IK, Dec. 1996

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Notes on Ternary Forms

I. Near misses relative to odd numbers

1. Introduction. This is the first of a series of notes, intended for limited distribution. Whether any of these will be polished and submitted for publication is problematical. At present the idea is to try to ensure that things of permanent value here (if any) do not vanish identically and hopefully will be available to investigators who go further some day.

I begin with an autobiographical note. When in the Fall of 1992 I ended my term as MSRI Director I decided to return to a childhood love: elementary number theory. The idea was to try to fill in some of the numerous gaps in my knowledge. I started by going back to Dickson, for whom I have the utmost admiration. His book "Studies in the Theory of Numbers" is mostly about ternary forms; I found it somewhat inpenetrable. Much better was the portion on ternary forms in his last book "Modern Elementary Theory of Numbers". In fact, I fell in love with ternary forms and haven't recovered yet. The subject had just the right combination of accessible facts and challenges where some progress by elementary methods looked possible.

Starting in the Fall of 1994 I was fortunate in having Will Jagy as a collaborator. He wrote a substantial number of dandy programs and contributed many ideas to the theory. Starting in December, 1995, tables furnished by Alexander Schiemann were vital in the classification of regular forms.

Final remark: in the Fall of 1996 I briefly studied indefinite ternaries. Correspodence with Andrew Earnest convinced me that my ignorance of spinor genera was too big a handicap, so I aborted the project. Thus: in these notes all forms are positive definite.

2. Background. In [3] I executed the project of finding the ternaries that represent all odd positive integers. As explained there, three forms accidentally showed up that seemed to miss exactly one odd number. I cautiously noted that "there may be others". Partly because this aroused my curiosity, and partly because of the similarity to Halmos's admirable Jugendarbeit [1], I looked into the matter and have now found all of them. The answer is that there are 41 more, for a total of 44. More honestly, there are at most 44: one for sure and 43 plausible candidates. This is a subject where, at present, proofs are rarer than hens' teeth.

Two things were needed to carry out this investigation: a priori bounds for the discriminant and an effective program written by Jagy. I proceed to explain the bounds.

3. A priori bounds. The idea is simple. Let f be an odd form with discriminant D. Suppose that the form f represents a, b, c at vectors u, v, w that are linearly independent over the integers. We first note that the submodule B spanned by u, v, w may not be all of the free 3-dimensional Z-module on which f operates. But this works in our favor. Write g for the restriction of f to B. The discriminant of g has the form (s^2)D. So a bound for (s^2)D is, a fortiori, a bound for D. The doubled up matrix attached to g has the form

$$\begin{pmatrix} 2a & ? & ? \\ ? & 2b & ? \\ ? & ? & 2c \end{pmatrix}.$$

A known theorem on the determinant of a positive definite symmetric matrix qsserts that is is bounded by the product of the diagonal elements. So the determinant of the above matrix is bounded by 8abc. This has to be divided by 2, after which we get the bound 4abc for D.

We proceed to the problem at hand. The worst scenario is where the "forgiven" number is l. We have the following estimate.

Proposition. Let D be the discriminant of an odd form (posititve definite ternary, of course) that represents 3, 5, 7, 9, and 11. Then $D \leq 660$.

Proof. Take u and v with f(u) = 3, f(v) = 5, where f is the given form. Necessarily u and v are linearly independent. The restriction of f to the submodule spanned by u and v has the form h = 3(x^2) + rxy + 5(y^2). We can take r = 0. Positive definiteness implies that $r \leq 7$. Thus there are 8 cases to handle: r = 0, 1, 2, ..., 6, 7. We take w with f(w) = 7, 9, or 11. We need to have u, v, and w linearly independent. Failure means that h represents s^2(7, 9, or 11) for some nonzero s. The following assertions meet the need.

3(x^2) + rxy + 5(y^2) does not represent 7(s^2) for r = 0, 2, 3, 4, 6, 7,

3(x^2) + xy + 5(y^2) does not represent 11(s^2),

3(x^2) + 5xy + 5(y^2) does not represent 9(s^2).

Proofs of these assertions are routine exercises; I leave them to the reader. In the worst case we get the estimate 4.3.5.11 = 660.

Remarks. (a) I likewise leave to th reader handling the remaining cases for odd forms and verifying that 660 (indeed, a smaller bound) always works.

(b) When the missing number is larger than 7 we have the following result: an odd form that represents 1, 3, 5, and 7 has discriminant ≤ 77. This bound is the one that came up in [3], 4

but it was not explicitly mentioned there. Incidentally, 77 is best possible, as witnessed by the form x^2 + xy + 3(y^2) + 7(z^2). For even forms representing 1,3,5,7; disc ≤ 15, Max achieved: $x^2+3y^2+5z^2$.

(c) For even forms a simpler version of the argument yields the bound 105.

This 105 is for even forms representing {3,5,7}. Here 9,11 are ignored.

4. The computation. Jagy wrote a suitable program. On odd forms it tested all of them in the available table -- up to discriminant 1000, going well beyond 660 to play it safe. Many forms could have been deleted a priori, but it was simpler to test them all. The run on even forms went to discriminant 150, beyond the bound of 105. The 44 that survived are listed in the table.

In conclusion there are four remarks.

(a) As remarked above, at present there is only one proof! This is for number 8, and is due to Jagy [2].

(b) All forms were verified on odd numbers up to at least 5000.

(c) There are two interconnections that have been proved. I will give the details in a later note. (i) Numbers (1) and (3) represent exactly the same numbers (even or odd). (ii) Number 15 represents exactly the same odd numbers as number 1 (or, equivalently, number 3). Thus, only 41 proofs remain to be found. At present, I know of no other interconnections.

(d) I thank Alexander Schiemann, who sent me his version of the Brandt-Intrau tables. Having this available on line was a great convenience.

References

1. P. R. Halmos, Note on almost-universal forms, Bull. Amer. Math. Soc. 44(1938), 141-144.

2. W. C. Jagy, Five regular or nearly-regular ternary quadratic forms, to appear in Acta Arithmetica.

3. I. Kaplansky, Ternary positive quadratic forms that represent all odd positive integers, Acta Arithmetica, 70(1995), 209-214.

Near misses relative to odd numbers

No.	Disc.	No. missed	Form
			Even
1.	26	5	1 3 9 2 0 0
2.	46	1	3 4 5 -2 2 2
			Odd
3.	26	5	1 1 9 1 1 1
4.	34	13	1 2 5 2 1 0
5.	42	1	2 3 3 3 2 2
6.	54	17	1 3 5 1 0 1
7.	58	1	2 3 3 1 0 2
8.	72	1	3 3 3 1 2 3
9.	78	1	3 3 3 1 1 3
10.	82	1	2 3 5 3 2 2
1.	86	3	1 5 5 3 0 1
2.	88	1	3 3 3 -1 1 2
3.	94	7	1 3 9 2 1 1
4.	102	1	3 3 3 0 1 1
5.	104	5	1 3 9 1 1 0
6.	106	7	1 3 10 2 0 1
7.	106	1	2 3 5 1 2 0
8.	118	1	2 3 5 1 1 0

19.	120	7	1	3	11	1	0	1
20.	126	3	1	5	7	3	1	0
21.	128	1	3	3	4	0	2	1
22.	142	1	3	3	5	2	3	1
23.	158	1	3	3	5	-1	2	1
24.	160	1	3	4	5	4	3	2
25.	162	1	3	3	5	1	2	1
26.	184	1	3	4	5	2	3	2
27.	190	1	3	5	5	5	2	3
28.	198	1	3	3	6	0	2	1
29.	200	1	3	4	5	-2	1	2
30.	214	1	3	5	5	4	1	3
31.	218	1	3	3	7	1	3	1
32.	226	1	3	5	5	3	2	3
33.	232	1	3	3	7	1	2	1
34.	232	1	3	5	5	3	1	3
35.	242	1	3	3	7	0	1	1
36.	278	1	3	5	6	-4	2	1
37.	282	1	3	3	9	2	3	1
38.	286	1	3	5	6.	2	2	3
39.	288	1	3	3	9	1	3	1

40.	288	1	3	4	7	-2	1	2
41.	296	1	3	4	7	2	1	2
42.	302	1	3	5	7	5	3	2
43.	328	1	3	5	7	3	2	3
44.	354	1	3	5	7	-3	1	2

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II. Near misses relative to all numbers

My interest in the near misses discussed in this note was stimulated by two ternary forms treated by Jones and Pall in [3]: they proved that 4 8 9 0 4 0 and 8 12 21 12 0 0 represent all eligible integers with a single exception. These are numbers 54a and 89a in the table of strong near misses below. A second stimulus arose from a remark by Watson at the end of his thesis [4]: "The method can be extended to four or more variables, or to forms which are regular with one exception; but in either case the calculations are formidable." I have no thoughts at present about higher dimensional forms, but I agree that ternaries admitting just one exception can be classified by Watson's methods and that this will require much more work than the classification of regular ternary forms achieved in [2]. I do not plan to undertake the job at this time. Being unwilling to forget about it entirely, I decided to exhibit a partial list, obtained by using tools and tables currently available.

Before describing what I have done I make a distinction between "strong" and "weak" near misses. By a strong near miss I mean just what was mentioned above: a form that misses exactly one eligible integer. Now there exist numerous forms with the following property: for a certain prime p (p = 2 is OK) the form represents an integer A if and only if it represents (p^2)A. Such a form cannot be a strong near miss: as soon as it misses A it misses all (p^2n)A. So I call f a weak near miss if, for a suitable prime p and a suitable eligible integer A, f represents all eligible integers except for the set {(p^2n)A}.

Now for a word about how the list was compiled. The program with command word "comp", written by Jagy for the work in [2], is well adapted for the purpose. The input is a pair f, g of forms and a preset maximum target T. The output gives all numbers $\leqq$ T whcih are represented by f(resp. g) but not by g (resp. f). For strong near misses we exclude f as soon as we find g in its genus with the output for f having at least two numbers. For weak near misses we need more: numbers a and b in the output (a < b) such that b/a is not an even power of a prime.

There was a complete search of the Brandt-Intrau tables, in the version sent to me by Schiemann. These tables go up to discriminant 1000 for odd forms and 250 for even forms. Beyond that, there was an additional search of most of the discriminants furnished by Schiemann in connection with [2]. These discriminants are as follows.

Odd: 1014, 1029, 1058, 1080, 1089, 1125, 1134, 1176, 1188, 1200, 1215, 1250, 1296, 1323, 1331, 1350, 1452, 1458, 1500, 1512, 1620, 1750, 1764, 1800, 1875, 1944, 2000, 2025, 2058, 2106, 2160, 2197, 2250, 2430, 2450, 2646, 2662, 2700, 2744, 3000.

Even: 252, 256, 270, 288, 294, 300, 320, 324, 336, 350, 360, 368, 378, 384, 392, 396, 400, 432, 448, 450, 480, 486, 490, 500, 504, 512, 540, 560, 576, 588, 600, 624, 640, 648, 672, 676, 686, 702, 720, 736, 750, 756, 768, 784, 810, 832, 864, 896, 900, 960, 972, 1000, 1008, 1024, 1029, 1080, 1120, 1125, 1134, 1152, 1188, 1200, 1225, 1248, 1280, 1296, 1344, 1350, 1440 ,1452, 1500, 1521, 1536, 1600, 1620, 1728, 1764, 1792, 1800, 1872, 1920, 1936, 2000, 2048, 2112, 2160, 2304, 2352, 2560, 2592, 2700, 2800, 2808, 2816, 2880, 3000, 3024, 3072, 3136, 3240, 3456, 3600, 3840, 3888, 3920, 4000, 4032, 4050, 4056, 4096, 4224, 4232, 4320, 4356, 4500, 4536, 4608, 4704, 4752, 4800, 5120, 5184, 5292, 5376, 5400, 5488, 5600, 5616, 5760, 5808, 5888, 6000, 6048, 6144, 6174, 6272, 6336, 6400, 6480, 6912, 7056, 7168, 7200, 7680, 7840, 7938, 8000.

I conclude with some remarks concerning the tables.

(a) There are 117 strong near misses (note the 5 interpolations) of which 35 are odd and 82 even. There are 186 weak near misses of which 85 are odd and 101 even. The grand total is 303.

(b) All forms were verified to at least 1000; the majority were verified considerably higher.

(c) There is some overlap between these near misses and the near misses with respect to odd numbers of Note I: on the strong list numbers 1,2,3,4,5,8,39; on the weak list numbers 9, 15, 25, 27, 30, 38.

(d) On the strong list there are no square-free discriminants. If this turns out to be provable, the classification of strong near misses will get off to a flying start.

(e) For the 303 forms only 8 proofs exist in addition to the two mentioned in the opening sentence: numbers 1, 19, 36, 37, 38, 44a on the strong list and numbers 7, 87 on the weak list. We need 293 proofs! Proofs for 1, 36, and 87 are in [1]. Numbers 37, 38 are readily deducible from 36; 7 is readily deducible from 87. I proved 19 and 44a; these proofs are not yet written up. Numbers 1, 19, and 648 are mates of regular forms, as are several others on the strong and weak lists.

References

1. W. C. Jagy, Five regular or nearly-regular ternary quadratic forms, Acta Arithmetica 70(1996), 361-367.

2. W. C. Jagy, I. Kaplansky, and A. Schiemann, There are 913 regular ternary forms, submitted to Mathematika.

3. B. W. Jones and G. Pall, Regular and semi-regular positive ternary quadratic forms, Acta Math. 70(1939), 165-191.

4. G. L. Watson, Some problems in the theory of numbers, Ph. D. thesis, University of London, 1953.

Strong near (5) misses

Odd

No.	Disc.	No. missed	Form	WCJ
1.	72	1	3 3 3 1 2 3	
2.	88	1	3 3 3 -1 1 2	
3.	120	7	1 3 11 1 0 1	
4.	128	1	3 3 4 0 2 1	
5.	160	1	3 4 5 4 3 2	
6.	162	29	2 2 11 1 1 1	
7.	162	29	2 5 5 4 1 1	
8.	184	1	3 4 5 2 3 2	
9.	200	1	3 3 7 1 3 2	
10.	243	1	2 4 8 1 1 1	
11.	288	13	1 5 16 4 0 1	
12.	288	1	4 5 5 3 2 4	
13.	324	1	4 4 6 0 3 2	
14.	360	21	3 5 8 4 0 3	
15.	486	87	2 5 14 5 2 1	
16.	504	1	4 5 7 -1 2 2	
17.	567	1	4 6 7 3 2 3	
18.	625	2	3 7 8 2 1 2	
19.	648	1	4 7 7 5 2 2	: KAP
20.	648	1	4 7 7 1 2 4	

			(6)					
21.	729	1	4	6	9	0	3	3
22.	800	5	3	7	12	6	2	3
23.	972	5	3	8	11	2	3	0
24.	1000	19	1	9	29	3	1	1
25.	1000	1	4	9	9	-7	2	2
26.	1080	63	5	5	11	1	1	1
27.	1125	1	4	9	11	9	4	3
28.	1188	7	1	13	24	6	0	1
29.	1215	2	5	8	9	3	3	4
30.	1944	1	4	7	19	1	4	2
31.	2000	3	7	7	12	4	6	1
32.	2000	1	4	9	15	5	0	2
33	2025	1	4	9	16	6	1	3
34.	2744	1	4	9	21	7	0	2
35.	2744	3	5	12	13	6	1	4

even STRONG NEAR MISSES

(7)

36.	18	1	2	2	5	0	2	0	: WCJ
37.	36	2	1	4	10	4	0	0	} Follow from WCJ
38.	36	2	1	4	9				
39.	46	1	3	4	5	-2	2	2	
39a.	64	1	4	5	5	4	0	4	
40.	72	1	4	4	5	2	2	0	
41.	72	4	2	7	7	6	2	2	
42.	96	13	3	3	11	0	2	0	
43.	96	47	3	6	6	0	2	2	
44.	96	1	4	5	6	2	4	0	: KAP
44a.	108	1	4	5	8	4	4	4	
45.	126	1	4	5	7	2	2	2	
46.	128	2	5	6	6	4	4	4	
7.	128	2	3	7	7	4	0	2	
8.	162	1	4	6	7	0	2	0	
9.	162	1	4	7	7	-4	2	2	
0.	184	4	3	6	11	2	0	2	
1.	192	94	3	6	12	4	0	2	
2.	192	2	3	7	10	2	2	2	
2a.	243	4k.6	2	11	14	10	2	2	
3.	252	2	5	7	8	4	0	2	
4.	252	2	7	7	7	0	4	6	: Jones + PaH
a.	256	1	4	8	9	0	4	0	
[illegible]	256	3	5	6	10	4	4	0	
[illegible]	288	3	7	7	7	2	4	4	
7	288	21	1	9	33	6	0	0	
8.	288	4	7	7	8	4	4	6	
9.	324	2	5	8	11	8	2	4	
0.	324	2	5	8	9	0	0	4	
1.	324	2	5	5	14	-2	2	2	

STRONG

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62.	324	1	4	7	13	-4	2	2
63.	368	11	3	7	19	-2	2	2
64.	432	13	1	21	21	6	0	0
65.	432	1	4	9	13	0	4	0
6.	504	4	7	7	11	2	4	0
7.	512	161	1	16	32			
8.	576	6	5	9	14	6	2	0
9.	576	8	4	5	37	2	4	4
0.	576	8	4	11	16	8	0	4
1.	576	8	1	16	40	16	0	0
2.	648	1	4	7	25	-4	2	2
.	648	4	7	7	16	-4	4	4
.	648	4	7	10	10	-4	2	2
.	~~7??~~ 832	41	3	11	27	6	2	2
.	864	63	3	11	27	6	0	0
.	864	9	5	14	14	-8	2	2
.	864	13	1	24	36	0	0	0
.	864	26	2	11	42	6	0	2
.	864	2	8	11	11	4	4	4
1.	1024	3	8	11	12	4	0	0
.	1024	4	3	11	35	-10	2	2
	1024	3	11	11	11	-2	2	10
.	1152	4	7	11	16	4	4	2
	1296	5	8	12	17	12	4	0
	1296	8	5	14	20	-4	4	2
7	1620	47	8	11	20	4	4	4
	1728	18	7	10	27	6	6	2
	1728	77	4	5	109	2	4	4 : Jones + Pall
	1728	5	8	12	21	12	0	0
	1729	4	5	13	29	10	2	2

1.	1764	14	5	10	38	-8	2	2
2.	2048	5	12	13	20	12	8	12
3.	2048	8	7	15	23	14	2	2
4.	2048	~~473~~ 3	11	12	19	-4	10	4
5.	2304	473	1	16	144			
6.	2304	35	3	11	75	-10	2	2
7.	2592	1	7	16	27	12	6	4
8.	2592	233	5	14	38	4	2	2
9.	2592	5	14	17	17	14	10	14
0.	4032	8	11	19	20	-4	4	2
1.	4032	8	7	20	31	4	6	4
2.	4500	7	8	23	28	16	4	4
3.	4500	4	11	16	31	-12	2	8
4.	4608	16	11	19	27	18	6	2
5.	4608	16	11	16	31	-12	2	8
6.	5184	8	5	29	44	-28	4	2
7.	5184	8	11	20	27	-12	6	4
8.	5184	8	9	17	41	-14	6	6
9.	5184	7	12	16	31	8	12	0
0.	7056	5	13	17	40	16	8	10
.	8000	803	3	27	107	-26	2	2
.	8000	3	12	27	28	-12	8	4

Weak ~~near~~ (10) misses

Odd

No	Disc.	Misses	Form
1.	14	$49^k.5$	1 2 2 0 0 1
2.	23	$23^{ik}.23$	1 2 3 1 0 0
3.	35	$49^k.22$	1 3 4 3 0 1
4.	38	$4^k.14$	1 2 5 0 1 0
5.	39	$9^k.2$	1 3 4 3 0 0
6.	45	$25^k.1$	2 3 3 −3 1 1
7.	50	$4^k.2$	1 3 5 −1 1 1
8.	51	$9^k.2$	1 4 4 3 0 1
1.	58	$4^k.1$	2 3 3 1 0 2
.	60	$25^k.1$	3 3 3 −1 2 3
1.	62	$4^k.26$	1 3 6 2 0 1
	68	$17^{2k}.119$	1 3 7 3 0 1
	75	$9^k.3$	1 2 11 0 1 1
	75	$9^k.2$	1 3 7 3 0 0
	78	$4^k.1$	3 3 3 4 1 3
	84	$9^k.3$	1 5 5 −2 1 1
7.	90	$25^k.5$	1 2 13 1 0 1
	90	$4^k.2$	1 5 5 1 1 1
	93	$9^k.1$	2 3 4 0 1 0
	98	$49^k.35$	1 2 14 1 0 1
1.	112	$49^k.7$	1 3 11 3 1 1
2.	112	$49^k.1$	3 3 4 −2 2 1
3.	117	$9^k.6$	1 3 10 0 1 0

24.	123	$9^k.1$	2	4	5	4	1	1
25.	142	$4^k.1$	3	3	5	2	3	1
26.	153	$9^k.6$	2	2	11	2	2	1
27.	158	$4^k.1$	3	3	5	-1	2	1
28.	162	$4^k.1$	3	3	7	3	3	3
9.	180	$25^k.3$	1	7	7	2	1	1
0.	190	$4^k.1$	3	5	5	5	2	3
1.	192	$9^k.3$	1	5	11	4	1	1
2.	198	$4^k.2$	1	5	11	3	1	1
3.	204	$9^k.8$	1	5	11	1	1	1
4.	225	$9^k.6$	1	3	19	0	1	0
5.	225	$9^k.1$	3	4	6	3	3	0
6.	225	$25^k.5$	2	3	12	3	1	2
7.	225	$25^k.2$	3	3	7	2	2	1
8.	232	$4^k.1$	3	3	7	1	2	1
9.	232	$4^k.1$	3	5	5	3	1	3
0.	234	$4^k.3$	1	7	9	3	0	1
1.	245	$49^k.14$	3	5	5	3	2	2
2.	252	$9^k.1$	3	4	7	4	3	0
3.	279	$9^k.3$	1	6	12	3	0	0
4.	300	$25^k.5$	1	9	9	3	1	1
5.	351	$9^k.2$	3	5	8	4	3	3
6.	369	$9^k.3$	2	5	11	5	2	1
7.	405	$25^k.10$	1	7	16	5	1	1
48.	405	$25^k.1$	4	4	7	-2	2	1
49.	405	$25^k.1$	3	4	9	3	0	0

50,	450	$25^k.1$	4	7	7	5	5	5
51,	450	$4^k.10$	2	5	13	5	2	0
2.	459	$4^k.2$	5	5	6	0	3	4
3,	486	$4^k.2$	3	5	11	5	3	3
4.	486	$4^k.3$	1	7	19	5	1	1
5.	528	$9^k.1$	4	5	7	1	0	2
6,	540	$25^k.9$	3	5	11	5	3	0
7,	567	$49^k.14$	2	5	15	3	0	1
8.	576	$9^k.1$	3	4	13	2	3	0
9.	612	$9^k.24$	3	5	11	1	3	0
0.	648	$4^k.1$	3	7	9	3	0	3
1,	675	$9^k.3$	2	8	11	-2	1	1
2.	675	$9^k.2$	3	8	9	6	0	3
3,	702	$4^k.9$	3	5	14	2	0	3
4,	720	$25^k.1$	4	7	7	3	0	2
5.	756	$9^k.3$	5	5	9	-3	3	2
6.	784	$49^k.7$	1	9	23	4	1	1
7.	784	$49^k.1$	4	7	9	7	2	0
.	810	$25^k.5$	2	8	14	5	2	1
.	810	$4^k.2$	5	5	9	3	3	1
.	810	$4^k.2$	5	5	11	1	2	5
	900	$25^k.5$	2	7	17	4	1	1
2.	900	$25^k.15$	3	5	17	5	3	0
3.	960	$9^k.1$	4	5	13	-1	2	2

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74.	1200	$9^{k}.3$	5	7	12	6	0	5
75.	1250	$4^{k}.2$	3	7	17	1	2	3
76.	1458	$4^{k}.1$	7	9	9	9	3	6
77.	1512	$49^{k}.91$	1	13	31	8	1	1
18.	1512	$49^{k}.1$	4	9	13	9	2	0
9.	1764	$9^{k}.7$	3	13	13	5	3	3
0.	2025	$25^{k}.5$	2	17	17	-11	1	1
1.	2025	$25^{k}.2$	5	8	17	7	5	5
2.	2025	$25^{k}.5$	2	15	17	0	1	0
3.	2700	$25^{k}.45$	1	15	49	15	1	0
4.	3000	$9^{k}.3$	7	12	13	12	3	6
5.	3000	$9^{k}.1$	4	11	19	1	2	4

Even (?) weak near misses

86.	19	$4^k \cdot 7$	1	2	10	2	0	0	
87.	25	$4^k \cdot 1$	2	3	5	2	2	0	: PROVED WCJ paper
88.	29	$4^k \cdot 2$	1	5	6	2	0	0	
9.	31	$4^k \cdot 13$	2	3	6	2	2	0	
?	39	$4^k \cdot 2$	3	3	5	2	2	0	
1.	45	$4^k \cdot 1$	2	5	5	0	0	2	
l.	48	$9^k \cdot 21$	1	1	48				
1.	~~[illegible]~~ 71	$4^k \cdot 2$	3	5	6	4	2	2	
4	79	$4^k \cdot 2$	3	5	6	0	2	2	
~~5~~	80	$25^k \cdot 11$	2	3	16	0	0	2	
6	81	$4^k \cdot 2$	5	5	5	4	4	4	
7.	84	$9^k \cdot 23$	3	4	8	4	0	0	
?.	95	$4^k \cdot 2$	3	5	7	2	0	2	
?.	99	$4^k \cdot 1$	2	7	9	6	0	2	
0.	117	$4^k \cdot 6$	2	5	14	4	2	2	
1.	144	$9^k \cdot 7$	3	3	16				
2.	162	$4^k \cdot 1$	3	7	9	6	0	0	
3.	180	$25^k \cdot 1$	4	7	8	4	4	2	
4.	192	$9^k \cdot 21$	1	4	48				
5.	192	$9^k \cdot 5$	1	12	16				
6.	225	$4^k \cdot 5$	1	10	25	10	0	0	
7.	2~~43~~	$4^k \cdot 6$	2	11	14	10	2	2	
8.	243	$4^k \cdot 1$	6	7	9	6	0	6	

109.	252	$9^k.69$	1	12	24	12	0	0
110.	320	$25^k.35$	3	3	40	0	0	2
111.	320	$25^k.3$	4	8	11	0	4	0
112.	320	$25^k.11$	3	7	16	0	0	2
113.	320	$25^k.15$	3	6	19	2	0	2
114.	324	$4^k.2$	5	6	11	0	2	0
115.	324	$4^k.2$	5	5	17	-2	4	4
116.	384	$9^k.2$	7	8	10	8	6	4
117.	400	$25^k.55$	2	3	80	0	0	2
118.	400	$4^k.4$	3	8	19	8	2	0
119.	432	$9^k.21$	1	9	48			
'20.	432	$9^k.21$	2	5	48	0	0	2
'21.	448	$49^k.1$	4	9	16	8	0	4
22.	576	$9^k.7$	3	12	16			
23.	576	$9^k.15$	3	4	48			
24.	576	$9^k.15$	3	7	28	4	0	0
25.	576	$9^k.3$	7	10	10	8	2	2
26.	624	$4^k.8$	5	12	12	0	4	4
27.	648	$4^k.4$	3	7	31	2	0	0
28.	720	$4^k.4$	7	7	20	-4	4	6
9.	756	$9^k.23$	3	8	32	4	0	0
30.	768	$9^k.21$	4	5	48	0	0	4
31.	768	$9^k.5$	4	13	16	0	0	4
32.	768	$9^k.19$	3	8	32			
33.	768	$9^k.13$	5	5	32	0	0	2

134.	810	$25^k.1$	4	7	31	-4	2	2
35.	900	$25^k.5$	7	8	17	2	4	2
36.	960	$9^k.1$	4	6	17	8	4	0
37.	1152	$9^k.6$	5	14	17	-2	2	2
38.	1280	$25^k.11$	3	16	27	0	2	0
39.	1280	$25^k.35$	3	11	43	-10	2	2
40.	1280	$25^k.3$	11	12	12	-8	4	4
41.	1296	$4^k.8$	11	11	11	-2	2	2
42.	1296	$4^k.8$	5	17	17	8	2	4
43.	1344	$49^k.35$	3	11	43	6	2	2
44.	1536	$9^k.33$	1	16	96			
45.	1600	$25^k.7$	7	7	10	0	0	6
6.	1600	$25^k.15$	8	15	15	10	0	0
7.	1600	$25^k.55$	3	7	80	0	0	2
8.	1600	$25^k.3$	7	12	22	-4	6	4
9.	1620	$25^k.2$	8	14	17	10	4	4
0.	1726	$9^k.21$	5	8	48	0	0	4
1.	1728	$9^k.9$	5	14	29	14	4	2
2.	1728	$9^k.5$	9	14	17	10	6	6
3.	1872	$4^k.24$	8	15	20	12	8	0
4.	2304	$9^k.15$	7	7	48	0	0	2
5.	2304	$9^k.7$	12	15	16	0	0	12
6.	2304	$9^k.39$	4	7	96	0	0	4
7.	2304	$9^k.57$	1	24	96			

58.	2880	$9^k.3$	11	11	32	8	8	10
59.	3072	$9^k.13$	5	20	32	0	0	4
0.	3072	$9^k.21$	5	13	52	-12	4	2
1.	3072	$9^k.5$	13	16	20	16	4	8
2.	3136	$49^k.7$	15	15	16	8	8	2
3.	3240	$25^k.4$	7	16	31	-4	4	4
4.	3600	$4^k.20$	4	11	91	2	4	4
5.	3888	$4^k.24$	8	17	33	6	0	8
6.	3888	$4^k.24$	8	11	47	-2	-4	4
7.	3888	$4^k.1$	9	24	25	24	6	0
8.	4032	$49^k.105$	9	17	32	16	0	6
9.	4032	$9^k.1$	4	25	48	24	0	4
0.	4050	$25^k.5$	11	20	26	20	8	10
1.	4608	$9^k.11$	3	32	48			
2.	4800	$9^k.5$	13	13	32	8	8	6
~~[illegible]~~	~~[illegible]~~	~~$4^k.72$~~	~~13~~	~~13~~	~~37~~	~~10~~	~~10~~	~~2~~
1.	~~64~~5184	$4^k.8$	5	24	44	0	4	0
4	5184	$4^k.8$	11	20	24	0	0	4
5.	5184	$4^k.8$	17	17	20	4	4	10
	5616	$4^k.72$	13	13	37	10	10	2
	6400	$25^k.55$	7	12	80	0	0	4
	6400	$25^k.15$	7	23	47	-22	6	2
	6400	$25^k.7$	15	23	23	14	10	10

180.	6480	$4^k \cdot 4$	7	31	31	-10	2	2
181.	6480	$25^k \cdot 185$	8	17	53	-14	4	4
182.	6480	$4^k \cdot 4$	7	31	39	-30	6	2
183.	6912	$9^k \cdot 19$	3	40	64	32	0	0
184.	6912	$9^k \cdot 13$	12	21	37	16	12	12
185.	6912	$9^k \cdot 21$	5	29	48	0	0	2
186.	6912	$9^k \cdot 5$	20	21	21	12	12	8

Irving Kaplansky
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Notes on Ternary Forms

III. JP (Jones-Pall) forms

What is a JP form? In this note I shall not attempt a precise definition, but (in the famous words of a Supreme Court Justice) I know one when I see it.

In [1] Jones and Pall presented seven examples, each the mate of a diagonal regular form. (There was a slip in describing the integers missed by one of the seven, corrected independently by Lomadze [2] and Schulze-Pillot [3].) The first was f = 2 2 5 2 2 0, the genus mate of the diagonal form 1 1 16. The form f is regular on even numbers. On odd ones f is entitled to all numbers of the form 4n + 1, but it misses an infinite number. Which ones? When I first saw the answer I found it remarkable and I still do. In the first place the (4n + 1)'s missed by f are all squares. Secondly, every prime factor must be of the form 4n + 1. The first ten such numbers are

1, 25, 169, 289, 625, 841, 1369, 1681, 2809, 3721.

Jones and Pall used ≠ m^2 as a notation for this situation. Similarly, ≠ w^2 means that the prime factors must all be of the form 3m + 1. Later it turned out that JP forms can be understood better in terms of spinor genera.

When I carried out the search for near misses reported in Note II I simultaneously searched for JP forms. The result is listed below. These 18 JP forms, together with the 7 original ones, make a total of 25 known thus far.

Remarks. (a) Numbers 1 and 15 are not really new. The device of replacing x^2 + xy + y^2 by x^2 + 3y^2 cariries them into two of the original seven.

(b) By the methods of Jones and Pall I proved that number 2 does what it says, as reported in the appended letter. Other than this, I have not attempted any proofs. Spinor genera method~ will presumably give the best proofs.

(c) The exceptions for number 4 need a new description: the relevant primes are those for which -7 is a quadratic residue.

(d) There exist forms with the following property: they miss the numbers appropriate for a JP form, and some others as well. The appended letter sheds some light on this phenomenon. I have one example worthy of mention: 9 16 32 0 0 8 with discriminant 4096. This seems to miss m^2 and excatly one more number, namely 4 (a different kind of "near miss").

References

1. B. W. Jones and G. Pall, Regular and semi-regular positive ternary quadratic forms, Acta Math. 70(1939), 165-191.

2. G. A. Lomadze, Formulas for the number of representations of certain numbers by certain regular and semi-regular ternary quadratic forms belonging to two-class genera (Russian), Acta Arithmetica 34(1978), 131-162.

3. R. Schulze-Pillot, Darstellung durch Spinorgeschlechter ternärer quadratischer Formen, J. of Number Theory 12 (1980), 529-540.

(3) JP(1939)

225220 – BEHH 3.1
445040 – BEHH 3.5
349000 – BEHH 3.11
499244
4912000
4817040
91648000 – BEHH Prop 3.

JP forms

Odd

No.	Disc.	Form	Designation
1.	108	3 3 4 0 0 3 ✓	ω^2 ✓ BEHH 3.2
2.	108 ✓	3 4 4 4 3 3 ✓	ω^2 ✓, form ≅ 344403, BEHH 3.3
3.	324	1 7 12 0 0 1 ✓	$3\omega^2$ ✓ BEHH 3.7
4.	343	2 7 8 7 1 0 ✓	See remark (c) BEHH 3.8
5.	432	3 7 7 5 3 3 ✓	ω^2, $4\omega^2$ ✓ BEHH 3.10
		Even	
6.	32	1 4 9 4 0 0 ✓	$2m^2$ ✓ BEHH 3.4
7.	64	2 5 8 4 0 2 ✓	m^2, $4m^2$ ✓ BEHH 3.6
8.	108	4 4 9 0 0 4 ✓	ω^2 ✓ BEHH 3.12
9.	256	4 5 13 2 0 0 ✓	m^2 ✓
10.	256	5 8 8 0 4 4 ✓	m^2, $4m^2$ ✓
11.	~~[illegible]~~ 972	4 9 28 0 4 0 ✓	ω^2 ✓
12.	1024	4 9 32 0 0 4 ✓	m^2 ✓
13.	1024	5 13 16 0 0 2 ✓	$4m^2$ ✓
14.	1024	9 9 16 8 8 2 ✓	m^2, $4m^2$ ✓
15.	1728	9 16 16 16 0 0 ✓	ω^2, $4\omega^2$ ✓
16.	1728	13 13 16 −8 8 10 ✓	$4\omega^2$?
17.	4096	9 16 36 16 4 8 ✓	m^2, $4m^2$
18.	4096	9 17 32 −8 8 6 ✓	m^2, $4m^2$, $16m^2$

Notes on Ternary Forms

IV. Semi-regularity

I believe that the term "semi-regular" was first used in Jones's thesis [3]. It appears in the title of [4]. I didn't find a definition in either of these references, but the meaning is clear from the context: regularity on an arithmetic progression.

My interest in semi-regularity stems from a statement made without proof by Pall in [6, p. 344]: the forms 1 1 7 and 1 2 4 2 0 0 (they form a genus) both represent all eligible 4n's and (4n + 1)'s. For some time I was frustrated by my inability to prove this. Then Dennis Estes put me out of my misery; I incorporated his proof in [5]. Later I acquired a copy of [3] and found that Jones had given a proof for 1 1 7; this is acknowledged in the added in proof in [5].

Jones remarked that semi-regular forms are abundant. He gave a table (Table VII) of 38 such forms. I agree that they are abundant. For instance, each of the strong near misses in Note II is (conjecturally) regular on 4n's or (4n + 1)'s or both. A systematic investigation of all sem-regular forms does not look like an attractive enterprise. Nevertheless, regularity on 4n's and (4n + 1)'s retains a (somewhat irrational) special place in my heart.

But there is a special reason for writing this note: recently I made some progress on the 4n case. I now have a method that sometimes works for proving regularity on multiples of 4.

The setup to which the method applies is as follows. Let f be an even form of odd discriminant. (If f has even discriminant the method of descent to a smaller discriminant is likely to work even better.) Suppose there exists an odd form g with the same discriminant as f and satisfying three conditions:

(1) g is regular,

(2) f and g admit the same eligible numbers,

(3) there exists a homothety from f to g (necessarily with scale factor 2).

For conditions (2) and (3) there is a decision procedure. As for (1), a list of regular forms is available [2], with the slight flaw that 22 of the 913 are, as of this writing, still candidates. At any rate, when these conditions hold it follows that f represents all eligible 4n's. The proof is routine and I omit it for now (I will record it on a later occasion).

The table below records all forms with discriminant < 100 to which the method applies.

There is a closely related result which is definitive for the forms in question. Again I omit the routine proof.

Theorem. For any positive integer t the following statements are equivalent:
(a) 1, 1, 4t - 1 represents all eligible 4n's, (b) 1, 1, t, 1, 0, 0 is regular.

Now, somewhat in the spirit of using the classification of finite simple groups, I invoke the classification of regular ternary forms to learn that 1, 1, t, 1, 0, 0 is regular for exactly t = 1, 2, 3, 4, and 7. Therefore: 1, 1, 4t - 1 represents all eligible 4n's exactly for 4t - 1 = 3, 7, 11, 15, and 27.

As for representing all eligible (4n + 1)'s, I have no such method and have to fall back on ad hoc devices, case by case. Suitable devices have been found for forms

1, 2, 4, 5, 6, 7, 8, 9, 10, 17, 18, and 19. Note: the form 1, 1, 11 (number 3) does not represent the eligible number 33; but computation suggests that it represents all (4n + 1)'s that are not divisible by 11.

Here is a historical note. In [1] Dickson proved that the quaternary forms 1 1 7 7 and 1 1 11 11 both represent all multiples of 4. Concerning the latter he added that an attack by using ternary forms had not succeeded. But it is easy to deduce the result from the fact that the ternary form 1 1 11 represents all eligible 4n's. (Subsequently Will Jagy showed me an alternate method: use the quaternary form which is the direct sum of the binary form x^2 + xy + 3y^2 with itself.)

In concluding this note I stick in something not related to semi-regularity -- it is just one more problem that I have invented. Which forms represent all (4n + 2)'s? It turns out that such a form must be even. The problem then easily reduces to the forms that represent all odd numbers. The table below gives the answer. Note the three candidates.

References

1. L. E. Dickson, Quadratic functions of forms, some of whose values give all positive integers, J. Math. Pures Appl. Series 9, 7 (1928), 517-536.

2. W. C. Jagy, I. Kaplansky, and A. Schiemann, There are 913 regular ternary forms, submitted to Mathematika.

3. B. W. Jones, Representation by positive ternary quadratic forms, Ph. D. thesis, Univ. of Chicago, 1928..

4. B. W. Jones and G. Pall, Regular and semi-regular positive ternary quadratic forms, Acta Math. 70(1939), 165-191.

5. I. Kaplansky, The first nontrivial genus of positive definite ternary forms, Math. of Comp. 64 (1995), 341-345.

6. G. Pall, Representations by quadratic forms, Can. J. Math. 1(1949), 344-364.

(34)

Even non-regular forms that represent all eligible $4n$'s -- proved by the method of this note.

No.	Disc.	Form
1	7	1 1 7
2	7	1 2 4 2 0 0
3	11	1 1 11
4	11	1 3 4 2 0 0
5	13	1 2 7 2 0 0
	13	2 3 3 2 0 2
	15	1 1 15
	15	1 4 4 2 0 0
	15	1 2 8 2 0 0
	15	1 3 5
	17	1 3 6 2 0 0
	17	2 3 4 0 2 2
	21	1 3 7
	21	2 3 4 0 2 0
	21	1 2 11 2 0 0
	21	3 3 3 0 2 2
	27	1 1 27
	27	1 4 7 2 0 0
	27	2 4 5 4 2 2

No.	Disc.	Form
20	33	1 6 7 6 0 0
21	33	2 4 5 2 0 2
22	45	1 3 15
23	45	3 4 4 2 0 0
24	45	2 3 8 0 2 0
25	45	3 3 5
26	45	1 7 7 4 0 0
27	45	2 4 7 2 2 2
28	49	1 7 7
29	49	2 4 7 0 0 2
30	63	2 5 8 4 2 2
31	63	3 5 5 4 0 0
32	75	2 5 8 0 2 0
33	75	3 5 5
34	75	1 5 15
35	75	4 4 5 0 0 2
36	81	1 3 27
37	81	3 4 7 2 0 0
38	81	1 6 15 6 0 0
39	81	4 4 7 4 4 2

Forms representing all $(4n+2)$'s
(they are all even)

No.	Disc.	Form	
1.	1	1 1 1	
2.	3	1 2 2 2 0 0	
3.	4	1 2 2	
4.	5	1 1 5	
5.	7	2 2 3 0 2 2	
6.	9	1 2 5 2 0 0	
7.	11	1 2 6 2 0 0	
8.	12	1 2 6	
9.	12	2 3 3 0 2 2	
10.	15	2 3 3 0 0 2	
11.	16	1 2 8	
12.	16	2 3 3 2 0 0	candidate
13.	19	1 2 10 2 0 0	
14.	23	2 3 5 2 0 2	
15.	25	2 3 5 2 2 0	
16.	28	2 3 5 2 0 0	
17.	28	2 6 3 2 2 0	candidate
18.	31	2 3 6 2 2 0	candidate
19.	37	2 5 5 4 2 0	