Oct. 4/99

$f =$ Odd 108 : 1, 3, 10, 3, 1, 0. Two homotheties do it

I.K.

Until now we have relied on Hsia's proof via spinor genera. Here is a proof that comes right out of two homotheties.

The nonos for f are: $3n+2$, $9^k(9n+6)$. The first of the related forms ~~are~~ is the regular diagonal

$$g = 1, 3, 36: \quad 4n+2, 3n+2; \; 9^k(9n+6)$$

There is a homothety from f to g. Therefore we need only worry about $4n+2$'s. I shall actually catch all even eligibles, using the lower.

$$h = 225/122: \quad 3n+1, \; 9^k(9n+3).$$

There is a homothety from f to $2h$. Done

 Oct. 4/99

Nonspinor proof of the regularity of

odd 432, $f = 1, 3, 37, 3, 1, 0$

Nonos: $3n+2, 4n+3, 16n+8, 9^k(9n+6)$.

We have $4f = (2x+z)^2 + 3(2y+z)^2 + 144z^2$.
Thus it suffices, given an eligible A, to represent $4A$ by $g = u^2+3v^2+144w^2$ with u, v, w having the same parity. The nonos for the regular form $h = 1, 3, 36$ are the same as f, with $16n+8$ deleted.

Suppose A is odd. Then A is represented ~~by g. If w is even~~

by h: $p^2+3q^2+36r^2 = A$. If r is even, just multiply by 4. Suppose r odd. then p^2+3q^2 is also odd, p and q have opposite parity. We again multiply by 4, using the usual trick:

$$4(p^2+3q^2) = (p+3q)^2 + 3(p-q)^2.$$

The case where $A = 4^k \cdot \text{odd}$ follows since $A = 2 \cdot \text{odd}$ and $8 \cdot \text{odd}$ are nonos, it suffices to do $A = 32\cdot$odd.

Then $A/4$ is eligible for h:

$$p^2 + 3q^2 + 36r^2 = \frac{A}{4}$$

Multiply by 16:

$$(4p)^2 + 3(4q)^2 + 144(2r)^2 = 4A.$$

Done.

Oct. 4/99

Nonspinor proof of regularity of

odd 972 $f = (117)\, 36$

Nonos: $3n+2, 4n+2$, exactly div by 3, $9^k(9n+6)$

$4f = (2x+y)^2 + 27y^2 + \cancel{36z^2}\ 144z^2$

It suffices to show that $u^2 + 27v^2 + 144w^2$ represents $4A$, if A is eligible (u and v will automatically have the same parity.

The nonos for the regular 1, 3, 36 are $3n+2, 4n+2, 9^k(9n+6)$. So 1, 3, 36 represents A.

$$p^2 + 3q^2 + 36r^2 = A$$

~~Done if we~~ If q is div. by 3, just multiply by 4.

So assume q prime to 3.

If A is div. by 3 it is divisible by 9. Then $3 \mid p$, and $3 \mid q$, nix. So A is prime to 3.

So is q; Now multiply by 4:

$$4(p^2 + 3q^2) = (p \pm 3q)^2 + 3(p \mp q)^2.$$

By choice of ~~[illegible]~~ sign make $p \mp q$ div by 3. Done

Oct. 1999

(2 pages)

A note on homotheties.

I. K.

Let p be an odd prime. The odd ternaries with discriminant p form a single genus. The even ternaries with discriminant p comprise two genera. One of these "sits above" the odd forms. I have investigated homotheties from the odd ones to the even ones. I did a complete job on the existing tables. (these tables have now been slightly extended: for even 251, thanks to Will Jagy, I have the relevant genus).

Observations:

(1) For each odd form there is at least one and at most 3 even forms admitting a homothety.

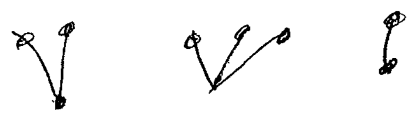
(2) Also the other way around: for each even form there is at least one and at most 3 odd forms admitting a homothety.

(3) The partially ordered set that arises is "connected" that is: by a sequence of moves up and down one can get from any form to any other form

In re (1) and (2): I suspect that a little honest toil would yield the two proofs. In re (3): I haven't a clue as to how to attack it. (There seems to be no reason to think that's true.)

There are two similar investigations

(a) From p to $4p$ for odd forms. But this is not new. It is the inverse of "going down by 4". So the partially ordered set might look like

 etc.

There is again the question of the bound of 3.

(b) $2p$: from odd forms to even forms. But this too is not new. It is the inverse of "going down by 4", obtained by "going down by 2" twice. Same thoughts.

Nov. 1/99

There is a 1-1 genus-preserving correspondence ~~[illegible]~~ between even forms of disc. 405 and odd forms of disc. 810. Hence there are 79 even forms of disc. 405 having 16 genera of sizes 3, 2, 2, 2, 3, 4, 4, 2, 4, 2, 1, 6, 10, 7, 17,

Probably the only genus that will be of interest to you is the one "sitting above" $\begin{cases} 2, 5, 11, 2, 2, 1 \\ 5, 5, 6, 3, 3, 5 \end{cases}$

There are two obvious lifts of 2, 5, 11, 2, 2, 1 : to $\boxed{8, 5, 11, 2, 4, 2}$ and ~~[illegible]~~ $\boxed{2, 20, 11, 4, 2, 2.}$

I took ~~[illegible]~~ 2, 20, 11, 4, 2, 2 and lowered it by 2, getting 1, 10, 22, 4, 2, 1. This siamed with 1, 10, 21, 3, 0, 1, the first form in a genus of 4 in the odd 810 table. I now had the genus that had to be lifted by 2.

8, 5, 11, 2, 4, 2 went down to 4, 10, 7, -8, 1, 2 which siamed 4, 7, 9, 6, 3, 1.

I switched the mate 5, 5, 6, 3, 3, 5 to 5, 5, 6, 0, 3, 5 and then lifted it to $\boxed{20, 5, 6, 0, 6, 10.}$ there is no homothety from 2, 5, 11, 2, 2, 1 to this. So this attempt to prove regularity failed

There is 1 more ~~[illegible]~~ form in the desired I could probably find it if you want it.

Added later: I think the 4^{th} member of the genus is $\boxed{5, 11, 11, -2, 2, 8}$ but I didn't check it

Feb. 1/96

Th. $(1,4,9)$ rep. all eligibles except 2.

Pf. Note that the generic ineligibles are $(9n\pm3)$, $4^k(8n+7)$ and $4n+3$. Suppose A eligible, $A\neq 2$.

1) Suppose A is div. by 3. Then it is divisible by 9, $A=9B$. $B=u^2+v^2+w^2$. u,v,w can not all be odd, for then $B\equiv 3\ (8)$ and so is A, contradiction. Multiply by 9. Henceforth A is prime to 3

2) Suppose A is div. by 4, $A=4C$. Then $C=u^2+v^2+w^2$. Since C is prime to 3, one of u,v,w is divisible by 3. Multiply by 4.

3) Suppose A is odd. notice $A\equiv 1\ (\text{mod } 4)$. In $A=u^2+v^2+w^2$, two of u,v,w are even, and at least one is divisible by 3. that gives us 1, 4, 9.

4) ~~Suppo~~ Finally, suppose A is twice odd., $A=2E$. $E\neq 1$, and it satisfies the conditions for representables by $2x^2+2y^2+2yz+5z^2$, so by Jagy's theorem it is representable. Then $A=2E$
$= 4x^2+(2y+z)^2+9z^2$. Done

Sept. 8/96

Proof that 1, 3, 36 and 1, 12, 36 are regular

1, 3, 36. As usual we can assume that the target A is prime to 2 and 3. If A is eligible for 1, 3, 36 it is certainly eligible for 1, 3, 9: $A = u^2 + 3v^2 + 9w^2$. If u, v, w are all odd we do the switch on $3v^2 + 9w^2$ and make v and w even; done. So one is ~~even and~~ odd and two even. We are done unless w is the odd one. Suppose A is 8n+5. Then $u^2 + 3v^2$ is 4. odd (since $9w^2 \equiv 1 \pmod 8$). That allows us to switch u and v to be odd, reverting to a case we have done. We have reached $A \equiv 1 \pmod 8$. Now since also $A \equiv 1 \pmod 3$ we have $A = 24n+1$. J & P's th. 5 does the trick. (thus: in the crucial case J & P's remarkable quaternion work does the trick)

1, 12, 36. The passage from 1, 3, 36 to 1, 12, 36 is ~~short~~ simple and elementary. We have $A = x^2 + 3y^2 + 36z^2$ and $A \equiv 1 \pmod 4$. If y is odd, x must be even and we find $A \equiv 3 \pmod 4$, a contradiction. So y is even. Done.

MR0106204 (21 #4938) 10.00
Brandt, Heinrich; Intrau, Oskar
Tabellen reduzierter positiver ternärer quadratischer Formen. (German)
Abh. Sächs. Akad. Wiss. Math.-Nat. Kl. **45** 1958 *no. 4* 261 *pp.*

The authors use the notation

$$f = a_1{x_1}^2 + a_2{x_2}^2 + a_3{x_3}^2 + a_4x_2x_3 + a_5x_3x_1 + a_6x_1x_2$$

for ternary quadratic forms, and

$$d = a_1{a_4}^2 + a_2{a_5}^2 + a_3{a_6}^2 - a_4a_5a_6 - 4a_1a_2a_3$$

is the formula for the discriminant. A form in which the a's are integers is called "primitive" if 1 is the g.c.d. of the a's. This table lists all reduced primitive positive ternary quadratic forms with integral coefficients with discriminants from -2 to -1000. There are over 36,000 forms listed. [Cf. the shorter tables of the reviewer, Nat. Res. Council Bull. no. 97 (1935)]. 36433

Two forms are of the same genus ("verwandt") if one may be taken into the other by a non-singular linear transformation with rational coefficients. The fundamental discriminant ("Stammdiskriminante") of a genus is the least discriminant among the forms of the genus with integral coefficients.

The adjugate form of f has the coefficients 4534 genera

$$\begin{gathered} {a_4}^2 - 4a_2a_3, \quad {a_5}^2 - 4a_3a_1, \quad {a_6}^2 - 4a_1a_2, \\ 4a_1a_4 - 2a_5a_6, \quad 4a_2a_5 - 2a_4a_6, \quad 4a_3a_6 - 2a_4a_5. \end{gathered}$$

The author denotes by I_1 the g.c.d. of these coefficients and defines I_2 by ${I_1}^2I_2 = 16d$. Two forms with the same invariants I_1, I_2 are said to be of the same order and I is defined by $I = I_1I_2/16$.

The basic conditions for a reduced form are

$$0 < a_1 \le a_2 \le a_3, \quad |a_6| \le a_1, \quad |a_5| \le a_1, \quad |a_4| \le a_2,$$

and, in case a_4, a_5, a_6 are all negative,

$$|a_4 + a_5 + a_6| \le a_1 + a_2.$$

These do not define in all cases a unique reduced form and the author merely sketches further considerations leading to unicity. He is not aware of or chooses to disregard the complete conditions obtained laboriously by L. E. Dickson [*Studies in the theory of numbers*, Chicago, 1930, Chap. IV].

In the table, forms for each discriminant are classified according to order and genus and the following invariants given: the number of automorphs, the number of forms in each genus, the prime factors of the discriminant, I_1, I_2, I and the related invariants of Minkowski, the fundamental discriminant and the characters.

This is a monumental piece of work and should be of great service to those working with quadratic forms.

Reviewed by *B. W. Jones*

TABLE 5

ALL POSITIVE INTEGERS NOT REPRESENTED BY
$a, b, c, = ax^2 + by^2 + cz^2$

1, 1, 1	A	1, 5, 8	$4n+3,\ 8n+2,\ I$
1, 1, 2	C	1, 5, 10	J
1, 1, 3	D	1, 5, 25	$5n\pm2,\ 25n\pm10,\ E$
1, 1, 4	$8n+3,\ A$	1, 5, 40	$4n+3,\ 8n+2,\ J$
1, 1, 5	E	1, 6, 6	$8n+3,\ G$
1, 1, 6	B	1, 6, 9	$3n+2,\ B$
1, 1, 8	$4n+3,\ 16n+6,\ C$	1, 6, 16	$8n\pm3,\ 16n\pm2,\ 64n+8,\ B$
1, 1, 9	$9n\pm3,\ A$	1, 6, 18	$3n+2,\ 9n+3,\ K$
1, 1, 12	$4n+3,\ D$	1, 6, 24	$8n\pm3,\ 32n+12,\ G$
1, 1, 16	$8n+6,\ 4n+3,\ 32n+12,\ A$	1, 8, 8	$4n+2,\ 4n+3,\ 8n+5,\ A$
1, 1, 21	$D,\ E,\ 49^k(49n+7r),\ r=1,\ 2,\ 4$	1, 8, 16	$4n+2,\ 4n+3,\ 8n+5,\ C$
1, 1, 24	$4n+3,\ 8n+6,\ B$	1, 8, 24	$4n+2,\ 4n+3,\ K$
1, 2, 2	A	1, 8, 32	$4n+2,\ 8n+3,\ A,\ 8n+5,\ 32n+20$
1, 2, 3	H	1, 8, 40	$4n+2,\ 4n+3,\ 8n+5,\ 32n+28,\ F$
1, 2, 4	C	1, 8, 64	$4n+2,\ 8n+3,\ C,\ 64n+40,\ 4^r(8n+5),\ 4^r(8n+7),\ r=0,\ 1$
1, 2, 5	I	1, 9, 9	$3n+2,\ 9n\pm3,\ A$
1, 2, 6	K	1, 9, 12	$3n+2,\ 4n+3,\ D$
1, 2, 8	$8n+5,\ A$	1, 9, 21	$3n+2,\ D,\ E,\ 49^k(49n+7r),\ r=1,\ 2,\ 4$
1, 2, 10	$8n+7,\ F$	1, 9, 24	$3n+2,\ 4n+3,\ 8n+6,\ B$
1, 2, 16	$8n+5,\ 8n+7,\ 16n+10,\ C$	1, 10, 30	$D,\ J,\ K$
1, 2, 32	$16n+14,\ A,\ 2^r(8n+5),\ r=0,\ 1,\ 2$	1, 12, 12	$4n+2,\ 4n+3,\ G$
1, 3, 3	G	1, 12, 36	$3n+2,\ 4n+2,\ 4n+3,\ D$
1, 3, 4	$4n+2,\ D$	1, 16, 16	$4n+2,\ 4n+3,\ A,\ 8n+5,\ 16n+8,\ 16n+12$
1, 3, 6	$3n+2,\ C$	1, 16, 24	$4n+2,\ 4n+3,\ B,\ 8n+5,\ 64n+8$
1, 3, 9	$3n+2,\ D$	1, 16, 48	$4n+2,\ 4n+3,\ 8n+5,\ 16n+8,\ 16n+12,\ D$
1, 3, 10	$D,\ F,\ 4^k(16n+2)$	1, 21, 21	$A,\ G,\ 49^k(7n+r),\ r=3,\ 5,\ 6$
1, 3, 12	$4n+2,\ G$		
1, 3, 18	$3n+2,\ 9n+6,\ H$		
1, 3, 30	$G,\ I,\ 4^k(16n+6)$		
1, 3, 36	$3n+2,\ 4n+2,\ D$		
1, 4, 4	$4n+2,\ 4n+3,\ A$		
1, 4, 6	$16n+2,\ B$		
1, 4, 8	$4n+2,\ 4n+3,\ C$		
1, 4, 12	$4n+2,\ 4n+3,\ D$		
1, 4, 16	$4n+2,\ 4n+3,\ 16n+12,\ A$		
1, 4, 24	$4n+2,\ 4n+3,\ B$		
1, 4, 36	$4n+2,\ 4n+3,\ 9n\pm3,\ A$		
1, 5, 5	$5n\pm2,\ A$		

Modern Elementary Theory of Numbers

LEONARD EUGENE DICKSON

TABLE 5—*Continued*

1, 24, 24	$4n+2,\ 4n+3,\ G,\ 8n+5,\ 32n+12$	3, 4, 36	$3n+2,\ 4n+1,\ 4n+2,\ D$
1, 24, 72	$3n+2,\ 4n+2,\ K,\ 4n+3,\ 9n+3$	3, 7, 7	$D,\ K,\ 49^k(7n+r),\ r=1,\ 2,\ 4$
1, 40, 120	$4n+2,\ 4n+3,\ D,\ J,\ K$	3, 7, 63	$3n+2,\ D,\ K,\ 49^k(7n+r),\ r=1,\ 2,\ 4$
1, 48, 144	$3n+2,\ 8n+5,\ D,\ 4^r(4n+2),\ 4^r(4n+3),\ r=0,\ 1$	3, 8, 8	$4n+1,\ 4n+2,\ D,\ 8n+7,\ 32n+4$
2, 2, 3	$8n+1,\ D$	3, 8, 12	$4n+1,\ 4n+2,\ L$
2, 3, 3	L	3, 8, 24	$3n+1,\ 4n+1,\ 4n+2,\ A$
2, 3, 6	$3n+1,\ A$	3, 8, 48	$4n+1,\ 4n+2,\ L,\ 8n+7,\ 64n+24$
2, 3, 8	$8n\pm1,\ 32n+4,\ D$	3, 8, 72	$3n+1,\ 4n+1,\ 4n+2,\ 8n+7,\ 32n+4,\ D$
2, 3, 9	$3n+1,\ 9n+6,\ H$	3, 10, 30	$A,\ G,\ N$
2, 3, 12	$16n+6,\ L$	3, 16, 48	$4n+1,\ 4n+2,\ 8n+7,\ 16n+4,\ 16n+8,\ G$
2, 3, 18	$3n+1,\ 8n+1,\ D$	3, 40, 120	$4n+1,\ 4n+2,\ A,\ G,\ N$
2, 3, 48	$8n\pm1,\ 16n\pm6,\ 64n+24,\ L$	5, 6, 15	$C,\ J,\ L$
2, 5, 6	$B,\ I,\ M$	5, 8, 24	$4n+2,\ 4n+3,\ B,\ I,\ M$
2, 5, 10	$8n+3,\ N$	5, 8, 40	$4n+2,\ 4n+3,\ N,\ 8n+1,\ 32n+12$
2, 5, 15	$B,\ H,\ N$	8, 9, 24	$3n+1,\ 4n+2,\ K,\ 4n+3,\ 9n+3$
2, 6, 9	$3n+1,\ 9n+3,\ K$	8, 15, 24	$4n+1,\ 4n+2,\ E,\ F,\ L$
2, 6, 15	$E,\ F,\ L$		
3, 3, 4	$4n+1,\ G$		
3, 3, 7	$G,\ M,\ 49^k(49n+7r),\ r=3,\ 5,\ 6$		
3, 3, 8	$4n+1,\ 8n+2,\ L$		
3, 4, 4	$4n+1,\ 4n+2,\ D$		
3, 4, 12	$4n+1,\ 4n+2,\ G$		

Diagonal

Table 5 gives each of the 102 regular forms and all the positive integers not represented by it. Use will be made of the abbreviations

$$A = 4^k(8n+7), \qquad B = 9^k(9n+3),$$
$$C = 4^k(16n+14), \qquad D = 9^k(9n+6),$$
$$E = 4^k(8n+3), \qquad F = 25^k(25n\pm5),$$
$$G = 9^k(3n+2), \qquad H = 4^k(16n+10),$$
$$I = 25^k(25n\pm10), \qquad J = 25^k(5n\pm2),$$
$$K = 4^k(8n+5), \qquad L = 9^k(3n+1),$$
$$M = 4^k(8n+1), \qquad N = 25^k(5n\pm1).$$