July 7, 1995

To: John Hsia and Rainer Schulze-Pillot

Dear colleagues:

I just noticed something. The proof is a triviality, but at least it applies to an infinite family of forms. And it resembles the phenomenon in the Jones-Pall Acta paper, which later turned out to be explainable by spinor genera. So I decided to send it to you.

Theorem. Let F be $2x^2 + 2y^2 + (4r^2 + 1)z^2 + 2xz + 2yz$. Then (a) If r is even, F does not represent s^2 for any prime s of the form 4n + 3. (b) If r is odd, $F \not\to m^2$ in the Jones-Pall notation, that is, F does not represent any m^2 where every prime factor of m is of the form 4n + 1.

Remarks. 1. For r = 1, F is the genus mate of $x^2 + y^2 + 16z^2$. For r > 1, I have no further information on F.

2. The proof is similar to the proof of the theorem on page 5 of the enclosed paper by Jagy and myself (to appear in Experimental Mathematics).

Proof. We have

(1) $$2F = (2x + z)^2 + (2y + z)^2 + 8r^2z^2.$$

Suppose that F represents an odd number A. Then 2F = 2A and we see from (1) that z is odd. We have

(2) $$u^2 + v^2 + 4r^2z^2 = A$$

for suitable u and v.

(a) Suppose $A = s^2$ with s a prime of the form 4n + 3. Then (s + 2rz)(s - 2rz) is a sum of two squares. Since r is even both factors are of the form 4n + 3. It follows that some prime q of the form 4n + 3 divides both factors. But then q divides 2s, so q = s. Furthermore s divides rz. Then in (2) s divides u and v. Divide (2) by s^2. Then the right side is 1, but $4r^2z^2/s^2 > 1$ (remember that z is odd and hence nonzero)

(b) Suppose that $A = m^2$ is divisible only by primes of the form $4n + 1$. We again have $(m + 2rz)(m - 2rz) = u^2 + v^2$. Since r and z are odd, both factors are again of the form $4n + 3$, so that again a prime q of the form $4n + 3$ divides both factors. Then q divides m, a contradiction.

While I am at it I'll add three notes concerning forms John treated by spinor genera.

(i) BI discriminant 108. Both $x^2 + xy + y^2 + 36z^2$ and its mate reduce, by the trick of replacing $x^2 + xy + y^2$ by $x^2 + 3y^2$, to forms treated by Jones and Pall.

(ii) Same discriminant. $f = 3x^2 + 4y^2 + 4z^2 + 3xy + 4yz$ is the mate of a form proved regular by John. I proved rather easily that $f \neq w^2$ in the Jones-Pall notation. Known?

(111) Discriminant 81. $g = 3x^2 + 3y^2 + 4z^2 + 3xy + 3xz$ is another such mate. I couldn't prove anything but I checked up to 100,000 that g represents all eligible integers except 1 and 58.

Best regards,

Irving Kaplansky

William C. Jagy, January 2009. Compare page 312 of Schulze-Pillot "Survey" 2004, Theorem 4.3 and his illustrative example on that page. The computer printed the square root of any exception that was a perfect square, then a colon. Kap proved that $2x^2+2y^2+17z^2+2yz+2zx \neq q^2$, prime $q \equiv 3 \bmod 4$. It is easy to show, using Theorem 4 and Theorem 5 of Jones and Pall (1939) and identities $(W^2+X^2+Y^2+Z^2)^2 = (W^2-X^2-Y^2+Z^2)^2+(2WX-2YZ)^2+(2WY+2ZX)^2$ and $(X^2+Y^2+Z^2)^2 = (X^2-Y^2-Z^2)^2+(2XY)^2+(2ZX)^2$, that $2x^2+2y^2+17z^2+2yz+2zx$ represents 4 and p^2 for primes $p \equiv 1 \bmod 4$, also q^4 for primes $q \equiv 3 \bmod 4$, and q^2r^2 for primes $q, r \equiv 3 \bmod 4$. Therefore, all squares are represented **except** 1 and q^2 for primes $q \equiv 3 \bmod 4$.

```
=====Discriminant  256  ==Genus Size==   4 {two spinor genera}
  Spinor genus misses square classes    {no exceptions}
  256 :    1   1    64    0   0   0
  256 :    1   4    17    4   0   0
  256 :    2   2    17    2   2   0
--------------------------size 3
  Spinor genus misses square classes    1
  256 :    2   5     8    4   0   2   {Spinor Regular!}
--------------------------size 1
=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=
        256:     2     2         17      2    2    0
  misses, compared with full genus (up to 250,000)
    1:       1             5    3:        9           13          37
            42    7:      49             73           85          93
           109   11:     121            177          322         345
           357   19:     361            397   23:    529         613
           697   31:     961           1285         1738   43:  1849
          2101   47:    2209           2797   59:   3481   67:  4489
   71:    5041   79:    6241   83:     6889  103:  10609  107: 11449
  127:   16129  131:   17161  139:    19321  151:  22801  163: 26569
  167:   27889  179:   32041  191:    36481  199:  39601  211: 44521
  223:   49729  227:   51529  239:    57121  251:  63001  263: 69169
  271:   73441  283:   80089  307:    94249  311:  96721  331: 109561
  347:  120409  359:  128881  367:   134689  379: 143641  383: 146689
  419:  175561  431:  185761  439:   192721  443: 196249  463: 214369
  467:  218089  479:  229441  487:   237169  491: 241081  499: 249001
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       256:    1     1         64      0    0    0
misses, compared with full genus (up to 250,000)
           21            33            42            57            133
          141           253           322           385            553
         1738
=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=
       256:    1     4         17      4    0    0
misses, compared with full genus (up to 250,000)
            2            10            58            82            130
          282           298          3298
=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=
       256:    2     5          8      4    0    2
  misses, compared with full genus (up to 250,000)
    1:      1    2:      4    5:     25   10:    100   13:    169
   17:    289   25:    625   26:    676   29:    841   34:   1156
   37:   1369   41:   1681   50:   2500   53:   2809   58:   3364
   61:   3721   65:   4225   73:   5329   74:   5476   82:   6724
   85:   7225   89:   7921   97:   9409  101:  10201  106:  11236
  109:  11881  113:  12769  122:  14884  125:  15625  130:  16900
  137:  18769  145:  21025  146:  21316  149:  22201  157:  24649
  169:  28561  170:  28900  173:  29929  178:  31684  181:  32761
  185:  34225  193:  37249  194:  37636  197:  38809  202:  40804
  205:  42025  218:  47524  221:  48841  226:  51076  229:  52441
  233:  54289  241:  58081  250:  62500  257:  66049  265:  70225
  269:  72361  274:  75076  277:  76729  281:  78961  289:  83521
  290:  84100  293:  85849  298:  88804  305:  93025  313:  97969
  314:  98596  317: 100489  325: 105625  337: 113569  338: 114244
  346: 119716  349: 121801  353: 124609  362: 131044  365: 133225
  370: 136900  373: 139129  377: 142129  386: 148996  389: 151321
  394: 155236  397: 157609  401: 160801  409: 167281  410: 168100
  421: 177241  425: 180625  433: 187489  442: 195364  445: 198025
  449: 201601  457: 208849  458: 209764  461: 212521  466: 217156
  481: 231361  482: 232324  485: 235225  493: 243049
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  Genus represents: 4^k ( 4 n + 1), 4^k ( 8 n + 2),
                    4^k ( 128 n + 96), 4^k ( 512 n + 192).
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genera in the genus (the spinor exceptions of the genus) has been determined in [53], the same problem for primitive representations has been solved in [13], both of these exceptional sets are in general infinite.

Writing

$$\vartheta(\text{spn } L, z) := \sum_{a=0}^{\infty} r(\text{spn } L, a) \exp(2\pi i a z), \tag{4.15}$$

one has by [55] that

$$\vartheta(L, z) - \vartheta(\text{spn } L, z) \tag{4.16}$$

is a cusp form of weight $\frac{3}{2}$ whose Shimura lifting is cuspidal. For such cusp forms of weight 3/2 the growth of the Fourier coefficients can be estimated using [11] and the Shimura lifting, which gives an asymptotic formula of the same type as in Theorem 3.1 (with an error term $O(a^{\frac{1}{2}-\frac{1}{28}+\epsilon})$) for all a outside the exceptional square classes [12].

If there is only one spinor genus in the genus, one is done at this point; since it is known [44, 23] that this is the case if the discriminant is not divisible by 2^7 and not by any p^3 for odd primes p, we are finished here for L of small discriminant. If there are several spinor genera, a more detailed analysis [12, 54, 57] yields the following result inside the exceptional square classes:

THEOREM 4.3. [57] *If* $\text{rk}(L) = 3$ *and a is restricted to numbers in* $q(\text{gen } L)$ *not divisible by* p^r *(r fixed) for the primes for which* L_p *is anisotropic, one has: If a is sufficiently large, then a is represented by all lattices in the genus of* (L, q) *unless one of the following holds.*

- *a is a spinor exception. In this case a is represented by exactly half the spinor genera in the genus of L, and it is represented by all the classes in these spinor genera.*
- *a is of the form $a'p^2$, where a' is a spinor exception and p is a prime that is inert in the imaginary quadratic extension*

$$E = \mathbb{Q}(-2a \det L) \tag{4.17}$$

of $\mathbb{Q}$. In this case a' is represented by exactly half the spinor genera in the genus of L and $a = a'p^2$ is represented precisely by those classes in this half of the spinor genera that represent a' and by all lattices in the other half of the spinor genera.

In particular, if there is a spinor exceptional integer a' for the genus of L that is represented by $\text{spn}(L)$ *but not by L (so a' is below the bound for being sufficiently large), then there are infinitely many integers $a'p^2$ with p prime that are not represented by L.*

An example for the behaviour of this theorem is the quadratic form

$$4x^2 + 48y^2 + 49z^2 + 48yz + 4xz \tag{4.18}$$

discussed in [57]; it does not represent any p^2 where $p \equiv -1 \bmod 3$ is a prime although the form $x^2 + 48y^2 + 144z^2$ in the same spinor genus represents all these numbers (but not primitively).

We will come back to the proof of this theorem in the next section where we discuss

Compare Kap i $\langle 2, 2, 16t^2+1 ; 2, 2, 0 \rangle \neq q^2$, PRIME $q \equiv 3 \bmod 4$

$\Delta = 256t^2$. / $t=1$ $\left\{\begin{matrix} 1 & 1 & 64 & 0 & 0 & 0 \\ 1 & 4 & 17 & 4 & 0 & 0 \\ 2 & 2 & 17 & 2 & 2 & 0 \\ 2 & 5 & 8 & 4 & 0 & 2 \end{matrix}\right\}$ -spinor regular

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Representation by integral quadratic forms - a survey

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ABSTRACT. In this article we give a survey of results on representation of numbers by an integral quadratic form (or more generally representation of quadratic forms by quadratic forms). Particular emphasis is put on definite forms and there on recent work about forms of rank 3 over number fields ond on questions of effectivity.

Introduction

An integral symmetric matrix $S = (s_{ij}) \in M_m^{\mathrm{sym}}(\mathbb{Z})$ with $s_{ii} \in 2\mathbb{Z}$ gives rise to an integral quadratic form $q(\mathbf{x}) = \frac{1}{2}\ {}^t\mathbf{x}S\mathbf{x}$ on $\mathbb{Z}^m$. If S is positive definite, the number $r(q,t)$ of solutions $\mathbf{x} \in \mathbb{Z}^m$ of the equation $q(\mathbf{x}) = t$ is finite, and it is one of the classical tasks of number theory to study the qualitative question which numbers t are represented by q or the quantitative problem to determine the number $r(q,t)$ of representations of t by q either exactly or asymptotically.

Starting with the work of Euler, Legendre–Gauß and Lagrange–Jacobi on the number of ways in which an integer can be represented as a sum of two, three and four integral squares, many deep and beautiful results have been obtained concerning these problems, as well in this classical setting as in generalized settings like the study of representations with congruence conditions, representation numbers of forms q' of rank $n \leq m$ by q, representation numbers or measures by definite or indefinite forms over the ring of integers of a number field.

In this article I want to give a survey of what is known (and what is not known) about these questions. In particular we will discuss and slightly extend some recent results about representation of numbers by totally definite forms of rank 3 over the integers of a totally real number field in Section 5. We will also discuss some recent progress concerning effectivity of results. Another recent survey is [20]

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