A SECOND GENUS OF REGULAR TERNARY FORMS

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§1. *Introduction.* In the paper [**2**] Hsia noted that the forms $x^2+xy+y^2+9z^2$ and $x^2+3y^2+3yz+3z^2$ constitute a genus and that both forms are regular; he asked whether there exist any other genera containing two or more regular forms. In this note it is proved that the forms

$$f=x^2+y^2+7z^2+yz \quad \text{and} \quad g=x^2+2y^2+4z^2+xy+yz$$

are regular. They constitute a genus with discriminant 27 (in the normalization used by Brandt and Intrau in [**1**]). It is noteworthy that Hsia's genus has the same discriminant.

§2. *Regularity of f.* There is no systematic method for checking whether a ternary form is regular, but there do exist devices that have proved useful. The device used here is to diagonalize a suitable scalar multiple of the form, make a connection with the form $h=x^2+y^2+27z^2$, and then work back to the original form. The two proofs are elementary.

We begin by noting that the integers forbidden to the genus are those exactly divisible by 3, together with those of the form $9^k(9n+6)$. All other integers (we call them eligible) are represented by f or g, and our task is to prove that any eligible integer is actually represented by both f and g.

The integers $9^k(9n+6)$ are precisely those not represented by $x^2+y^2+3z^2$. It follows that any eligible integer is represented by $x^2+y^2+3z^2$.

LEMMA 1. *Let A be a positive integer. Then f represents A if, and only if, h represents $4A$.*

Proof. We have

$$4f=4x^2+(2y+z)^2+27z^2. \tag{1}$$

The implication in the "only if" direction is immediate. Conversely, suppose that $u^2+v^2+27w^2=4A$. It cannot be the case that u and v are both odd for then $u^2+v^2\equiv 2 \pmod 4$, w is even, and we get a contradiction. Say u is even. To make the desired identification with (1) we need to find x, y, z satisfying $2x=u$, $2y+z=v$, $z=w$. This can be done as soon as we note that v and w have the same parity.

THEOREM 1. *The form $f=x^2+y^2+7z^2+yz$ is regular.*

Proof. By Lemma 1 it suffices to prove that if A is eligible then h represents $4A$. We first note that A can be written $p^2+q^2+3r^2$, as remarked above. If r

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is divisible by 3 we are done (we even have that A is represented by h and multiply by 4). So assume that r is prime to 3. It cannot be the case that both p and q are divisible by 3 for then $4A=p^2+q^2+3r^2$ is exactly divisible by 3 and so is A, contrary to the assumption that A is eligible. Say q is prime to 3. We now use a well known trick. We have

$$4(q^2+3r^2)=(q\pm 3r)^2+3(q\mp r)^2.$$

Pick the sign so as to make $q\mp r$ divisible by 3. Then

$$4A=4p^2+(q\pm 3r)^2+3(q\mp r)^2$$

gives us what is needed.

§3. *Regularity of g.* Three lemmas precede the proof of Theorem 2.

LEMMA 2. *Let A be a positive integer. Then g represents A if, and only if, 16A can be written as* $p^2+q^2+27r^2$ *with* $q\equiv r$ (mod 8).

Proof. We have

$$16g=4(2x+y)^2+(y+8z)^2+27y^2. \tag{2}$$

The implication in the "only if" direction is immediate. To handle the "if" direction we begin by noting that p is even, say $p=2t$. From $q\equiv r$ (mod 8) we deduce $q^2\equiv r^2$ (mod 16). Thus $4t^2+28r^2$ is divisible by 16, t^2+7r^2 is divisible by 4, whence t and r have the same parity. We set $y=r$, $z=(q-r)/8$, $x=(t-r)/2$ and find from (2) that $16g=16A$.

LEMMA 3. *If* $p^2+q^2+27r^2=16A$ *with r odd then g represents A.*

Proof. It is not possible for both p and q to be even. Say q is odd. We claim that either $q+r$ or $q-r$ is divisible by 8; by Lemma 2 this provides what is needed. The alternative is that either $q+3r$ or $q-3r$ is divisible by 8. From this one deduces that $q^2\equiv 9r^2$, $q^2+27r^2\equiv 36r^2$, both mod 16. Since $r^2\equiv 1$ or 9 (mod 16) we get $q^2+27r^2\equiv 4$ (mod 16). Note that p is even, so that $p^2\equiv 0$ or 4 (mod 16). Then $p^2+q^2+27r^2\equiv 4$ or 8 (mod 16), a contradiction.

We move down to $p^2+q^2+27r^2=4A$ and find an analogous result.

LEMMA 4. *If* $p^2+q^2+27r^2=4A$ *with r odd then g represents A.*

Proof. Again either p or q is odd, say q. By a change of sign, if necessary, we arrange $q\equiv r$ (mod 4). In

$$(2p)^2+(2q)^2+27(2r)^2=16A$$

we have $2q\equiv 2r$ (mod 8) and cite Lemma 2.

THEOREM 2. *The form* $g=x^2+2y^2+4z^2+xy+yz$ *is regular.*

Proof. Let A be an eligible integer. Again we begin by writing $A = u^2+v^2+3w^2$. There is a major case distinction depending on whether w is divisible by 3, and then several subcases.

Case I. $3|w$. Write $w=3s$. If v and w have the same parity then v and s also have the same parity. We note that in

$$16A=(4u)^2+(4v)^2+27(4s)^2$$

we have $4v \equiv 4s \pmod 8$. We cite Lemma 2. Thus in the rest of Case I we may assume that u and v both differ in parity from w.

Case Ia. $3|v$. Then

$$4(v^2+3w^2)=(v+3w)^2+3(v-w)^2 \tag{3}$$

and so

$$4A=(2u)^2+(v+3w)^2+27[(v-w)/3]^2$$

with $(v-w)/3$ odd. We cite Lemma 4.

Case Ib. u and v prime to 3. Again we use the identity (3) and find

$$4A=(2u)^2+(v+3w)^2+3(v-w)^2.$$

This means that $4A=p^2+q^2+3r^2$ with p, q, r all prime to 3, p even, and q and r odd. Arrange that $p-r$ is divisible by 3 and use the basic trick once more, getting

$$16A=4q^2+(p+3r)^2+27[(p-r)/3]^2.$$

Here $(p-r)/3$ is odd and Lemma 3 applies.

We are finished with Case I. We are also thereby done with $3|A$. For if $3|A$ then $9|A$ (since A is eligible), and $A=u^2+v^2+3w^2$ forces $3|u$ and $3|v$ and then $3|w$.

Case II. w prime to 3. Since A is not divisible by 3 we can assume that v is prime to 3 and then (by a change of sign) that $v-w$ is divisible by 3.

Case IIa. v and w differ in parity. We use the basic trick yet again, getting

$$4A=(2u)^2+(v+3w)^2+27[(v-w)/3]^2.$$

Here $(v-w)/3$ is odd and Lemma 4 applies.

Case IIb. v and w have the same parity. We have

$$A=u^2+[(v+3w)/2]^2+3[(v-w)/2]^2.$$

This reverts the argument to Case I. The proof of Theorem 2 is complete.

Added in proof (October 16, 1995). There is a third genus consisting of two regular forms: the forms are $x^2+4y^2+7z^2+xz$ and $x^2+5y^2+7z^2+xy+5yz$

with discriminant 108. I have no plans to submit this for publication. A preprint containing the two proofs of regularity is available on request.

References

1. H. Brandt and O. Intrau. Tabelle reduzierten positiver ternärer quadratischer Formen. *Abh. Sächs. Akad. Wiss. Math.-Nat. Kl.*, 45 (1958), no. 4. MR 21, 11493.
2. J. S. Hsia. Regular positive ternary quadratic forms. *Mathematika*, 28 (1981), 231–238.

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