THERE ARE 913 REGULAR TERNARY FORMS

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Dedicated, with admiration, to G. L. Watson

§1. *Introduction.* The forms under discussion are integral positive definite quadratic forms in three variables. Such a form g is called *regular* if g represents every integer represented by the genus of g. This can be recast in elementary terms: g is regular if the solvability of $g \equiv a \pmod{n}$ for every n implies the solvability of $g = a$.

The study of regular forms was initiated by Dickson in [**2**]. Jones made a big advance in his 1928 thesis [**4**] by finding that there were precisely 102 diagonal regular forms. (We are assuming, as we may without loss of generality, that the forms being studied are primitive, *i.e.*, the greatest common divisor of the coefficients is 1.) Subsequently there was further work by Jones and Pall, Hsia and his colleagues, Schulze-Pillot, and above all Watson. What we are announcing here completes the work of Watson, using and extending his methods. A followup paper is planned; it will present detailed proofs and descriptions of the computations.

The title of this announcement is not quite accurate. There are *at most* 913 regular ternary forms, but for 22 of them regularity has not yet been proved. For each of these "candidates" it has been checked that it represents all eligible numbers up to a million (an integer is eligible for g if it is represented by some form in the genus of g).

§2. *Notation and terminology.* We use the abbreviation $a\ b\ c\ d\ e\ f$ for the form

$$ax^2 + by^2 + cz^2 + dyz + exz + fxy.$$

There is an important distinction, depending on whether or not d, e, and f are all even. The designations "classical" and "nonclassical" used by the Dickson school appear to have fallen into disuse. Gauss's term "improperly primitive" seems also to be disappearing. With some hesitation we introduce still another terminology. We shall call $a\ b\ c\ d\ e\ f$ *even* if d, e, and f are all even and *odd* otherwise.

There are two tables of regular forms below, the first for odd forms and the second for even ones. Indentation of a form signifies that it is not alone in its genus. (Note that any form alone in its genus is necessarily regular.) An asterisk attached to a form means that there is no proof of regularity. The first table contains 248 forms of which 15 are candidates and the second contains 665 forms of which 7 are candidates.

[MATHEMATIKA, **44** (1997), 332–341]

Table 1. The regular odd forms with their discriminants. When the form is indented, it is not alone in its genus. An asterisk before a form means that there is as yet no proof of its regularity.

2: 1 1 1 1 1 1	45: 1 3 4 0 1 0	125: 2 3 7 3 1 2
3: 1 1 1 0 0 1	45: 1 2 7 2 1 1	126: 1 5 7 2 1 1
5: 1 1 2 1 1 1	45: 2 2 3 0 0 1	126: 3 3 5 3 3 0
6: 1 1 2 0 0 1	46: 1 3 5 3 1 1	132: 1 5 7 1 0 1
6: 1 1 2 1 1 0	48: 1 3 5 3 1 0	135: 1 3 12 3 0 0
7: 1 1 2 0 1 0	49: 1 2 7 0 0 1	135: 2 2 9 0 0 1
8: 1 1 3 1 1 1	50: 1 2 7 2 1 0	135: 2 5 5 5 1 2
9: 1 1 3 0 0 1	50: 1 4 4 3 1 1	144: 1 3 13 3 1 0
10: 1 1 3 1 1 0	50: 2 3 3 1 2 2	147: 1 2 21 0 0 1
10: 1 2 2 2 1 1	52: 1 3 5 1 1 1	147: 3 3 5 −2 2 1
11: 1 1 3 0 1 0	54: 1 1 18 0 0 1	150: 1 5 9 5 1 0
12: 1 1 4 0 0 1	54: 1 4 4 2 1 1	150: 2 5 5 5 0 0
12: 1 2 2 1 1 1	54: 2 2 5 1 2 2	162: 1 6 7 0 1 0
13: 1 2 2 1 0 1	54: 2 3 3 3 0 0	162: 1 7 7 5 1 1
14: 1 1 5 1 1 1	56: 1 3 5 1 1 0	162: 2 5 5 1 2 2
15: 1 2 2 1 0 0	60: 1 4 5 4 1 0	168: 1 4 11 2 1 0
15: 1 1 4 0 1 0	63: 1 3 6 3 0 0	169: 2 5 5 −3 1 1
17: 1 2 3 2 1 1	63: 2 2 5 2 2 1	180: 1 7 7 3 0 1
18: 1 1 5 1 1 0	70: 1 2 9 0 1 0	180: 3 5 5 5 3 3
18: 1 1 6 0 0 1	72: 1 3 7 2 1 1	189: 2 3 8 0 1 0
18: 1 2 3 2 1 0	72: 1 3 7 3 1 0	189: 1 4 13 2 1 1
18: 2 2 2 1 2 2	72: 1 4 5 2 1 0	189: 2 5 5 1 1 1
20: 1 1 7 1 1 1	72: 2 2 5 1 1 1	196: 1 7 9 7 1 0
20: 1 2 3 1 0 1	75: 1 4 5 0 0 1	200: 3 3 7 3 3 1
21: 1 2 3 0 0 1	75: 2 2 5 0 0 1	216: 1 3 19 3 1 0
21: 1 2 3 1 1 0	78: 1 5 5 4 1 1	216: 3 5 5 2 3 3
22: 1 2 3 0 1 0	80: 1 3 7 1 1 0	216: 2 5 6 3 0 1
24: 1 3 3 3 1 1	81: 1 3 7 0 1 0	225: 2 2 15 0 0 1
24: 1 2 4 2 1 1	81: 1 4 6 3 0 1	225: 1 4 15 0 0 1
25: 2 2 2 −1 1 1	84: 1 3 8 2 0 1	225: 2 5 7 5 1 0
27: 1 1 7 0 1 0	90: 1 1 30 0 0 1	234: 2 3 11 3 2 0
27: 1 1 9 0 0 1	90: 2 3 5 3 2 0	* 240: 1 5 13 2 1 1
27: 1 2 4 1 0 1	98: 3 3 3 −1 1 1	242: 1 3 22 0 0 1
27: 1 3 3 3 0 0	99: 2 3 5 3 1 0	243: 1 7 9 0 0 1
27: 2 2 2 1 1 1	100: 2 2 7 −1 1 1	243: 2 3 11 3 1 0
28: 1 3 3 2 1 1	100: 3 3 3 1 1 1	243: 2 5 8 5 1 2
30: 1 1 10 0 0 1	108: 1 1 36 0 0 1	250: 1 9 9 8 1 1
30: 1 3 3 1 1 1	108: 1 3 10 3 1 0	250: 2 3 13 1 2 2
32: 1 3 3 1 0 1	108: 1 4 7 0 1 0	252: 3 5 5 1 0 3
33: 1 2 5 1 1 1	108: 1 5 7 5 1 1	270: 1 6 13 6 1 0
36: 1 1 12 0 0 1	108: 2 2 8 2 2 1	270: 3 3 11 3 3 3
36: 1 3 4 3 1 0	108: 3 3 5 3 3 3	270: 5 5 5 4 5 5
40: 1 3 4 2 0 1	120: 1 3 11 3 1 0	289: 3 5 6 1 2 3
42: 1 1 11 1 1 0	121: 1 3 11 0 0 1	294: 5 5 5 −3 3 4
44: 1 3 4 0 0 1	125: 1 4 9 3 1 1	* 297: 1 6 13 3 1 0

Table 1. *Continued.*

297: 2 5 8 −2 1 1	882: 2 11 11 1 2 2	3528: 8 11 11 1 4 4
300: 3 5 7 5 3 0	882: 5 5 10 2 2 3	4050: 5 11 21 3 0 5
324: 1 7 13 5 1 1	900: 1 15 19 15 1 0	4356: 7 13 13 −7 1 1
350: 3 3 10 0 0 1	900: 5 7 8 2 0 5	4500: 5 11 24 6 0 5
360: 1 3 31 3 1 0	972: 1 7 36 0 0 1	* 4500: 7 8 23 6 7 2
363: 2 7 7 3 1 1	972: 5 8 9 −6 3 4	* 4536: 5 9 27 0 3 3
375: 2 7 7 −1 1 1	1000: 3 7 13 −3 1 2	5292: 5 17 17 1 2 5
378: 1 9 13 9 1 0	1014: 1 13 23 13 1 0	5292: 7 13 19 8 7 7
378: 2 5 11 1 2 2	1058: 5 7 10 2 4 5	5400: 7 7 28 −2 2 1
392: 3 3 12 −2 2 1	1080: 3 9 11 3 3 0	6174: 11 11 15 3 3 8
396: 3 5 8 2 0 3	1089: 6 7 10 7 3 6	6750: 3 17 35 5 0 3
400: 3 3 12 2 2 1	* 1125: 1 10 29 5 1 0	6750: 5 17 23 7 5 5
* 405: 2 5 11 2 2 1	* 1125: 2 7 22 −6 1 1	6750: 7 13 22 2 4 7
405: 2 8 8 7 1 1	1125: 5 6 11 3 5 0	6750: 9 11 21 3 9 6
432: 1 3 37 3 1 0	1134: 5 9 9 9 3 3	7938: 5 17 26 2 4 5
432: 3 5 9 3 0 3	1176: 5 5 13 1 1 3	8100: 7 13 27 3 6 7
441: 3 6 7 0 0 3	1188: 5 8 9 6 3 2	* 8232: 5 13 33 −6 3 1
441: 2 8 8 −5 1 1	1323: 2 8 21 0 0 1	9126: 9 14 23 14 3 6
450: 3 7 7 4 3 3	1323: 5 5 17 2 5 4	* 10125: 9 11 29 −4 3 6
450: 5 5 6 0 0 5	1323: 6 7 10 7 3 0	10584: 5 17 33 9 3 2
484: 1 3 44 0 0 1	1350: 1 19 19 8 1 1	12150: 7 13 37 13 1 2
486: 1 7 18 0 0 1	1350: 5 9 11 9 5 0	13068: 7 13 39 −9 6 1
490: 3 3 14 0 0 1	1350: 7 7 7 −1 1 1	* 24696: 11 15 39 −3 6 3
500: 1 9 15 5 0 1	1452: 7 7 8 2 2 3	
500: 3 7 7 1 2 3	1500: 7 7 8 −2 2 1	
504: 3 5 11 4 3 3	1512: 5 8 11 4 1 4	
540: 5 5 8 2 4 5	* 1620: 5 8 11 −4 1 2	
588: 3 3 17 −1 1 1	1620: 8 9 9 9 6 6	
600: 5 7 7 6 5 5	1764: 8 9 9 −3 6 6	
648: 1 7 25 5 1 1	1800: 5 11 11 7 5 5	
648: 5 5 8 4 4 1	2025: 7 7 13 2 7 4	
675: 1 4 45 0 0 1	2058: 5 10 13 10 1 2	
675: 5 6 6 3 0 0	* 2160: 5 9 15 9 3 3	
675: 3 7 10 5 0 3	2250: 1 19 30 0 0 1	
676: 5 5 8 −2 2 3	2250: 3 7 30 0 0 3	
686: 1 9 21 7 0 1	2250: 9 9 11 9 9 3	
686: 3 5 13 1 3 2	2430: 9 9 11 3 6 9	
702: 5 5 9 3 3 4	2450: 1 9 70 0 0 1	
* 720: 3 5 15 3 3 3	* 2592: 5 9 17 6 5 3	
750: 1 10 19 0 1 0	2646: 5 5 33 −3 3 4	
750: 3 3 22 2 2 1	2646: 6 7 19 7 6 0	
750: 3 7 10 0 0 3	2700: 3 7 37 4 3 3	
756: 1 13 15 3 0 1	3042: 3 17 17 8 3 3	
756: 5 5 8 2 2 1	3267: 7 10 13 5 1 4	
810: 1 7 31 5 1 1	* 3375: 2 15 32 15 1 0	

Table 2. The regular even forms with their discriminants. When the form is indented, it is not alone in its genus. An asterisk before a form means that there is as yet no proof of its regularity.

1:	1 1 1 0 0 0	24:	1 4 6 0 0 0	49:	3 5 5 −4 2 2
2:	1 1 2 0 0 0	24:	1 4 7 4 0 0	50:	1 5 10 0 0 0
3:	1 1 3 0 0 0	24:	1 5 5 2 0 0	52:	3 3 7 2 2 2
3:	1 2 2 2 0 0	24:	2 2 7 2 2 0	54:	1 3 18 0 0 0
4:	1 1 4 0 0 0	24:	3 3 3 0 0 2	54:	1 6 9 0 0 0
4:	1 2 2 0 0 0	25:	1 2 13 2 0 0	54:	2 3 9 0 0 0
5:	1 1 5 0 0 0	25:	1 5 5 0 0 0	54:	2 5 6 0 0 2
5:	1 2 3 2 0 0	25:	2 3 5 0 0 2	56:	1 5 12 4 0 0
6:	1 1 6 0 0 0	27:	1 3 9 0 0 0	56:	3 4 6 4 2 0
6:	1 2 3 0 0 0	27:	1 6 6 6 0 0	60:	1 4 16 4 0 0
7:	2 2 3 2 2 2	27:	2 2 9 0 0 2	60:	2 5 6 0 0 0
8:	1 1 8 0 0 0	27:	2 3 5 0 2 0	60:	2 6 7 6 2 0
8:	1 2 4 0 0 0	28:	1 4 8 4 0 0	63:	2 5 7 2 2 0
8:	1 3 3 2 0 0	28:	2 3 5 2 0 0	63:	3 3 7 0 0 0
8:	2 2 3 2 2 0	28:	2 3 6 2 0 2	64:	1 2 32 0 0 0
9:	1 1 9 0 0 0	30:	1 3 10 0 0 0	64:	1 4 16 0 0 0
9:	1 2 5 2 0 0	32:	1 2 16 0 0 0	64:	1 5 13 2 0 0
9:	1 3 3 0 0 0	32:	1 4 8 0 0 0	64:	1 8 8 0 0 0
9:	2 2 3 0 0 2	32:	1 6 6 4 0 0	64:	2 3 11 2 0 0
10:	1 2 5 0 0 0	32:	2 3 7 2 2 2	64:	3 3 8 0 0 2
11:	1 2 6 2 0 0	32:	2 4 5 4 0 0	64:	3 5 5 2 -2 2
12:	1 1 12 0 0 0	32:	3 3 4 0 0 2	64:	4 5 5 2 4 4
12:	1 2 6 0 0 0	32:	3 3 5 −2 2 2	72:	1 9 9 6 0 0
12:	1 3 4 0 0 0	35:	1 2 18 2 0 0	72:	2 3 12 0 0 0
12:	1 4 4 4 0 0	36:	1 3 12 0 0 0	72:	3 3 8 0 0 0
12:	2 2 3 0 0 0	36:	1 6 6 0 0 0	72:	3 4 7 4 0 0
12:	2 3 3 2 2 2	36:	2 3 6 0 0 0	72:	3 5 5 2 0 0
14:	1 3 5 2 0 0	36:	2 5 5 4 2 2	72:	5 5 5 −2 4 4
15:	2 2 5 0 0 2	36:	3 3 4 0 0 0	75:	1 10 10 10 0 0
15:	2 3 3 0 0 2	36:	3 4 4 4 0 0	75:	2 3 15 0 0 2
16:	1 1 16 0 0 0	39:	2 3 7 0 2 0	80:	1 8 12 8 0 0
16:	1 2 8 0 0 0	40:	1 5 8 0 0 0	80:	3 3 11 −2 2 2
16:	1 4 4 0 0 0	40:	3 3 6 −2 2 2	80:	3 4 7 0 2 0
16:	1 4 5 4 0 0	45:	1 6 9 6 0 0	80:	3 6 6 −4 2 2
16:	2 3 3 2 0 0	45:	2 2 15 0 0 2	80:	4 4 7 4 4 0
16:	3 3 3 −2 2 2	48:	1 4 12 0 0 0	81:	1 9 9 0 0 0
18:	1 3 6 0 0 0	48:	1 4 13 4 0 0	81:	2 3 14 0 2 0
18:	2 3 3 0 0 0	48:	1 8 8 8 0 0	81:	2 5 9 0 0 2
20:	1 2 10 0 0 0	48:	2 3 10 2 0 2	84:	3 3 11 2 2 2
20:	1 3 7 2 0 0	48:	2 3 8 0 0 0	90:	1 3 30 0 0 0
20:	2 3 4 0 0 2	48:	2 5 5 2 0 0	96:	1 10 10 4 0 0
20:	3 3 3 2 2 2	48:	3 3 6 2 2 0	96:	1 4 24 0 0 0
21:	1 1 21 0 0 0	48:	3 3 7 −2 2 2	96:	1 6 16 0 0 0
23:	2 3 5 2 0 2	48:	3 4 4 0 0 0	96:	2 7 8 4 0 2
24:	1 1 24 0 0 0	48:	4 4 5 4 4 0	96:	4 4 7 0 4 0

Table 2. Continued.

96: 4 5 6 0 0 4	144: 3 4 12 0 0 0	* 224: 3 6 14 4 2 2
96: 4 5 7 2 4 4	144: 3 7 7 2 0 0	225: 3 10 10 10 0 0
98: 3 5 7 0 0 2	144: 3 8 8 8 0 0	225: 5 6 9 6 0 0
100: 2 3 20 0 0 2	144: 4 4 11 4 4 0	240: 2 7 19 2 2 2
100: 2 5 10 0 0 0	144: 4 6 7 0 4 0	240: 5 5 12 −4 4 2
100: 3 5 7 0 2 0	144: 4 7 7 2 4 4	240: 5 8 8 8 0 0
100: 3 7 7 −6 2 2	144: 5 5 8 −4 4 2	240: 6 7 8 4 0 6
108: 1 12 12 12 0 0	144: 5 5 8 4 4 4	240: 7 7 8 −4 4 6
108: 1 3 36 0 0 0	147: 3 7 7 0 0 0	243: 2 9 14 0 2 0
108: 1 4 28 4 0 0	150: 2 5 15 0 0 0	245: 6 6 7 0 0 2
108: 1 6 18 0 0 0	160: 3 6 11 6 2 2	252: 5 8 8 4 4 4
108: 1 9 12 0 0 0	175: 5 6 6 2 0 0	256: 1 16 16 0 0 0
108: 2 3 18 0 0 0	176: 1 8 24 8 0 0	256: 1 8 32 0 0 0
108: 2 5 12 0 0 2	180: 2 6 15 0 0 0	256: 3 3 32 0 0 2
108: 2 6 11 6 2 0	180: 2 6 17 6 2 0	256: 3 11 11 −10 2 2
108: 2 6 9 0 0 0	180: 3 8 8 4 0 0	256: 3 8 11 0 2 0
108: 3 4 10 4 0 0	180: 4 7 8 4 4 0	256: 4 5 16 0 0 4
108: 3 5 8 4 0 0	189: 1 9 21 0 0 0	256: 5 5 12 4 4 2
108: 4 6 7 6 4 0	189: 2 5 21 0 0 2	270: 3 9 11 6 0 0
108: 5 5 5 −2 2 2	192: 1 8 24 0 0 0	288: 2 12 15 12 0 0
112: 3 6 7 2 2 2	192: 1 16 16 16 0 0	288: 2 3 48 0 0 0
112: 3 7 7 6 2 2	192: 2 5 20 4 0 0	288: 3 8 12 0 0 0
112: 5 5 5 2 2 2	192: 3 3 24 0 0 2	288: 3 8 14 8 0 0
117: 1 6 21 6 0 0	192: 3 7 11 6 2 2	288: 5 5 12 0 0 2
120: 1 12 13 12 0 0	192: 3 8 8 0 0 0	288: 5 5 14 −4 4 2
120: 3 4 11 4 0 0	192: 4 5 13 2 4 4	288: 5 5 14 2 2 4
121: 2 6 11 0 0 2	192: 4 7 8 0 0 4	300: 1 10 30 0 0 0
125: 1 5 25 0 0 0	192: 5 5 8 0 0 2	300: 4 10 11 10 4 0
125: 2 5 13 0 2 0	192: 5 7 7 6 2 2	300: 5 8 8 4 0 0
128: 1 8 16 0 0 0	192: 7 7 7 −2 6 6	320: 1 8 40 0 0 0
128: 3 3 16 0 0 2	196: 3 5 14 0 0 2	320: 3 11 11 6 2 2
128: 3 4 11 0 2 0	196: 4 7 8 0 4 0	320: 4 8 13 8 4 0
128: 3 5 10 4 0 2	196: 5 5 10 −2 2 4	320: 7 7 7 −2 2 2
128: 3 7 7 −2 2 2	200: 1 5 40 0 0 0	324: 3 4 28 4 0 0
128: 4 5 8 0 0 4	200: 4 6 11 2 4 4	324: 4 7 15 6 0 4
128: 4 7 7 6 4 4	216: 1 12 21 12 0 0	336: 4 4 23 4 4 0
132: 4 7 7 6 0 4	216: 1 9 24 0 0 0	343: 2 11 18 8 2 2
135: 2 3 23 0 2 0	216: 2 11 11 4 2 2	343: 3 5 26 −4 2 2
135: 3 7 7 4 0 0	216: 2 5 24 0 0 2	351: 3 7 18 6 0 0
135: 6 6 7 6 6 6	216: 3 4 19 4 0 0	360: 1 12 33 12 0 0
144: 1 4 36 0 0 0	216: 3 8 11 8 0 0	360: 3 4 31 4 0 0
144: 1 12 12 0 0 0	216: 4 7 10 2 4 4	368: 5 8 12 8 4 0
144: 1 6 24 0 0 0	216: 5 5 9 0 0 2	375: 2 5 38 0 2 0
144: 2 5 17 4 2 2	216: 5 6 8 0 4 0	375: 5 6 14 6 0 0
144: 3 3 19 −2 2 2	216: 5 8 8 8 2 4	375: 6 6 11 2 2 2

Table 2. Continued.

384: 1 16 24 0 0 0	512: 3 11 16 0 0 2	768: 7 7 20 4 4 6
384: 4 11 11 6 4 4	512: 3 11 19 −10 2 2	784: 3 19 19 −18 2 2
384: 4 5 24 0 0 4	512: 4 7 23 6 4 4	784: 5 10 17 6 2 2
384: 4 7 15 6 0 0	512: 5 12 12 −8 4 4	784: 5 12 17 12 2 4
384: 4 7 16 0 0 4	512: 5 8 13 0 2 0	800: 6 11 14 −6 4 2
384: 5 7 13 −6 2 2	512: 7 7 12 −4 4 2	832: 7 12 12 8 4 4
384: 7 8 8 0 4 4	512: 7 7 15 −2 6 6	864: 5 14 14 4 4 4
392: 3 7 19 0 2 0	529: 5 10 14 10 2 4	864: 5 6 29 0 2 0
392: 5 10 10 −8 2 2	540: 5 8 17 8 2 4	864: 5 8 24 0 0 4
396: 5 5 17 −2 2 2	540: 6 7 13 2 0 0	864: 6 9 17 6 0 0
400: 3 3 51 −2 2 2	540: 6 7 18 6 0 6	864: 7 10 15 −6 6 2
400: 3 7 20 0 0 2	560: 1 8 72 8 0 0	864: 8 11 11 −2 4 4
400: 3 7 22 −6 2 2	567: 7 7 13 −2 2 4	864: 9 11 11 −2 6 6
400: 4 11 11 2 4 4	576: 3 8 24 0 0 0	900: 10 12 13 12 10 0
400: 5 8 12 8 0 0	576: 1 24 24 0 0 0	900: 3 10 30 0 0 0
400: 7 7 12 −4 4 6	576: 3 16 16 16 0 0	900: 4 15 16 0 4 0
405: 2 14 17 10 2 2	576: 4 13 13 2 4 4	900: 7 7 23 −2 2 6
432: 1 12 36 0 0 0	576: 4 7 24 0 0 4	960: 5 8 24 0 0 0
432: 1 24 24 24 0 0	576: 5 5 24 0 0 2	960: 4 5 61 2 4 4
432: 2 11 23 10 2 2	576: 5 8 17 −4 2 4	960: 7 7 24 0 0 6
432: 3 10 16 8 0 0	576: 6 7 15 6 0 0	960: 7 8 19 4 2 4
432: 3 4 36 0 0 0	576: 7 10 10 −4 4 4	960: 8 11 11 2 0 0
432: 3 8 20 8 0 0	576: 7 7 15 −6 6 2	972: 5 8 29 8 2 4
432: 4 12 13 12 4 0	576: 8 9 9 6 0 0	* 1008: 7 8 20 0 4 4
432: 4 7 19 2 4 4	588: 5 12 12 −4 4 4	1008: 12 12 13 12 12 0
432: 5 5 20 −4 4 2	600: 5 8 17 8 0 0	1008: 5 8 28 8 4 0
432: 5 8 12 0 0 4	600: 7 7 15 0 0 6	1024: 3 11 32 0 0 2
432: 5 8 14 8 2 4	624: 5 5 28 −4 4 2	1024: 5 12 20 8 4 4
432: 6 7 15 6 6 6	675: 2 15 23 0 2 0	1024: 5 13 20 −12 4 2
432: 6 8 11 4 6 0	675: 7 7 15 0 0 4	1029: 5 10 21 0 0 2
432: 7 7 10 −2 2 4	675: 9 11 11 −8 6 6	1080: 3 11 35 10 0 0
432: 7 7 10 4 4 2	676: 7 8 15 8 2 4	1080: 9 12 14 12 6 0
432: 8 8 11 4 8 8	720: 2 17 24 12 0 2	1125: 2 15 38 0 2 0
432: 8 8 9 0 0 8	720: 4 7 31 2 4 4	1125: 6 14 15 0 0 6
441: 1 21 21 0 0 0	720: 6 8 17 4 6 0	1125: 7 13 13 −4 2 2
441: 5 10 10 6 2 2	720: 8 8 15 0 0 8	1152: 3 8 48 0 0 0
448: 5 8 12 0 4 0	720: 8 8 17 4 8 8	1152: 5 12 20 0 4 0
448: 3 8 19 0 2 0	756: 4 15 16 12 4 0	1152: 5 14 20 −8 4 4
448: 4 5 29 2 4 4	756: 8 11 11 2 4 8	1152: 5 17 17 −14 2 2
448: 7 7 12 4 4 6	768: 1 16 48 0 0 0	1152: 5 5 48 0 0 2
450: 5 6 15 0 0 0	768: 3 11 27 −10 2 2	1152: 8 12 15 12 0 0
500: 3 7 27 −6 2 2	768: 5 13 13 −6 2 2	1152: 8 12 17 12 8 0
500: 4 11 15 10 0 4	768: 5 8 20 0 4 0	1188: 8 11 15 6 0 4
507: 2 7 39 0 0 2	768: 7 12 12 −8 4 4	1200: 1 40 40 40 0 0
512: 1 8 64 0 0 0	768: 7 7 16 0 0 2	1200: 10 11 16 8 0 10

Table 2. Continued.

1200: 11 11 14 −6 6 8	1728: 7 15 19 −6 2 6	3024: 4 31 31 26 4 4
1200: 4 11 31 2 4 4	1728: 7 7 39 −6 6 2	3024: 8 20 23 4 8 8
1200: 8 12 17 4 8 8	1728: 8 11 23 10 4 4	3072: 7 20 23 −4 2 4
1215: 7 13 18 −12 6 2	1728: 8 11 24 0 0 8	3072: 7 23 23 −18 2 2
1225: 2 18 35 0 0 2	1728: 8 12 23 12 8 0	3087: 7 15 30 6 0 0
1280: 7 12 16 0 0 4	1728: 8 15 15 6 0 0	3136: 3 19 56 0 0 2
1280: 4 13 29 10 4 4	1728: 9 17 17 −14 6 6	3136: 11 11 32 −8 8 6
1280: 7 15 15 −2 6 6	1764: 8 11 23 2 8 4	3136: 12 17 20 4 8 12
1296: 5 8 36 0 0 4	1792: 7 12 23 −4 2 4	3136: 5 12 56 0 0 4
1296: 3 8 56 8 0 0	1800: 11 11 15 0 0 2	3375: 10 19 21 6 0 10
1296: 4 19 19 2 4 4	1800: 5 21 21 18 0 0	3375: 11 14 26 14 2 4
1296: 8 11 20 4 8 8	1872: 4 7 79 2 4 4	3375: 6 19 34 8 6 6
1323: 3 14 35 14 0 0	1936: 8 11 24 0 8 0	3375: 7 13 42 −12 6 2
1323: 3 7 63 0 0 0	2000: 4 20 31 20 4 0	3456: 11 11 32 −8 8 2
1344: 12 12 15 12 12 8	2000: 7 7 52 −4 4 6	3456: 11 15 23 −6 2 6
1350: 7 13 15 0 0 2	2025: 10 13 22 8 10 10	3456: 15 15 20 12 12 6
1452: 8 13 17 2 4 8	2048: 7 12 28 −8 4 4	3456: 5 24 29 0 2 0
1500: 8 13 17 2 8 4	2048: 7 15 23 10 2 6	3456: 8 11 44 8 8 4
1521: 6 13 21 0 6 0	* 2112: 7 15 23 −6 2 6	3456: 9 17 24 0 0 6
1536: 11 11 16 −8 8 2	2112: 11 11 19 2 2 6	3600: 10 13 37 8 10 10
1536: 11 11 16 8 8 6	2160: 12 13 21 6 12 12	3600: 11 11 35 −10 10 2
1536: 4 11 40 8 0 4	2160: 13 13 13 2 2 2	3600: 12 13 33 6 12 12
1536: 5 13 24 0 0 2	2160: 5 5 92 −4 4 2	3600: 13 13 22 −2 2 4
1536: 5 13 28 −12 4 2	2160: 7 18 19 6 2 6	3600: 3 40 40 40 0 0
1536: 7 15 16 0 0 6	2160: 7 19 19 14 2 2	3840: 11 19 19 6 2 2
1536: 7 15 20 −12 4 6	2304: 3 16 48 0 0 0	3840: 7 23 31 −22 6 2
1600: 3 27 27 −26 2 2	2304: 11 16 16 0 8 8	3888: 8 9 56 0 8 0
1600: 5 8 40 0 0 0	2304: 13 13 16 8 8 2	3920: 7 24 24 8 0 0
1600: 7 12 23 12 2 4	2304: 4 13 48 0 0 4	3969: 10 13 34 2 10 4
1600: 8 17 17 14 8 8	2304: 5 20 29 20 2 4	4032: 5 29 29 10 2 2
1620: 8 11 23 2 4 8	2304: 7 15 24 0 0 6	4356: 7 19 39 −18 6 2
1728: 1 24 72 0 0 0	2304: 7 15 28 −12 4 6	4500: 11 11 39 6 6 2
1728: 1 48 48 48 0 0	2304: 9 17 17 2 6 6	4563: 7 18 39 0 0 6
1728: 5 5 72 0 0 2	2352: 12 17 17 6 12 12	4608: 12 17 32 16 0 12
1728: 8 9 24 0 0 0	2700: 11 11 26 −2 2 8	4608: 15 20 20 8 12 12
1728: 1 16 112 16 0 0	2700: 12 13 25 10 0 12	4608: 17 17 20 −4 4 14
1728: 4 13 37 2 4 4	2700: 9 11 30 0 0 6	4608: 5 20 48 0 0 4
1728: 12 13 16 8 0 12	2800: 5 24 24 8 0 0	4608: 5 20 53 20 2 4
1728: 3 16 40 16 0 0	* 2880: 11 16 19 8 2 8	4608: 8 15 39 6 0 0
1728: 3 8 72 0 0 0	2880: 8 15 24 0 0 0	4608: 8 17 41 10 8 8
1728: 4 19 24 0 0 4	2880: 12 15 23 6 12 12	4800: 1 40 120 0 0 0
1728: 4 7 72 0 0 4	2880: 7 16 31 16 6 0	4800: 11 16 31 8 2 8
1728: 5 20 20 −8 4 4	2880: 8 17 24 0 0 8	4800: 17 17 25 −10 10 14
1728: 7 10 28 8 4 4	2880: 8 17 24 12 0 4	4800: 4 11 120 0 0 4
1728: 7 13 21 6 6 2	2880: 8 21 21 18 0 0	4800: 4 31 40 0 0 4

Table 2. Continued.

5120:	13	16	29	8	10	8
5184:	3	16	112	16	0	0
5184:	7	15	55	6	−2	6
5184:	16	19	27	18	0	16
5292:	13	13	33	−6	6	2
5292:	5	17	68	−16	4	2
5376:	16	16	23	8	8	0
5400:	13	22	22	−16	2	2
5400:	7	15	52	0	4	0
5488:	5	12	101	12	2	4
5488:	8	21	37	14	8	0
5616:	12	21	28	12	0	12
6000:	21	21	21	2	18	18
6000:	5	24	56	24	0	0
6000:	5	8	152	8	0	0
6075:	9	11	71	−8	6	6
6144:	11	16	43	−8	10	8
6144:	11	19	32	−8	8	2
* 6336:	5	20	68	−8	4	4
6336:	17	20	20	−8	4	4
6400:	3	27	80	0	0	2
6400:	11	16	44	16	4	8
6400:	17	17	32	16	16	14
6480:	8	23	39	6	0	8
6912:	1	48	144	0	0	0
6912:	9	17	48	0	0	6
6912:	5	20	77	20	2	4
6912:	11	27	32	−24	8	6
6912:	12	23	32	16	0	12
6912:	13	16	37	8	2	8
6912:	13	21	28	12	4	6
6912:	15	23	23	14	6	6
6912:	16	19	28	4	16	8
6912:	7	15	76	−12	4	6
6912:	7	28	39	−12	6	4
7056:	19	19	27	−6	6	18
7056:	4	43	43	2	4	4
* 8000:	11	16	51	8	2	8
8000:	15	16	39	16	10	0
8000:	7	12	103	12	2	4
8100:	7	27	52	−24	4	6
8112:	8	28	41	4	8	8
8464:	11	19	51	−14	6	10
8640:	13	24	28	0	4	0
8640:	19	19	28	4	4	14
8640:	5	29	68	−28	4	2
8640:	7	24	52	0	4	0
9072:	13	28	28	−16	4	4
9216:	13	16	52	16	4	8
9216:	13	21	37	6	10	6
9408:	17	20	33	20	2	4
10800:	11	26	39	6	6	2
10800:	11	35	39	−30	6	10
10800:	25	25	28	−20	20	10
10800:	8	17	92	4	8	8
10800:	9	41	41	−38	6	6
10816:	11	19	59	−18	10	2
11520:	17	32	32	32	8	16
11520:	21	29	29	26	18	18
12096:	15	16	55	16	6	0
12096:	15	23	44	−20	12	6
12544:	17	20	41	−12	10	4
13068:	13	21	52	12	8	6
13824:	11	32	44	−16	4	8
13824:	11	35	44	28	4	10
13824:	15	20	56	16	0	12
13824:	15	23	44	−4	12	6
14400:	3	40	120	0	0	0
* 14400:	7	23	92	12	4	2
14400:	12	13	120	0	0	12
14400:	12	33	40	0	0	12
14400:	13	33	37	−18	2	6
14400:	16	19	64	8	16	16
14400:	27	28	28	−24	12	12
15552:	9	17	113	−14	6	6
16128:	13	37	37	−22	2	2
16464:	20	31	31	−22	4	4
18000:	15	24	56	24	0	0
18000:	7	52	52	−16	4	4
18000:	8	15	152	0	8	0
18432:	17	20	57	12	6	4
18432:	17	32	41	16	10	16
19008:	11	32	59	8	10	8
19200:	11	16	124	16	4	8
19200:	16	31	44	4	16	8
19440:	13	28	61	28	2	4
19600:	8	35	72	0	8	0
20736:	16	19	76	4	16	8
21168:	12	17	129	6	12	12
21168:	12	28	73	28	12	0
23232:	17	32	52	32	4	8
24000:	13	32	68	32	4	8
24336:	19	19	84	−12	12	14
25600:	17	32	57	−16	6	16
25920:	23	32	44	32	4	8
27648:	23	32	47	16	22	16
28224:	11	32	92	32	4	8
32000:	16	31	71	22	8	8
32400:	27	40	43	40	18	0
34560:	19	28	67	−4	10	4
37632:	17	41	68	−36	12	10
43200:	9	41	120	0	0	6
43200:	11	35	120	0	0	10
43200:	25	48	52	48	20	0
43200:	36	39	44	12	24	36
48384:	16	31	103	−10	8	8
49392:	28	37	60	12	0	28
54000:	13	28	157	28	2	4
54000:	21	29	101	−26	18	6
54000:	21	40	76	40	12	0
54000:	24	45	61	30	24	0
57600:	13	37	132	−36	12	2
57600:	33	48	52	48	12	24
63504:	19	27	136	−24	16	6
69696:	19	28	139	28	2	4
72000:	39	44	44	8	12	12
73008:	28	57	57	−42	12	12
84672:	17	20	257	20	2	4
84672:	33	52	52	−8	12	12
97200:	9	41	281	−38	6	6
112896:	16	43	172	4	16	8
129600:	27	28	187	28	6	12
172800:	39	44	111	−36	6	12
209088:	52	57	84	−12	24	36
338688:	48	73	112	56	0	24

The tables are in ascending order according to the "discriminant". Over time, usage has varied in the literature. For an odd form, the choice

$$\frac{1}{2}\begin{vmatrix} 2a & f & e \\ f & 2b & d \\ e & d & 2c \end{vmatrix}$$

has become fairly standard (it is always an integer). For an even form we use

$$\begin{vmatrix} a & f/2 & e/2 \\ f/2 & b & d/2 \\ e/2 & d/2 & c \end{vmatrix}$$

honoring the Gaussian tradition. (For an even form the discriminant in **[1]** and in Watson's papers is this determinant multiplied by 4.)

§3. *Forms alone in their genera.* In **[9]** Watson reported on his work on finding all forms alone in their genera. He gave the total as 790; 68 were listed and the reader was told how to derive the others. Our total is 794. This is close enough to be considered a moderately good confirmation.

§4. *Diagonal forms.* Jones's proof of the regularity of the 102 diagonal forms he discovered was completed and polished by Jones and Pall in **[5]**. But there remains the proof that there are no other diagonal regular forms. This occupies 47 pages in his thesis and was never published. Our work provides an independent confirmation.

§5. *Watson's thesis.* Part 7 of Watson's thesis **[8]** contains much more on regular ternary forms than he subsequently published in **[10]**. When our work was nearly complete we obtained a copy through the courtesy of C. A. Rogers of University College, London. Some proofs are sketchy, but his two key ideas are clearly set forth: *a priori* bounding of the discriminant in the case of regular odd forms with square-free discriminant, and a method of descent for handling the remaining forms. At the end he presented a list of regular odd forms with discriminant not divisible by 4. The corresponding part of our list differs slightly; it is planned to set forth the details in the followup paper.

§6. *Genera containing more than one regular form.* The regular odd forms 1 1 9 0 0 1 and 1 3 3 3 0 0 of discriminant 27 constitute a genus, and the same is true for 1 1 7 0 1 0 and 1 2 4 1 0 1. These appear in Watson's thesis, anticipating Hsia **[3]** and Kaplansky **[6]**. There is a third example: 1 4 7 0 1 0 and 1 5 7 5 1 1 of discriminant 108. This is mentioned in **[6]** as an added in proof; the two proofs of regularity are contained in **[7]**. There are no other genera that contain more than one regular form.

§7. *More tables.* Two more tables exist; they are available by ordinary mail or electronically, on request to WCJ or IK. The first lists all other forms ("genus mates") in the genera containing the 119 regular forms or candidates that are not alone in their genera. The second lists all exceptional numbers up to a million for these genus mates, that is, eligible numbers that are not represented.

This paper and its tables are likewise available electronically.

Added in proof (September 8, 1997). 1. It has now been checked that 22 candidates represent all eligible numbers up to two million. 2. A document has been prepared that accounts for the 97 $(=913-794-22)$ proofs of regularity. 3. A table has been compiled that lists all non-represented eligible integers up to a million for non-regular forms up to discriminant 22 for odd forms and discriminant 18 for even forms. (These bounds were chosen because after that there are genera with more than two forms.) Items 2 and 3 are available on request.

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11E20: *NUMBER THEORY: Forms and linear algebraic groups; General ternary quadratic forms.*

Received on the 2nd of October, 1996.