IK, May, 1994

Memo

More on ternary forms

I have now been looking at ternary forms for about a year and a half. I plan to put them aside, at least for a while. To keep what I have from being scattered and/or forgotten, I am assembling this memo. I will send out a few copies.

50

1. The form $h = x^2 + 5y^2 + 5z^2 + 3xy + xz$. In [5] it was proved that h represents all odd numbers. The argument is easily extended to identify all numbers it represents.

I think it is a fair statement that very few non-regular forms are known for which there is complete information on the numbers represented. I am glad to add one more.

The genus of h represents everything except $4^k(16n + 14)$.

Theorem 1. h represents all eligible numbers except the odd powers of 2.

Proof. (a) I attack first the non-representability of odd powers of 2.

Suppose that h(x, y, z) is even. Write $t = y + z$. Mod 2, h is $x^2 + xt + t^2$, which vanishes only if x and t vanish. In other words, if h is even then x is even and y and z have the same parity. Note that in that case x^2 and $3xy + xz = x(3y + z)$ are both divisible by 4, and $5y^2 + 5z^2 \equiv 2 \pmod 4$. Hence: if h(x, y, z) is divisible by 4 then x, y, and z are all even. This reduces our problem to the non-representability of 2. It simplifies matters to diagonalize. If h(x, y, z) = 2 then

$$20h(x, y, z) = (10y + 3x)^2 + (10z + x)^2 + 10x^2 = 40. \tag{1}$$

As noted, x must be even. Then (1) shows that x = 0 or 2. Both possibilities are easily ruled out by inspection.

(b) Let A be an eligible number (i. e., not of the form $4^k(16n + 14)$) which is not an odd

$(1, 5, 5, 0, 1, 3) \simeq (1, 3, 5, -1, 1, 1)$

power of 2. It is to be shown that h represents A. I ask the reader to examine the proof in [6] and confirm that it works provided 2A is not a square and A is not divisible by 25. Write $A = r^2 B$ with B square-free. Eligibility of A implies eligibility of B. We will be finished by induction unless B = 2. Furthermore, induction can again be applied unless r = 5. It remains to represent 50 and here it is: take x = 0, y = 3, z = 1.

54 2. The form $k = x^2 + 3y^2 + 5z^2 + xy + yz$. This form is one of the potential "near misses" 54 of [4]; it seems to represent all odd numbers except 17. In [4] some facts about odd numbers represented by k were stated without proof. Here I shall supply proofs and extend the results to even numbers.

I begin by noting that the integers missed by the genus of k are $4^k(16n + 10)$; these are the same as those missed by $x^2 + 2y^2 + 3z^2$.

Theorem 2. k represents every eligible number which is congruent to 0 or 1 mod 3.

Theorem 3. Let A be an eligible number which is congruent to 2 mod 3. Then k represents A if and only if A admits a representation by $x^2 + 2y^2 + 3z^2$ with z prime to 3.

The following remarks apply to the proofs of both theorems.

We make a change of variable, replacing z by z - x and then changing the sign of x. This yields $6x^2 + 3y^2 + 5z^2 + 10xz + yz$. We shall abuse notation and still call this form k. We have:

(2) $$12k = (6y + z)^2 + 2(6x + 5z)^2 + 9z^2.$$

Let A be an eligible number. Then (2) shows that k represents A if and only if 12A can be written $u^2 + 2v^2 + 9w^2$ with u, v, and w congruent mod 6. Now if $u^2 + 2v^2 + 9w^2$ is divisible by 4 then u, v, and w have the same parity. For, firstly u and w must have the same parity. If they are even, then v must be even. If they are odd, then $u^2 + 9w^2 \equiv 2 \pmod 4$ whence v must be odd. Thus our task simplifies to writing $12A = u^2 + 2v^2 + 9w^2$ with u, v, and w congruent mod 3. Of course, it is equally good to achieve $3A = u^2 + 2v^2 + 9w^2$ with u, v, and w congruent mod 3 (just multiply by 4).

Here is the general plan. Given an eligible A we begin by writing it as $A = r^2 + 2s^2 + 3t^2$. Multiply by 3 and absorb the factor 3 in $r^2 + 2s^2$, to reach $3A = u^2 + 2v^2 + 9w^2$. If u, v, and w are congruent mod 3, there is success. If not, we try to change them so as to achieve this, something which will not always be possible.

Proof of Theorem 2. I. $A \equiv 0 \pmod 3$. In $A = r^2 + 2s^2 + 3t^2$, r^2 and s^2 are congruent mod 3. We write

(3) $$3(r^2 + 2s^2) = (r \pm 2s)^2 + 2(r \mp s)^2 = u^2 + 2v^2.$$

Subcase (a): $t \equiv 0 \pmod 3$. If r and s are divisible by 3, all is well. If not, by making the right choice of sign in (3) we arrange that u and v are divisible by 3. Subcase (b): t prime to 3. We dispose first of the possibility $r = s = 0$. Then

$$12A = 36t^2 = (5t)^2 + 2t^2 + 9t^2$$

achieves our goal. So we may assume that $r^2 + 2s^2 \neq 0$. By Lemma 3 of [4] we can rewrite $3(r^2 + 2s^2)$ as $p^2 + 2q^2$ with p and q both prime to 3. Now $3A = p^2 + 2q^2 + 9t^2$ with p, q, and t all prime to 3. By changing signs, if necessary, we make them congruent mod 3.

II. $A \equiv 1 \pmod 3$. In $A = r^2 + 2s^2 + 3t^2$ it has to be the case that r is prime to 3 and s is divisible by 3. Subcase (a): t prime to 3. On the right side of (3), u and v are prime to 3 no matter which sign we pick. Subcase (b): $t \equiv 0 \pmod 3$. This subcase calls for a different strategy. We first multiply $A = r^2 + 2s^2 + 3t^2$ by 4 and use

(4) $$4(r^2 + 3t^2) = (r \pm 3t)^2 + 3(r \mp t)^2.$$

Note that r must be prime to 3 and s divisible by 3. The terms on the right of (4) are prime to 3 and we have $4A = p^2 + 2s^2 + 3q^2$ with p and q prime to 3 and s divisible by 3. Now we multiply by 3 and again use (3), with p and q replacing r and s. With either choice of sign we get u and v prime to 3. This concludes the proof of Theorem 2.

Proof of Theorem 3. I. We are given an eligible A and an expression $A = r^2 + 2s^2 + 3t^2$ with t prime to 3. From $A \equiv 2 \pmod 3$ we deduce that r is divisible by 3 and s is prime to 3. When we use (3) we get u and v prime to 3, just what is needed.

II. We are given $A \equiv 2 \pmod 3$ and $12A = u^2 + 2v^2 + 9w^2$ with u, v, and w congruent mod 3; our task is to work back to a representation $A = x^2 + 2y^2 + 3z^2$ with z prime to 3. We have $u^2 \equiv v^2 \pmod 3$. Since 12A is not divisible by 9 we cannot have u and v divisible by 3; thus u and v are prime to 3 and so is w. One knows that $(u^2 + 2v^2)/3$ is expressible as $p^2 + 2q^2$. Then $4A = p^2 + 2q^2 + 3w^2$. Note that $4A \equiv 2$, $p^2 \equiv 0$ or 1, $2q^2 \equiv 0$ or 2, all mod 3. It follows that p is divisible by 3 and q is prime to 3. Another consequence is that $p^2 + 3w^2$ is exactly divisible by 3. The usual parity argument shows that q is even and that $p^2 + 3w^2$ is divisible by 4. One knows that $(p^2 + 3w^2)/4$ can be wrtten $a^2 + 3b^2$. Here $a^2 + 3b^2$ inherits the property of being exactly divisible by 3, from which it follows that b is prime to 3. The representation $A = a^2 + 2(q/2)^2 + 3b^2$ achieves our goal and concludes the proof of Theorem 3.

As reported in [5], up to 16,383 the only exceptional odd integer that occurs in Theorem 3 is 17. As for even integers $\equiv 2 \pmod 4$ the exceptional eligible integers that showed up in a hand computation up to 2,000 were 2, 38, and 482. (Confession: in the past such hand computations were not too reliable.) At any rate, I will now ask the first of several questions. (Please note: these are questions, not conjectures.)

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Question 1. Does k represent all eligible integers except $4^k(2, 17, 38, \text{and } 482)$? Alternatively, are these the only eligible integers whose representations by $x^2 + 2y^2 + 3z^2$ all have z divisible by 3?

3. Partial results on two more forms in [5]. With a big assist from Gordon Pall [7] I obtain partial results on two forms that arose in [5].

34 The form $f = x^2 + 2y^2 + 5z^2 + xz + 2yz$ is one of the near misses in [5]: it seems to represent all odd numbers except 13 and does so for sure up to 16,383. The numbers $4^k(16n + 14)$ are the ones excluded by its genus. 34

Theorem 4. f represents all eligible numbers congruent to 0 or 2 (mod 3).

Proof. We have

$$4f = (2x + z)^2 + 2(2y + z)^2 + 17y^2.$$

If the form $d = u^2 + 2v^2 + 17w^2$ is a multiple of 4 then u, v, and w have the same parity. It follows that f represents A if and only if d represents 4A. Now Pall [7, 56] proved that d represents all eligible numbers congruent to 0 or 2 (mod 6). It therefore represents all eligible numbers congruent to 0 or 8 (mod 12). Dividing by 4 proves the theorem.

An exploratory computation leads me to ask:

Question 2. Does f represent all eligible numbers except $4^k(10 \text{ or } 13)$?

38 The form $g = x^2 + 2y^2 + 5z^2 + xz$ is the first of the four unsettled candidates in [5] for representing all odd numbers. The numbers $4^k(16n + 10)$ are the ones excluded by its genus.. 38

Theorem 5. g represents all eligible numbers congruent to 0 or ±2 (mod 5).

Proof. The proof follows the same pattern. We have

$$4g = (2x + z)^2 + 8y^2 + 19z^2.$$

If the form $e = u^2 + 2v^2 + 19w^2$ is a multiple of 4 then u and w have the same parity and v is even. It follows that g represents A if and only if e represents 4A. Now Pall [7, p. 57] proved that e represents all eligible numbers congruent to 0 or ±2 (mod 10). It therefore represents all eligible numbers congruent to 0 or ±8 (mod 20). Dividing by 4 proves the theorem. (38)

The eligible even numbers not represented by g ~~are fairly numerous~~. $4^k\{14\}$

4. A fourth near miss. I began a systematic search for near misses. This requires a good deal of computation and I have put it aside for a while. A fourth possible near miss emerged: $b = x^2 + 3y^2 + 2yz + 9z^2$. To help with computations (and perhaps to help in future proofs) here is a diagonalization.

(26) even (26)

Theorem 6. b represents A if and only if $c = u^2 + 3v^2 + 26w^2$ represents 3A.

Proof. We have

$$3b = 3x^2 + (3y + z)^2 + 26z^2.$$

If c is divisible by 3 then u^2 and w^2 are congruent mod 3. By changing a sign we can make u and w themselves congruent mod 3. This enables us to work back from c to b.

For quite a ways up b represents all odd numbers except 5. The eligible integers not represented by b which are even but not divisible by 4 start as follows: 2, 46, 62, 122. Perhaps they die at this point.

5. Even forms with small discriminant. The idea behind this investigation is the hope that patterns will emerge that will suggest conjectures that may some day be theorems.

I have become used to calling a form even if its cross product coefficients are even, odd otherwise. I realize that nobody else is using this terminology.

In [4] and [6] I carried the theory of even forms as far as I could for discriminant ≤ 10 and began looking at discriminant 11. I now continue.

Discriminant 11. There are some points of similarity between the even forms of discriminant 7 and those of discriminant 11.

(a) In each case there are three forms falling into two genera. The genera with just one form do not concern us any further. Here are the genera with two forms:

Discriminant 7: $f = x^2 + y^2 + 7z^2$, $g = x^2 + 2y^2 + 2yz + 4z^2$,

Discriminant 11: $h = x^2 + y^2 + 11z^2$, $k = x^2 + 3y^2 + 2yz + 4z^2$.

(b) This point concerns representing eligible numbers of the form 4n or 4n + 1. Gordon Pall stated without proof that f and g both represent all eligible 4n's and (4n + 1)'s; Dennis Estes supplied a proof that was incorporated in [6]. But the resemblance is imperfect, working for only three out of the four statements.

Theorem 7. k represents all eligible 4n's and (4n + 1)'s. 15

Proof. Let A be an eligible number congruent to 0 or 1 mod 4. If h does not represent A then k does. So we may assume that h represents A, say $A = u^2 + v^2 + 11w^2$. We now use the identity

$$u^2 + v^2 + 11w^2 = u^2 + 3(2w)^2 + 2(2w)[(v - w)/2] + 4[(v - w)/2]^2.$$

If v and w have the same parity we get an integral representation of A by k. By symmetry it is equally good for u and w to have the same parity. So we are defeated only if u, v are odd and w even, in which case $A \equiv 2 \pmod 4$, or u, v are even and w odd, in which case $A \equiv 3 \pmod 4$.

Theorem 8. h represents all eligible 4n's. 16

Proof. The plan is similar but the identity is different. Let B be an eligible multiple of 4, B = 4C. The number C is again eligible. If h represents C, all is well. So we can assume that k represents C, say $C = u^2 + 3v^2 + 2vw + 4w^2$. Now we have

$$B = 4C = (2u)^2 + (v + 4w)^2 + 11v^2,$$

a representation of B by h.

The fourth of the statements under scrutiny is false: 33 is an eligible number of the form 4n + 1 but h does not represent it. One example is enough to sink a theorem but here are three more: 11.67, 11.235, and 11.427. Yes, these are all multiples of 11.

Question 3. Does $h = x^2 + y^2 + 11z^2$ represent all (4n + 1)'s prime to 11? 11

If my computation stands up, this is true up to 2,000.

(c) Each of these genera has a close connection to a genus with a different discriminant: f and g with an even genus of discriminant 14 (see below)*; h and k with an odd genus of discriminant 22 (see under odd discriminants).

(d) In both cases the non-diagonal form seems to come much closer to regularity than the diagonal one. (This also seems to be true for the two form genus of even discriminant 10.) I shall spell out some details.

(i) List II in [4] presents 26 numbers prime to 7 not represented by $x^2 + y^2 + 7z^2$, the largest being 4759; there are no more up to 100,000. Up to 700 there are 20. For $x^2 + y^2 + 11z^2$, I have found 43 numbers prime to 11 up to 700 not represented. (This list is probably not quite accurate, and so I am not publicizing it.) So h is running ahead of f, better than two to one.

(ii) According to list III of [4], adjusted so as to apply to $g = x^2 + 2y^2 + 2yz + 4z^2$, the only non-represented numbers prime to 7 and less than 50,000 are 10 and 79. For

* and also to the odd genus of discriminant 28

$k = x^2 + 3y^2 + 2yz + 4z^2$, I present a question.

Question 4. Does k represent all numbers prime to 11 except 2?

My computation says that this is true up to 667. This was actually done by running $x^2 + 3y^2 + 11z^2$ up to 2,001, based on an easy statement which I shall call a theorem.

Theorem 9. $x^2 + 3y^2 + 2yz + 4z^2$ represents A if and only if $x^2 + 3y^2 + 11z^2$ represents 3A. (11)

Proof. The method is the usual one, based on the identity

$$3(x^2 + 3y^2 + 2yz + 4z^2) = 3x^2 + (3y + z)^2 + 11z^2.$$

One notes that if $u^2 + 3v^2 + 11w^2$ is a multiple of 3, then $u^2 \equiv w^2 \pmod 3$, and a change of sign, if necessary, can make $u \equiv w \pmod 3$.

(iii) To discuss representations of multiples of 7 (resp. 11) I prefer to make an appropriate switch of forms.

According to list IV of [4] the numbers 2, 74, and 506 are the only ones up to 100,000 which are prime to 7, squares mod 7, and not represented by $x^2 + 7y^2 + 14z^2$. The corresponding investigation has suggested the following question.

Question 5. Are 3 and 174 the only numbers which are multiples of 3, prime to 11, squares mod 11, and not represented by $x^2 + 11y^2 + 33z^2$?

My computation says that this is true up to 2,007.

Discriminant 12. The even forms of discriminant 12 are all alone in their genera.

Before proceeding further, I make a remark on regular even forms of discriminant ≤ 20. In Jones's thesis [2], in addition to his well known list of 102 primitive regular diagonal forms, he presented (among other things) a list of all even primitive regular non-diagonal forms of discriminant ≤ 20. The two lists combine to show that there is only one even regular form of discriminant ≤ 20 lying in a genus with more than one form: $x^2 + y^2 + 16z^2$. I believe that to this day it is difficult to prove that this form is regular.

Discriminant 13. There are four even forms of discriminant 13 and they comprise two genera containing two forms each. All four forms are non-regular (this is covered by the remark just made).

13 At present I have examined only $x^2 + y^2 + 13z^2$. I have proved nothing and will only ask a four part question. 13

First I note that if $x^2 + y^2 + 13z^2$ is divisible by 4, then x, y, and z are all even. We can therefore confine our considerations to odd number and doubles of odd numbers. The numbers excluded by the genus are then all $(8n+3)$'s I found it convenient to divide the work into four cases.

Question 6. (a) Does $x^2 + y^2 + 13z^2$ represent all $(4n + 1)$'s except 721? (b) Does it represent all $(8n + 2)$'s? (c) Does it represent all $(8n + 6)$'s except 6 and 46? (d) Does it represent all $(8n + 7)$'s except 7, 55, 79, 271, and 439?

All four computations went up to 2,000. I found it a little startling to see the exception 721 sticking out like a sore thumb in a sea of 500 represented numbers.

Discriminant 14. There are three even forms of discriminant 14, including the genus

14 consisting of $x^2 + y^2 + 14z^2$ and $x^2 + 2y^2 + 7z^2$. In [4] it was shown that the theory of this genus is equivalent to that of the genus above of discriminant 7, but part of the proof was omitted. I shall make a record of that part here. 14

Theorem 10. Let A be an odd number. Then $x^2 + y^2 + 14z^2$ (resp. $x^2 + 2y^2 + 7z^2$) represents A if and only if $x^2 + y^2 + 7z^2$ (resp. $x^2 + 2y^2 + 2yz + 4z^2$) represents 2A.

Proof. From [4, Lemmas 1 and 2] we know the following: $x^2 + y^2 + 7z^2$

(resp. $x^2 + 2y^2 + 2yz + 4z^2$) represents a number B if and only if $x^2 + y^2 + 14z^2$ (resp. $x^2 + 2y^2 + 7z^2$) represents 2B. Furthermore it is trivially true of any form that if it represents A then it represents 4A. That covers the implications in one direction. The other direction comes down to this: assume that $x^2 + y^2 + 14z^2$ (resp. $x^2 + 2y^2 + 7z^2$) represents 4A and prove that it represents A. For $x^2 + y^2 + 14z^2$ the argument goes as follows: x and y have the same parity. If they are even, z must be even. If they are odd, $x^2 + y^2 \equiv 2 \pmod 8$, z must be odd, $14z^2 \equiv 6 \pmod 8$, and $x^2 + y^2 + 14z^2$ is divisible by 8, a contradiction. For $x^2 + 2y^2 + 7z^2$ the argument is a little different. This time x and z have the same parity, and if they are even, y is even. If they are odd, $x^2 + 7z^2 \equiv 0 \pmod 8$, y must again be even and $x^2 + 2y^2 + 7z^2$ is divisible by 8, same contradiction.

Remark. We really need to assume that A is odd: both $x^2 + y^2 + 14z^2$ and $x^2 + 2y^2 + 7z^2$ represent 24 but fail to represent 6.

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Discrininant 15. I am commenting briefly on the form $x^2 + 3y^2 + 5z^2$ because it will come up in treating an odd form below. Here is what I know. It is in a genus of two forms. The forbidden integers are $25^k(25n \pm 10)$. All eligible multiples of 3 are represented; this is proved in [2] by the standard method of relating it to the regular form $x^2 + 2y^2 + 2yz + 3z^2$ via the identity

$$3(x^2 + 2y^2 + 2yz + 3z^2) = 3x^2 + 5y^2 + (y + 3z)^2.$$

Up to 701 I found that the form fails to represent the following numbers prime to 5:
168th 276th
2, 11, 22, 34, 38, 74, 158, 179, 287, and 298. The first few numbers prime to 5 and squares mod 5 not represented by $x^2 + 5y^2 + 15z^2$ are 11, 26, 34, 91, and 119.

6. Odd forms with small discriminant. I have become used to used to using the term "discriminant" for the determinant of the doubled up form. This is twice the discriminant as used in [1].

Discriminant 22. I fast forward to 22 because up to then all genera have only one form. Results in [3, pp. 173-4] are decisive. Of the two forms, one is regular, and the other represents exactly the same numbers as the form $x^2 + 3y^2 + 2yz + 4z^2$ discussed above. (This last point was stated without proof. A proof supplied by Dennis Estes was incorporated into [6].)

Discriminant 24. Two forms, each in its own genus.

Discriminant 26. Just one form.

Discriminant 28. There are three forms. The two forms

$$j = x^2 + y^2 + 4z^2 + xz + yz, \quad m = x^2 + xy + 2y^2 + 2z^2$$

constitute a genus. Their theory is reducible to the even genus of discriminant 14 discussed above. This is based on the identities

$$4j = (2x + z)^2 + (2y + z)^2 + 14z^2, \quad 4m = (2x + y)^2 + 7y^2 + 8z^2$$

and observations made above: if $u^2 + v^2 + 14w^2$ is divisible by 4, then u, v, and w have the same parity; if $u^2 + 2v^2 + 7w^2$ is divisible by 4, then v is even and u and w have the same parity.

Discriminant 30. There are three forms. The two forms

$$p = x^2 + xy + y^2 + 5z^2, \quad q = x^2 + 2y^2 + yz + 2z^2$$

constitute a genus. The form q is regular; I am not sure who first proved this -- it is not in [2].

As for the form p, it represents exactly the same numbers as $x^2 + 3y^2 + 5z^2$ (because of the well known fact that that the binary forms $x^2 + xy + y^2$ and $x^2 + 3y^2$ represent the same numbers). The form $x^2 + 3y^2 + 5z^2$ was discussed above.

Discriminant 32. There are no odd forms with discriminant an odd power of 2 (and all other discriminants are possible). John Hsia sent me a proof of this.

Discriminant 34. There are two forms and they form a genus. Just one is regular. I am breaking off at this point and may return some day.

References

1. H. Brandt and O. Intrau, Tabelle reduzierten positiver ternärer quadratischer Formen, Abh. Sächs. Akad. Wiss. Math.-Nat. Kl. 45(1958), no. 4. MR 21, 11493.

2. B. W. Jones, Representation by positive ternary quadratic forms, Ph. D. thesis, Univ. of Chicago, 1928.

3. B. W. Jones and G. Pall, Regular and semi-regular positive ternary quadratic forms, Acta Math. 70(1940), 165-191.

4. I. Kaplansky, The first nontrivial genus of positive definite ternary forms, to appear in Math. of Computation.

5. ______, Ternary positive forms that represent all odd positive integers, submitted to Acta Arithmetica.

6. ______, The next two nontrivial genera, memo.

7. G. Pall, The completion of a problem of Kloosterman, Amer. J. of Math. 68(1946), 47-58