumption. So $(p^2 + 3r^2)/4$ is odd. We now apply Lemma 2. To simplify notation, I shall continue to write p and r for the revised versions. To summarize: p, q, and r are all odd, p is prime to 3, q is divisible by 3 and r is prime to 3. We make a second switch, this time on q and r, using Lemma 1, and again change notation. We have achieved the following: p is odd and prime to 3, q is even and divisible by 3, r is even and prime to 3. We are ready for multiplication by 4, using Lemma 3 on $4(p^2 + 3r^2)$.

III. $A \equiv 1 \pmod{24}$. I quote Theorem 5 on page 177 of [3] to arrange $A = p^2 + q^2 + 3r^2$ with p divisible by 6 (the case where A is a square can of course be ignored). Note that q is necessarily prime to 3. We multiply by 4, using Lemma 3 on $4(q^2 + 3r^2)$.

Exactly as at the foot of page 3 we are done with A odd and prime to 3 and the rest of the argumemt is easy.

IV. Assume A is prime to 3 and divisible by 4, $A = 4B$, $B = p^2 + q^2 + 3r^2$. It suffices to arrange that p or q agrees with r. This fails only if p and q are prime to 3 and r is divisible by 3. To overcome this use Lemma 4.

V. Assume that A is divisible by 9, $A = 9C$, $C = p^2 + q^2 + 3r^2$. Via Lemma 1 arrange that p or q is even.

1. Brandt and Intrau

2. Hsia, Mathematika 28

3. Jones and Pall, Acta 70

4. IK, Mathematika, to appear