VITA

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Representation by Positive
Ternary Quadratic Forms

Abstract of a Dissertation
Submitted to the Graduate Faculty
In Candidacy for the Degree of
Doctor of Philosophy

Department of Mathematics

By
Burton Wadsworth Jones

INTRODUCTION

The problem of this thesis is to find the totality of positive integers represented by certain positive ternary quadratic forms $f = ax^2 + by^2 + cz^2 + ryz + sxz + txy$, i.e. (a, b, c, r, s, t), with integral coefficients where x, y and z range over all integers.

Dirichlet[1] proved that every positive integer not of the form $4^k(8n+7)$ (k and n positive integers or 0) can be represented as the sum of three squares, that is, that the positive integers represented by the form $x^2+y^2+z^2$ are exclusively those not of the form $4^k(8n+7)$. He also applied the same method to prove that (1,1,3) represents all positive integers prime to 3.

Ramanujan[2] in finding the positive quaternary quadratic forms without cross products which represent all positive integers made use of certain results for ternaries

[1] Journal für Mathematik: 40 (1850), pp.228-32.
[2] Proc. Cambridge Philosophical Society, 19, (1916-1919), pp.11-21.

which he stated but did not prove. He noticed that in the case of the form $x^2+y^2+10z^2$, i.e. (1,1,10), the odd integers not represented did not seem to follow a definite law. He could find no formula or formulae even empirically which included all and only the integers not represented.

J. G. A. Arndt[1] proved certain facts he needed with regard to representation by certain ternaries, using Dirichlet's method and elementary transformations.

L. E. Dickson gave a modification of Dirichlet's method necessary for certain types of forms[2], proved results for certain ternaries he needed dealing with certain quaternary forms representing all positive integers[3], and in *The Annals of Mathematics*, (2), 28, (1927), p.333, applied Dirichlet's method and certain elementary transformations to prove results for certain ternaries. In this last article he gave system to dealing with integers represented by forms $f=(a,b,c)$ (a,b,c positive integers) as follows: he called attention to the irregularity noted by Ramanujan in the case of the form (1,1,10) and made the following definition: "All integers not represented by a regular form f coincide with all the positive integers given by certain arithmetical progressions". Otherwise a form is

1. "Ueber die Darstellung ganzer Zahlen als Summen von [illegible]en Kuben", Dissertation, Gottingen, 1925.
2. Bulletin of the American Mathematical Society, 33 (1927), p. 63.
3. American Journal of Mathematics, 49, (1927), p.39.

said to be irregular. Dickson then proceeded to prove the following theorem for a=1 and b and c relatively prime:

Theorem: The form f=(a,b,c), where it is understood that no two of a,b,c have an odd prime factor in common and not all are even, is irregular if there exists a positive odd integer k prime to abc such that k is not represented by f and f≡k(mod 8) is solvable.

This amounts to finding conditions on a positive integer k not represented by f sufficient to assure us that every arithmetical progression containing k contains also positive integers represented by f.

By means of this and other theorems he proved that not more than seventeen forms f are regular where a=1 and b and c are relatively prime and less than certain large integers.

IRREGULAR FORMS

In this section it is noted first that Dickson proved in effect the theorem above and proofs are given for certain additional basal theorems along lines suggested by his work: e.g. here are found the additional conditions on k sufficient to insure irregularity of f when one of its coefficients has a factor in common with k. Then, making use of these theorems and Bertrand's Postulate, it is proved that not more than seventeen forms f=(a,b,c) are regular when no two of a,b,c have a factor in common. Next, it is noted that all but a limited number of forms (a, b, c) are irregular by virtue of the above results, when two of a, b, c are even but no two have an odd prime factor in common, and the theorems are applied to this remaining limited number to prove many irregular. Following this, forms (a, b, c) are dealt with when two of a, b, c have a factor 3 in common but no two have a prime factor greater than 3 in common. With the aid of an additional theorem this process is carried through to prove finally that no regular form (a, b, c) has a prime greater than 7 as a factor of one of its coefficients and that not more than 103 forms (a, b, c) are regular when 1 is the greatest common divisor of a, b and c.

Certain forms (a, b, c, r, s, t) of Hessian less than 21 are proved irregular by referring them back to forms without cross products but no systematic treatment of irregular forms with cross products is attempted.

REGULAR FORMS

These 103* forms are next dealt with. The methods used are those of Dirichlet, Dickson and certain elementary transformations. One method which was found useful was due to Arndt and was based on the easily established fact that all integers represented by x^2+3y^2 with x and y odd are represented with x and y even. A generalization of this result is established. Another kind of elementary trans-

* Actually 102, as (1,5,200) is not regular. W. C. Jagy
MOST handwritten notes by I. Kaplansky.

formation led to the following type of result: if an integer 5n is represented by (1,1,1) then n is represented by (1,1,5) and conversely (if n is an integer), thus making the proof for (1,1,5) result from known facts for (1,1,1). Proofs for 23 regular forms have been published previous to this thesis. Here shorter proofs are given for some of these forms and 74 additional forms are proved regular. Thus the totality of integers represented by each of the following 97 forms has been found by proof:

(1,1,a) where a=1,2,3,4,5,6,8,9,12,16,21 or 24;

(1,2,a) where a=2,3,4,5,6,8,10 or 16;

(1,3,a) where a=3,4,6,9,10,12,18 or 30;

(1,4,a) where a=4,6,8,12,16,24 or 36;

(1,5,a) where a=5,8,10,25,40 or 200; * 9 ≠ 200. W.C. Jagy

(1,6,a) where a=6,9,16,18 or 24;

(1,8,a) where a=8,16,24 or 40;

(1,9,a) where a=9,12,21 or 24; (1,10,30); (1,12,12);

(1,16,a) where a=16,24, or 48; (1,21,21); (1,24,a) where a=24 or 72; (1,40,120); (2,2,3); (2,3,a) where a=3,6,8,9, 12,18 or 48;

(2,5,a) where a=6,10 or 15; (2,6,a) where a=9 or 15;

(3,3,a) where a=4, 7 or 8; (3,4,a) where a=4,12 or 36;

(3,7,a) where a=7 or 63; (3,8,a) where a=8,12,24,48 or 72;

(3,10,30); (3,16,48); (3,40,120); (5,6,15); (5,8,a) where a=24 or 40; (8,9,24) and (8,15,24).

Partial proofs are given for the six remaining forms without cross products not proven irregular: (1,2,32), (1,8,32), (1,8,64), (1,3,36), (1,12,36), (1,48,144). It remains to prove that the first form represents all positive integers of the form $8n+3$, the first three forms all $8n+1$, and the last three all $24n+1$. It is verified that such is the case for all positive integers less than 1000.

Using the above methods and previous results for certain forms, 61 forms with cross products are proved regular. This includes the proofs of the regularity of all reduced forms (a,b,c,r,s,t) (r,s,t not all 0) of Hessian less than 21 not previously proved irregular.

SEMI-REGULAR FORMS

The form (1,1,10) is regular as to evens since it represents exclusively all positive integers not of the form $4^k(16n+6)$[1] but it is irregular as to odds (use $k=3$ in the theorem quoted above). Numerous such semi-regular forms are dealt with in this section. The following theorem is found useful:

Theorem: If $f=(d,db,c)$ and $g=(d,db,cm)$, where all the prime factors of the positive integer m are represented by x^2+by^2, then g represents ma if and only if f represents the integer a, where b and d are positive integers prime to m; i.e. if f is regular, g is regular as to multiples of m.

Other methods of proof are illustrated.

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[1] The Annals of Mathematics, (2), 28, (1927), p. 341.

THE UNIVERSITY OF CHICAGO

REPRESENTATION BY POSITIVE TERNARY QUADRATIC FORMS

A DISSERTATION
SUBMITTED TO THE GRADUATE FACULTY
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY

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INTRODUCTION

The problem of this thesis is to find the positive integers represented (or not represented) by certain positive ternary quadratic forms $f = ax^2 + by^2 + cz^2 + ryz + sxz + txy$ with integral coefficients where x, y, and z range over all integers.

Dirichlet[1] proved that every positive integer not of the form $4^k(8n+7)$ can be represented as the sum of three squares, that is, that the positive integers represented by the form $x^2+y^2+z^2$ are exclusively those not of the form $4^k(8n+7)$. He also applied the same method to prove that $x^2+y^2+3z^2$ represents all positive integers prime to 3.

Ramanujan[2] in finding the positive quaternary quadratic forms without cross products which represent all positive integers made use of certain results for ternaries which he stated but did not prove. He noticed that in the case of the form $x^2+y^2+10z^2$ the odd integers not represented did not seem to follow a definite law. He could find no formula or formulae even empirically which included all and only the integers not represented.

J. C. A. Arndt[3] proved certain facts he needed with regard to representation by certain ternaries, using Dirichlet's methods and elementary transformations.

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3. "Uber die Darstellung ganzer Zahlen als Summen von sieben Kuben", Dissertation, Gottingen, 1925.

L.E.Dickson[1] gave a modification of Dirichlet's method necessary for certain types of forms (Bulletin) and proved results for certain ternaries. In the "Annals of Mathematics" he gave system to dealing with integers represented by forms: $f=ax^2+by^2+cz^2$ as follows: he called attention to the irregularity noted by Ramanujan in the case of the form $x^2+y^2+10z^2$ and made the following definition: "All integers not represented by a regular form $f=ax^2+by^2+cz^2$ coincide with all the positive integers given by certain arithmetical progressions". Otherwise a form is said to be irregular. Dickson established a method to prove forms irregular and applied it to prove that of the forms $x^2+by^2+cz^2$ where b and c are relatively prime and less than certain large integers, all but 17 are irregular.

In Part A this method is supplemented by certain modifications and additional theorems to prove that not more than 103 forms $f=ax^2+by^2+cz^2$ with no factor common to a, b and c are regular and several forms with cross products are proved to be irregular.

In Part B the methods of Dirichlet, Dickson and Arndt together with modifications and additional theorems are applied to prove most of the 103* forms without cross products and many with cross products to be regular.

In Part C certain semi-regular forms are dealt with.

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[1] American Journal of Mathematics, 49, (1927), p.39.
Bulletin of the American Mathematical Society, 33 (1927), p.63.
Annals of Mathematics (2), 28 (1927), p.333.

* Actually 102. Form 92 in Table IV, page 130, is not regular. Form 92 is (1,5,200). This note by W.C. Jagy. Faulty proof page 91.

NOTATIONS

1. We denote the form $f = ax^2 + by^2 + cz^2$ by (a,b,c) and $f = ax^2 + by^2 + cz^2 + ryz + sxz + txy$ by (a,b,c,r,s,t).
2. All letters assume only integral values unless the contrary is specifically stated.
3. $f = mF$ or $f/m = F$ where f and F are forms shall be taken to mean: the multiples of m represented by f coincide with m times the integers represented by F.
4. $f = g \equiv 1 \pmod 4$ shall mean that the integers $\equiv 1 \pmod 4$ represented by form f coincide with those integers $\equiv 1 \pmod 4$ represented by form g.
5. $f \neq k$ where f is a form and k an integer shall mean that f does not represent k.
6. The letters, f, F, g, h, φ, λ shall generally be used to denote forms.
7. a, b and c are positive integers unless the contrary is stated.

PART A

IRREGULAR FORMS

I. Theorems.

The following lemmas and theorems have been proved by L. E. Dickson:[1]

Lemma 1. If p is an odd prime dividing neither a nor b and if k is any integer, $ax^2+by^2\equiv k \pmod p$ has integral solutions.

Lemma 2. If no one of a, b, c is divisible by the odd prime p, $f\equiv k \pmod p$ has solutions with x and y not both divisible by p, where $f=ax^2+by^2+cz^2$.

Theorem 1. If p is an odd prime not dividing abc, $f\equiv k \pmod{p^n}$ has solutions when k and n are arbitrary.

Theorem 2. If an odd prime p divides c, but not ab, and if k is prime to p, $f\equiv k \pmod{p^n}$ is solvable.

Theorem 3. If k is odd and if $f\equiv k \pmod 8$ is solvable, then $f\equiv k \pmod{2^n}$ is solvable when n is arbitrary.

We state:

Theorem 4. $f=ax^2+by^2+cz^2$ (where no two of a, b, c have an odd prime factor in common and not all are even) is irregular if there exists a positive odd integer k prime to abc such that k is not represented by f and $f\equiv k \pmod 8$ is solvable. $(a, b, c > 0)$

1 Annals of Math. (2) vol. 28 (1927) p. 333.

The proof carries through exactly as in Dickson's paper taking the coefficient of x^2 to be a instead of 1.

Implicit in the proof of the above theorem are three sub-theorems which we will state for purposes of reference.

Theorem 4a. $f \equiv k \pmod{N}$ solvable for all N and $f \neq k$ implies that f is irregular.

Theorem 4b. $f \equiv k \pmod{N}$ is solvable for all odd N containing no factor common to two of a, b, c, k.

Theorem 4c. $f \equiv k \pmod{N}$ and $f \equiv k \pmod{N'}$ solvable implies that $f \equiv k \pmod{NN'}$ is solvable.

We prove:

Lemma 3. $f = ax^2 + by^2 + pc'z^2 \equiv pk' \pmod{p^n}$, where a and b are prime to p, a prime, and n is a positive integer (c' and k' may contain p as a factor) is solvable for n arbitrary if $f \equiv pk' \pmod{8}$ or $f \equiv pk' \pmod{p}$ for p even or odd respectively has a solution $x = \xi$, $y = \eta$, $z = \zeta$ where ξ and η are prime to p; i.e. has a solution with two of ξ, η, ζ prime to p for $\xi \not\equiv 0 \pmod{p}$ implies $\eta \not\equiv 0 \pmod{p}$.

Proof by induction:

1) If $p = 2$. Suppose $f \equiv pk' \pmod{p^m}$ $(m \geq 3)$ has a solution $x = \xi_1$, $y = \eta_1$, $z = \zeta_1$ where ξ_1 and η_1 are prime to p. We then know that $a\xi_1^2 + b\eta_1^2 + pc'\zeta_1^2 = pk' + rp^m$ where r is an integer.

Let $x = \xi_1 + p^{m-1}X$, $y = \eta_1 + p^{m-1}Y$, $z = \zeta_1 + p^m Z$. Then $f = a\xi_1^2 + 2a\xi_1 p^{m-1}X + ap^{2m-2}X^2 + b\eta_1^2 + 2b\eta_1 p^{m-1}Y + bp^{2m-2}Y^2 + c'\zeta_1^2 + 2c'p^{m+1}\zeta_1 Z + c'p^{2m+1}Z^2 \equiv pk' + p^m(r + a\xi_1 X + b\eta_1 Y) \pmod{p^{m+1}}$. Now $-r \equiv a\xi_1 X + b\eta_1 Y \pmod{p}$

is solvable for X and Y since a, b, ξ_1, η_1 are prime to p. If the last congruence has the solutions $X = X'$, $Y = Y'$, then $x = \xi_1 + p^{m-1}X'$, $y = \eta_1 + p^{m-1}Y'$, $z = \zeta_1 + p^m Z'$ where Z' is arbitrary are solutions of $f \equiv pk' \pmod{p^{m+1}}$ with x and y prime to p. Thus the induction is complete.

2) If p is odd we proceed in the same manner except that we take $m \geq 1$ and $x = \xi_1 + p^m X$, $y = \eta_1 + p^m Y$, $z = \zeta_1 + p^m Z$.

<u>Corollary 1.</u> If above ξ_1, η_1, ζ_1 are solutions for $n = 1$ or 3 according as p is odd or even, then $f \equiv pk' \pmod{p^n}$ is solvable with $x \equiv \xi_1 \pmod{p}$, $y \equiv \eta_1 \pmod{p}$, $z \equiv \zeta_1 \pmod{p}$ for n arbitrary. (This results directly from the manner of choice of x, y, and z solutions).

<u>Corollary 2.</u> If $f = ax^2 + by^2 + cz^2 \equiv 2^r k \pmod{8}$ is solvable when a, b and c are odd, with two of x, y, z odd, then $f \equiv 2^r k \pmod{2^n}$ is solvable. For suppose x and y are odd. Then $z = 2z'$ and $f = ax^2 + by^2 + 4cz'^2 \equiv 2^r k \pmod{2^n}$ is solvable from Lemma 3.

<u>Theorem 5.</u>[1] $f = ax^2 + by^2 + cz^2$ (where no two of a, b, c have an odd prime factor in common) is irregular if we can find a positive integer k having in common with abc the prime factors p_i $(i = 1, \ldots, r)$ with the following properties:

1) f does not represent k.
2) $f \equiv k \pmod{p_i}$ is solvable with two of x, y, z, prime to p_i $(i = 1, \ldots, r)$.

---

1 For k even cf. Theorem 13 <u>Annals of Math</u>. (2) Vol. 28 (1927) p. 338.

3) $f \equiv k(\text{mod } 8)$ is solvable.

4) k can be taken even only if just one of a, b, c is even and $f \equiv k(\text{mod } 8)$ is solvable with two of x, y, z odd.

Proof: Conditions 2) and 4) on k are sufficient from lemma 3 to assure us that $f \equiv k(\text{mod } p_i^n)$ is solvable for every p_i and n arbitrary. Then theorems 3, 4b, 4c, 4a in succession complete the proof in view of condition 1) on k.

__Lemma 4a.__ If $g = ax^2 + pby^2 \equiv pk'(\text{mod } p^3)$ where p is an odd prime, prime to a, has a solution $x = p\xi_1$, $y = \eta$ where ξ_1 is prime to p, then $g \equiv pk'(\text{mod } p^n)$ is solvable for n arbitrary and positive.

Proof by induction: Suppose that for $n = m \geq 3$ there exist $x = \xi = \xi_1 p$ and $y = \eta$ where ξ_1 is prime to p, solutions of $g \equiv pk'(\text{mod } p^m)$. Then $a\xi^2 + pb\eta^2 = pk' + rp^m$ where r is an integer.

Let $x = \xi + p^{m-1}X$, $y = \eta + p^m Y$. And substituting get $g = a\xi^2 + 2a\xi p^{m-1}X + ap^{2m-2}X^2 + pb\eta^2 + 2bp^{m+1}\eta Y + bp^{2m+1}Y^2 \equiv a\xi^2 + 2a\xi p^{m-1}X + pb\eta^2 = pk' + p^m(r + 2a\xi_1 X) \quad (\text{mod } p^{m+1})$.

Now $r + 2a\xi_1 X \equiv 0(\text{mod } p)$ has a solution $X = X'$ since a and ξ_1 are prime to p. Thus $g \equiv pk'(\text{mod } p^{m+1})$ has solutions $x = p(\xi_1 + {}^{m-2}X)$ and $y = \eta + p^m Y$ (Y arbitrary) where $x = px'$ with x' prime to p since ξ_1 is prime to p and $m \geq 3$. Thus the induction is complete.

__Lemma 4b.__ $F = ax^2 + pby^2 + cz^2 \equiv pk' \ (\text{mod } p^n)$ where a and c are prime to p, is solvable for n an arbitrary positive integer if there exists an integer t such that $bt^2 - k' = pr$ where r is an integer prime to p.

Proof: Let $x=px'$, $z=pz'$ and we have $F\equiv pk' \pmod{p^3}$ is solvable if $apx'^2+cpz'^2+by^2\equiv pk' \pmod{p^2}$ is solvable. Setting $y=t$ we see that the last congruence is solvable if $ax'^2+cz'^2\equiv -r \pmod p$. The last congruence is solvable by lemma 1 and furthermore, since r is prime to p, x' and z' are not both divisible by p. Suppose x' is prime to p. Then $F\equiv pk' \pmod{p^3}$ has a solution $x=px'$ where x' is prime to p and thus from lemma 4a, $F\equiv pk' \pmod{p^n}$ has a solution with $z=0$ for n arbitrary. We may proceed similarly if z' is prime to p, taking $x=0$.

Theorem 6.[1] $f=ax^2+by^2+cz^2$ (where no two of a, b, c have an odd prime factor in common) is irregular if we can find a positive odd integer k having in common with abc the prime factors p_i $(i=1,\ldots,v)$ with the following properties:

1) f does not represent k.
2) For π a prime factor common to k and c there exist integers r and t, where r is prime to π, such that

$$ct^2/\pi - k/\pi = \pi r$$

and similarly for other factors common to k and c, to k and b, to k and a.

3) $f\equiv k \pmod 8$ is solvable.

Proof: Condition 2) on k assures us from lemma 4b that $f\equiv k \pmod{p_i^n}$ is solvable for every p_i and n arbitrary. Then

---

1 Cf. Lemma 3, Annals of Math. (2) vol. 28, (1927), p.339.

theorems 3, 4b, 4a in succession complete the proof in view of conditions 1) and 3) on k.

<u>Lemma 5</u>. $f=ax^2+pb'y^2+pc'z^2 \equiv k \pmod{p^n}$, where k and a are prime to p, an odd prime, is solvable for n arbitrary if $f \equiv k \pmod p$ is solvable.

Proof: Set $z=0$ and to prove the theorem by induction, suppose there exists a ξ and an η such that $a\xi^2+pb\eta^2=k+rp^m$ $(m \geq 1)$. Let $x=\xi+p^mX$, $y=\eta+p^m$. Then $f=a\xi^2+2a\xi p^mX+ap^{2m}X^2+b\eta^2p+2\eta bp^{1+m}+bp^{2m+1} \equiv a\xi^2+2a\xi p^mX+bp\eta^2 \equiv k+p^m(r+2a\xi X) \pmod{p^{m+1}}$. Now ξ is prime to p since k is, a and 2 are prime to p and thus $r+2a\xi X \equiv 0 \pmod p$ has a solution $X=X'$. Then a solution of $f \equiv k \pmod{p^{m+1}}$ is $x=\xi+p^mX'$, (prime to p), $y=\eta+p^m$, $z=0$ and the induction is complete.

<u>Theorem 7</u>. $f=ax^2+by^2+cz^2$, where there is no factor common to a, b and c, and p_i $(i=1,\ldots,t)$ are all the odd prime factors common to any two of a, b, c, is irregular if there exists a positive odd integer k prime to abc such that $f \equiv k \pmod{p_i}$ is solvable $(i=1,\ldots,t)$, $f \equiv k \pmod 8$ is solvable and f does not represent k. If k is even we have the further condition on k that $f \equiv k \pmod 8$ be solvable with two of x, y, z odd, from lemma 3.

Proof: Lemma 5 applies to show that $f \equiv k \pmod{p_i^n}$ is solvable for any p_i and n arbitrary. Then theorems 3, 4c, 4b, 4a apply successively to prove the theorem.

Note: If $f=ax^2+by^2+cz^2$ is irregular as to multiples of a number m it is irregular, i.e. if $f=mg$ (see notations) where g is irregular, f is. For, since g is irregular there exists a k not represented by g such that $g \equiv k \pmod{N}$ is solvable for any N. Thus f does not represent mk and $f \equiv mk \pmod{N}$ is solvable for any N, thus proving by Theorem 4a that f is irregular.

II. $f=x^2+by^2+cz^2$, with b and c relatively prime.

We may without loss of generality take $b\leq c$. We prove that all forms f not given in Table I are irregular. In most cases we apply Theorem 4 and exhibit a positive integer k such that $f\neq k$, k is prime to b and c and $f\equiv k \pmod 8$ is solvable, thus proving the form to be irregular.

b = 1

We shall prove f is irregular unless c = 1, 2, 3, 4, 5, 6, 8, 9, 12, 16, 21, 24.

A. If $c\equiv 2 \pmod 4$, $c\neq 2, 6$, consider

(i) $c\equiv 6 \pmod 8$. Take $k=c/2+4\equiv 3 \pmod 4$. $f\neq k$ for $k<c$ since $c>8$, k is prime to c, and $f\equiv k \pmod 8$ is solvable.

(ii) $c\equiv 2 \pmod 8$. Take $k=c/2+2\equiv 3 \pmod 4$, $f\neq k$ for $k<c$ since $c>4$, k is prime to c, and $f\equiv k \pmod 8$ is solvable.

B. If $c\equiv 3 \pmod 4$, $c\neq 3$, take $k=c-4\equiv 3 \pmod 4$, for $k>0$ since $c>4$.

C. If $c\equiv 1 \pmod 4$, $c\neq 1, 5, 9, 21$, we know $f\equiv 2+c \pmod 8$ is solvable.

(i) $c\equiv 1 \pmod 8$.

a) If $c\not\equiv 0 \pmod 3$, take $k=3<c$, for $f\neq 3$ and $f\equiv 3 \pmod 8$ is solvable. This takes care of $c=17$.

b) If $c\equiv 0 \pmod 3$, one of $c/3+8$, $c/3+16$ is prime to 3. Choose k to be one which is prime to 3 and therefore to c. $k<c$ since $c>24$, $k\equiv 3 \pmod 8$ and thus $f\neq k$, $f\equiv k \pmod 8$ is solvable.

(ii) $c\equiv 5 \pmod 8$.

a) If $c \not\equiv 0 \pmod 7$, take $k=7<c$, for $f \neq 7, f \equiv 7 \pmod 8$ is solvable.

b) If $c \equiv 0 \pmod 7$, one of $c/7+4$, $c/7-4$ is prime to 7. Choose k to be one which is prime to 7 and therefore to c. $k<c$ since $c>5$, $k \equiv 7 \pmod 8$ and thus $f \neq k$, $f \equiv k \pmod 8$ is solvable.

D. If $c \equiv 0 \pmod 4$, $c \neq 4, 8, 12, 16, 24$.

(i) If $c=4C$ where C is odd, $f \equiv 0 \pmod 4$ implies $x=2X$, $y=2Y$ and $f/4 = X^2+Y^2+Cz^2$ which is irregular from above unless $C=1,3,5,9,21$. If $C=5,9,21$ use $k=6,22,22$ respectively with Theorem 5 to prove f irregular. Otherwise ($C \neq 1,3$) f/4 is irregular and by the note at the end of paragraph I, f is irregular.

(ii) If $c=8C$ where C is odd, $f/4 = X^2+Y^2+2Cz^2$ which is irregular since $C \neq 1,3$ and thus f is irregular.

(iii) If $c=16C$ where C is odd, $f/4 = X^2+Y^2+4Cz^2$ which is irregular, since $C \neq 1$, unless $C=3$. If $C=3$ apply Theorem 6 to f with $\pi = 3$ and $k=21$, put $t=2$ and note that $16 \cdot 2^2 - 7 = 3 \cdot 19$ and thus, since $f \neq k$, $f \equiv k \pmod 8$ is solvable, f is irregular.[1]

(iv) If $c=32C$, where C is odd, $f/4 = X^2+Y^2+8Cz^2$ which is irregular unless $C=1,3$. Then use $k=21,77$ respectively, to prove f irregular from theorem 4.

(v) If $c=64C$, where C is odd, $f/4 = X^2+Y^2+16Cz^2$ which is irregular unless $C=1$ in which case we use $k=21$ to prove

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1 See Annals of Math. (2) vol. 28, p. 339.

f irregular by theorem 4.

(vi) If $c=2^r C$ where $r\geq 7$ and C odd. Then $f/4^s$ is of the form 5) or 4) if $s=(r-6)/2$ or $(r-5)/2$ according as r is even or odd. Thus $f/4^s$ is irregular and therefore f is.

$b=2$

We shall prove f is irregular unless $c=3,5$.

A. If $c\equiv 1 \pmod 8$ take $k=c-2\equiv 7 \pmod 8$ for $f\neq k$ since $x^2+2y^2\not\equiv 7 \pmod 8$ but $f\equiv k \pmod 8$ is solvable.

B. If $c\equiv 3 \pmod 8$ take $k=c-4\equiv 7 \pmod 8$ for $f\neq k$ and $f\equiv k \pmod 8$ is solvable.

C. If $c\equiv 5 \pmod 8$ take $k=c-8\equiv 5 \pmod 8$ for $f\neq k$ since $x^2+2y^2\not\equiv 5 \pmod 8$ but $f\equiv k \pmod 8$ is solvable.

D. If $c\equiv 7 \pmod 8$ take $k=c-2\equiv 5 \pmod 8$ for $f\neq k$ and $f\equiv k \pmod 8$ is solvable.

$b>2$ and b or $c\equiv 1 \pmod 4$.

We shall prove f is irregular unless $b=5$, $c=8$.

A. If b or $c\equiv 1 \pmod 8$, if b or $c\equiv 5 \pmod 8$ and the other odd or $\equiv 4 \pmod 8$, then $f\equiv 2 \pmod 8$ is solvable with two of x, y, z odd, $f\neq 2$ and thus theorem 5 (with the corollary 2 to lemma 3) applies to prove f irregular.

B. If b or $c\equiv 5 \pmod 8$ and the other $\equiv 2 \pmod 8$. Then $f\equiv 8 \pmod 8$ is solvable with x, y, z odd and thus, since $f\neq 8$ we take $k=8$ to prove f irregular.

C. If $b\equiv 5 \pmod 8$ and $c\equiv 6 \pmod 8$ noting that $f\equiv 1,3,5$ or $7 \pmod 8$ is solvable and $x^2+by^2\not\equiv 3 \pmod 4$ we prove f irregular.

(i) If $c \equiv 6 \pmod{16}$

a) If $c > 2b$ take $k = (c-2b)/4 \equiv 3 \pmod 4$ for $0 < k < c$ and thus $f \neq k$, k is prime to b and c.

b) If $c/2 < b < 3c/4$ take $k = 2b - c/2 \equiv 7 \pmod 8$ for k is prime to b and c and $f \neq k$ since $0 < k < c$.

c) If $b > 3c/4$. Then, from Bertrand's Postulate[1] there exists a prime p such that $(b+1)/2 \leq p \leq b-1$. (p is odd since $b \geq 5$). Therefore p is prime to b since $2p > b$. Also p is prime to c unless $c = 2p$ for $3p > 3b/2 > 9c/8 > c$.

Thus if $c \neq 2p$ let $k = p$ for, since $2 < k < b$, $f \neq k$

If $c = 2p$, $b = 5$, then $p = 3$ and use $k = 13$.

If $c = 2p$, $b \neq 5$, then $(b+1)/2 \neq b-2$ and we take k one of the two: $(b+1)/2$, $b-2$ which is not p. Then k is prime to p, b, c; $k \equiv 3 \pmod 4$ and $f \neq k$.

d) None of the inequality signs of cases a), b), c) can be replaced by equalities.

(ii) If $c \equiv 14 \pmod{16}$, $f \equiv 2 \pmod 8$ implies $x = 2X$, $y = 2Y$ and $f/2 = g = 2X^2 + 2bY^2 + cz^2/2 \equiv 1 \pmod 4$ (see Notations). Now $g \equiv 1 \pmod 8$ is solvable and thus $g \equiv 1 \pmod N$ is solvable for all N by theorems 3 and 4b with 4c. Thus $f \equiv 2 \pmod N$ is solvable for all N, $f \neq 2$ and thus by theorem 4a, f is irregular.

D. If $b \equiv 6 \pmod 8$ and $c \equiv 5 \pmod 8$ noting that $f \equiv 1, 3, 5,$ or $7 \pmod 8$ is solvable we prove f irregular.

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1 "Verteilung der Primzahlen", Landau, vol.1, 1909, pp.89-92.

(i) Consider $b \equiv 6 \pmod{16}$.

a) If $4b<c$ take $k=c-4b \equiv 5 \pmod 8$ for k is prime to b and c and since $0<k<c$, $x^2+by^2 \not\equiv 5 \pmod 8$, $f \neq k$.

b) If $4b>c$. Then, from Bertrand's Postulate (see note on preceeding page) there exists a prime p such that $(b+1)/2 \leq p \leq b-1$. (p is odd since $b \geq 6$). p is prime to b and $f \neq p<b$ and thus we take $k=p$ to prove f irregular if p is prime to c. Now since $8p>4b>c$ p is prime to c unless $c=3p$, $c=5p$, $c=7p$.

If $c=3p$, then b is prime to 3 and thus $b>6$ and we take $k=b/2-6 \equiv 5 \pmod 8$ for k is prime to p since $k<p$, k is prime to 3 and thus to b and c. Also $f \neq k$ since $x^2 \not\equiv 5 \pmod 8$ and $k<b$.

If $c=5p$, then b is prime to 5 and we take $k=b/2-20 \equiv 7 \pmod 8$ proving f irregular as above unless $b=6$, 22 or 38 when we take $k=11$, 3 or 3 respectively, knowing that then k is prime to p and thus to c since $p \equiv 1 \pmod 8$.

If $c=7p$, then b is prime to 7 and we take $k=b/2-14 \equiv 5 \pmod 8$ proving f irregular as above unless $b=6$ when we take $k=5$ since $p \equiv 3 \pmod 8$.

(ii) If $b \equiv 14 \pmod{16}$ interchange b and c in C *ii*) above to prove f irregular.

E. Remove temporarily the condition $b \leq c$ and find there remains to consider $b \equiv 5 \pmod 8$ and $c \equiv 0 \pmod 8$, excluding $b=5$, $c=8$. Now $f \equiv 0 \pmod 4$ implies $x=2X$, $y=2Y$ and continuing this process we find $f/4^r = g_r = X^2+bY^2+Cz^2$ where $c=4^rC$,

$C\not\equiv 0 \pmod 4$. Reference to preceeding pages (which include consideration of all such g_r) shows that we need consider only the f's for which g_r has $b=5$, $C=1$ or 2 or $b=21$, $C=1$ since otherwise g_r and therefore f is irregular.

(i) $r=1$. Then $C\equiv 0 \pmod 2$ and g_1 is irregular unless $b=5$ and $C=2$ which is the case excluded.

(ii) $r=2$. Use $k=13$ for all three cases proving by theorem 4 that $F=x^2+by^2+16Cz^2$ where $C\not\equiv 0 \pmod 4$ is irregular.

(iii) $r>2$. Then $f=4^{r-2}F$ and thus f is irregular.

<u>b or $c\equiv 2 \pmod 4$ and the other $\equiv 3 \pmod 4$, $b>2$ $(c\geq b)$.</u>

We shall prove f is irregular, unless $b=3$, $c=10$. Note that $f\equiv 1,3,5,$ or $7 \pmod 8$ is solvable.

A. If b or $c\equiv 2 \pmod 8$ and the other $\equiv 7 \pmod 8$ or if b or $c\equiv 6 \pmod 8$ and the other $\equiv 3 \pmod 8$ we know that $f\equiv 2 \pmod 8$ is solvable with x, y, z odd and thus by theorem 5 that f is irregular since $f\neq 2$.

B. If $b\equiv 2 \pmod 8$ and $c\equiv 3 \pmod 8$ f is irregular.

(i) If $b<c/2$ take $k=c-2b\equiv 7 \pmod 8$ for k is prime to b and c and since $c>k>0$ and $x^2+by^2\not\equiv 7 \pmod 8$, $f\neq k$.

(ii) If $b>c/2$, then, as on page 11, there exists an odd prime p such that $(b+1)/2\leq p\leq b-1$. p is prime to b and $f\neq p$ and thus we take $k=p$ to prove f irregular if p is prime to c. Now, since $5p>5b/2>c$ and c is odd we know that p is prime to c unless $c=3p$.

If $c=3p$, take $k=b-3\equiv 7 \pmod 8$ for k is prime to b since 3 is, is prime to p since $p\equiv 1 \pmod 8$ and $7p>b$, and thus is prime to b and c, and $f \neq k < b$.

C. If $b \equiv 3 \pmod 8$ and $c \equiv 2 \pmod 8$, we prove f is irregular unless $b=3$, $c=10$. Note that $f \equiv 1,3,5,7 \pmod 8$ is solvable.

(i) If $4b<c$ take $k=c-4b \equiv 6 \pmod 8$ for $f \equiv 6 \pmod 8$ is solvable with x and y odd, $f \neq k$ since $x^2+by^2 \not\equiv 2 \pmod 4$ and theorem 5 applies to prove f irregular.

(ii) If $4b>c$ we may take $b>3$ since $b=3$ implies $c=10$ which is the case excluded. Then, as on page 11 there exists an odd prime p such that $(b+1)/2 \leq p \leq b-1$; p is prime to b and $f \neq p$ and thus we take $k=p$ to prove f irregular if p is prime to c. Now, since $8p>4b>c$ and $c \equiv 2 \pmod 4$ we know that p is prime to c unless $c=2p$ or $6p$.

If $c=2p$, take $k=b-4 \equiv 7 \pmod 8$ for k is prime to p since $p \equiv 1 \pmod 4$ and $3p>b>k$ and k is prime to b and c, $f \neq k<b$.

If $c=6p$, take $k=b-6 \equiv 5 \pmod 8$ for b is prime to 3 since $c \equiv 0 \pmod 3$; thus k is prime to b and 3. k is prime to p since $p \equiv 3 \pmod 4$ and $6p>b>k$, $f \neq k<b$. ($3p \neq k \not\equiv 0 \pmod 3$).

D. If one of b, c is $\equiv 6 \pmod 8$ and the other $\equiv 7 \pmod 8$, f is irregular for, as on page 11, $f \equiv 2 \pmod 8$ implies $x=2X$, $y=2Y$ considering first the case $b \equiv 7 \pmod 8$, $c \equiv 6 \pmod 8$ and $f/2 = g = 2X^2+2bY^2+cz^2/2 \equiv 1 \pmod 4$. Then $g \equiv 1 \pmod 8$ is solvable and as in the section referred to f is irregular. If $b \equiv 6 \pmod 8$ and $c \equiv 7 \pmod 8$ we interchange b and c and proceed as above.

$b \equiv c \equiv 3 \pmod 4$.

Then $f \equiv 2 \pmod 8$ is solvable with y and z odd, $f \neq 2$ and thus by theorem 5 we take $k=2$ to prove f irregular.

b or $c \equiv 3 \pmod 4$ and the other $\equiv 0 \pmod 4$.

We shall prove that f is irregular unless $b=3$, $c=4$. Note that $f \equiv 1, 3, 5,$ or $7 \pmod 8$ is solvable.

A. If $b \equiv 3 \pmod 8$ and $c \equiv 4 \pmod 8$, then $f \equiv 0 \pmod 8$ is solvable with x and y odd, $f \neq 8$ except in the case excluded and using theorem 5 we take $k=8$ to prove f irregular.

B. If $b \equiv 3 \pmod 8$ and $c \equiv 8 \pmod{16}$ consider

(i) $c=8$, then $b=3$ and take $k=5$.

(ii) $c \equiv 24 \pmod{32}$, then $f \equiv 0 \pmod 8$ implies $x=2X$, $y=2Y$ and $f/4 = X^2 + bY^2 + cz^2/4 = g$. $g \equiv 2 \pmod 8$ is solvable with X and Y odd and thus by Lemma 3 and theorems 4b and 4c we have $g \equiv 2 \pmod N$ is solvable for all N; thus $f \equiv 8 \pmod N$ is solvable for all N, $f \neq 8$ and theorem 4a applies to prove f irregular.

(iii) $c \equiv 8 \pmod{64}$, $c \neq 8$, then $g \equiv 2 \pmod 8$ implies $X=2x'$, $Y=2y'$ and $g/2 = F = 2x'^2 + 2by'^2 + cz^2/8 \equiv 1 \pmod 4$. Now $F \equiv 1 \pmod 8$ is solvable and thus from theorems 3, 4b, 4c we have $F \equiv 1 \pmod N$ is solvable for all N. Thus $g \equiv 2 \pmod N$ is solvable for all N and f is irregular as in the preceeding case.

(iv) $c \equiv 40 \pmod{64}$.

a) if $c > 16b$ take $k = 8k'$ where $k' = c/8 - 2b \equiv 7 \pmod 8$

for k' is prime to b and c and since $F \equiv k' \pmod 8$ is solvable, $F \equiv k' \pmod N$ is solvable for all N and $f \equiv k \pmod N$ is solvable for all N. Furthermore $f \neq k = 8k' \equiv 8 \pmod{16}$ for $k < c$ and $x^2 + by^2 = 4(X^2 + bY^2) \equiv 0 \pmod 8$ and thus $x^2 + by^2 \not\equiv 8 \pmod{16}$. Thus from theorem 4a, f is irregular.

b) $c < 16b$. Then, as on page 11, there exists an odd prime p such that $(b+1)/2 \leq p \leq b-1$ unless $b = 3$ when $c = 40$ and we may take $k = 11$. Now p is prime to b and $f \neq p$ and we take $k = p$ to prove f irregular unless $c \equiv 0 \pmod p$. Now, since $40p > 20b > c$ and $c \equiv 40 \pmod{64}$ we know that p is prime to c unless $c = 8p$ or $24p$.

If $8p = c$, take $k = b - 4 \equiv 7 \pmod 8$ for k is prime to p since $2p > b$ and $p \equiv 5 \pmod 8$ and thus k is prime to b and c, $f \neq k < b$.

If $24p = c$, take $k = b - 6 \equiv 5 \pmod 8$ for k is prime to p since $2p > b$ and $p \equiv 7 \pmod 8$, is prime to 3 since b is and thus is prime to b and c. Also $f \neq k < b$.

C. If $b \equiv 3 \pmod 8$ and $c = 4^r C$ where $r > 0$, $C \equiv 8 \pmod{16}$. Then $f = x^2 + by^2 + 4^r Cz^2 \equiv 8 \cdot 4^r \pmod{4^{r+2}}$ implies x and y are even and repeating this process r times we finally have $f/4^r = g_r = X^2 + bY^2 + Cz^2 \equiv 8 \pmod{16}$ implied by $f \equiv 8 \cdot 4^r \pmod{4^{r+2}}$. Now, in B above we showed for every g_r the existence of a $k \equiv 8 \pmod{16}$ such that $g_r \neq k$ and $g_r \equiv k \pmod N$ is solvable for every N. Thus $f \equiv 4^r k \pmod N$ is solvable for every N. Also $f \neq 4^r k$ since that would imply $g_r = k$. Thus by theorem 4a, f is irregular.

D. If $b \equiv 3 \pmod 8$ and $c=4^r C$ where $r>0$ and $C \equiv 4 \pmod 8$. Except in the case $b=3$, $C=4$ the same reasoning as that above may be carried through to prove f irregular for only when $b=3$, $C=4$ is g_r regular.

If $b=3$, $C=4$ take $k=5$ for any r.

E. If $b \equiv 7 \pmod 8$ and $c \equiv 0 \pmod 4$ consider

(i) $b=7$, $c \equiv 4 \pmod 8$. Then $f \equiv 0, 4 \pmod 8$ is solvable with x and y odd and $x^2+7y^2 \not\equiv 3, 5$ or $6 \pmod 7$. Choose k to be one of $c-4$, $c-8$, $c-16$, $c-28$ which is $\equiv 3, 5$ or $6 \pmod 7$ for $c>28$. This is possible since if $c \equiv 3, 5$ or $6 \pmod 7$, $c-28 \equiv 3, 5$ or $6 \pmod 7$; if $c \equiv 1 \pmod 7$, $c-16 \equiv 6 \pmod 7$; if $c \equiv 2 \pmod 7$, $c-4 \equiv 5 \pmod 7$; if $c \equiv 4 \pmod 7$, $c-8 \equiv 3 \pmod 7$; and we know c is prime to 7. Then $k \equiv 0 \pmod 4$ and thus $f \equiv k \pmod 8$ is solvable with x and y odd, $f \neq k$ since $0<k<c$ and thus by theorem 5 f is irregular.

It remains to consider $c=12$ or 20 when we take $k=5$ or 3 respectively to prove f irregular.

(ii) $b=7$, $c \equiv 0 \pmod 8$. Then $f \equiv 0 \pmod 8$ is solvable with x and y odd and as above we take k to be one of $c-8$, $c-16$, $c-56$, $c-32$ which is $\equiv 3, 5$ or $6 \pmod 7$ for $c>56$. Then $k \equiv 0 \pmod 8$ and $f \equiv k \pmod 8$ is solvable with x and y odd, $f \neq k$ since $0<k<c$ and thus by theorem 5, f is irregular.

It remains to consider $c=8, 16, 40$ when we take $k=3$ and $c=24, 32, 48$ when we take $k=5$ to prove f irregular by theorem 4.

(iii) $b>7$, take $k=8 \neq f$ and apply theorem 5.

F. If $b \equiv 0 \pmod 4$ and $c \equiv 3 \pmod 4$ take $k=c-b \equiv 3 \pmod 4$ since $x^2+by^2 \not\equiv 3 \pmod 4$ shows that $f \neq k < c$, and we apply theorem 4.

Thus we have proven that all forms $f=x^2+by^2+cz^2$, where b and c are relatively prime, are irregular except those appearing in Table I.

III. $f=ax^2+by^2+cz^2$ with $a>1$, $b\geq a\leq c$ and no two of a, b, c having a factor in common.

We prove that all such f are irregular.

A. If $a\equiv 1 \pmod 4$, then b or c is odd and $f\equiv 1 \pmod 8$ is solvable. We take $k=1\neq f$ and prove f irregular by theorem 4.

B. If $a\equiv 2 \pmod 4$, then $f\equiv 1 \pmod 8$ is solvable since both b and c are then odd and we take $k=1$ to prove f irregular.

C. If $a\equiv 3 \pmod 4$, f is irregular for

1). If b or c is $\equiv 1$ or $2 \pmod 4$ then $f\equiv 1 \pmod 8$ is solvable, $f\neq 1$ and thus by theorem 4, f is irregular.

2). If $b\equiv c\equiv 3 \pmod 4$ then $f\equiv 2 \pmod 8$ is solvable with y and z odd and since $f\neq 2$, theorem 5 applies to prove f irregular.

Otherwise b or c is even. Permute if necessary and take b as the odd coefficient and note that there remains

3). If $b\equiv 3 \pmod 4$ and $c=4^rC$ where $r>0$ and $C\not\equiv 0 \pmod 4$. Then $f\equiv 0 \pmod 4$ implies x and y are even and repeating this process we find $f/4^r = aX^2+bY^2+Cz^2 \equiv g_r$. If $C=1$ reference to table I shows that g_r is irregular since $b\geq a\equiv 3 \pmod 4$. If $C>1$ is the minimum of g_r reference to the above with <u>a</u> and C interchanged shows that g_r is irregular. If <u>a</u> is still the minimum we have the same result thus proving that f is irregular for every r.

D. If $a \equiv 0 \pmod 4$, we prove f is irregular. Now b and c are odd.

1). If b or $c \equiv 1 \pmod 4$ then $f \equiv 1 \pmod 8$ is solvable and we take k=1.

2). If $b \equiv c \equiv 3 \pmod 4$ by interchanging <u>a</u> and c in C 3) above we see that f is irregular.

IV. Processes 1 and 2.

Consider $f=ax^2+p^r by^2+p^s cz^2$ where $1\leq r\leq s$ and p is a prime dividing neither a, b nor c.

If $r\geq 2$, $f\equiv 0 \pmod p$ implies $x=px_1$ and $f/p^2 = ax_1^2+p^{r-2}by^2+p^{s-2}cz^2$

If $r\geq 4$, $f/p^2\equiv 0 \pmod p$ implies $x_1=px_2$ and $f/p^4 = ax_2^2+p^{r-4}by^2+p^{s-4}cz^2$

Continuing this process we come finally to

(1) $f/p^r = ax_{r/2}^2+by^2+p^{s-r}cz^2$ if r is even or

(2) $f/p^{r-1} = ax_{\frac{r-1}{2}}^2+bpy^2+p^{s-r+1}cz^2$ if r is odd.

In order to go from one form in the above sequence to that below we substituted $x_i = px_{i+1}$ and divided the form by p^2. The reverse process is

<u>Process 1.</u> Multiply through the lower form by p^2 and absorb it into the x (i.e. let $x_{i+1}=x_i/p$) to obtain the higher form.

Form (2) may be reduced further as follows:
$f/p^{r-1}\equiv 0 \pmod p$ implies $x_{\frac{r-1}{2}}=px_{\frac{r+1}{2}}$ and we have
$f/p^r = pax_{\frac{r+1}{2}}^2+by^2+p^{s-r}cz^2$. If $s-r>0$, let $y=py_1$ and have
$f/p^{r+1} = ax_{\frac{r+1}{2}}^2+bpy_1^2+p^{s-r-1}cz^2$ and so continuing we have finally

(3) $f/p^s = ax_{s/2}^2+bpy_{\frac{s-r+1}{2}}^2+cz^2$ or $f/p^s = apx_{\frac{s+1}{2}}^2+by_{\frac{s-r}{2}}^2+cz^2$ according as s is even or odd. Since in each case to obtain the lower form we let whichever of x and y had a coefficient prime to p be p times a similar term we reverse it and have

<u>Process 2.</u> Multiply through the lower form by p, absorb p^2 in the resulting coefficient of x or y into the variable to obtain the higher form. i.e. in the first form (3) above, the next higher form would be $apx_{s/2}^2+by_{\frac{s-r-1}{2}}^2+cpz^2$.

Note that process 2 does not apply except to a form where p appears only to the first power in one of the coefficients.

Now f will be irregular for certain powers of p and therefore irregular if a form (1), (2) or (3) resulting from it is irregular. Thus we need consider only those f's for which (1), (2) or (3) has not been proven irregular. That is, only those forms derived from an apparently regular form (1) by process 1 or from an apparently regular form (3) by process 2 and process 1 applied in any order or succession, need be considered. Furthermore if at any stage all forms of type (1), (2) or (3) or forms resulting from them are irregular, all higher forms derived from them will also be irregular. Thus at each stage only those forms which are not proven irregular need be carried to the next higher stage.

<u>Remark:</u> If for a certain r <u>all</u> forms $f=p^r g$ derived by processes 1 or 2 applied in any order from a form g in Table I are irregular, then all forms $f=p^r g'$ where g' is regular are irregular for from the nature of processes 1 and 2 reversed, for every g' there exists an m and a g in table I such that $g'=mg$, where m is a positive integer. Thus $f=mp^r g$ is irregular.

V. $f=ax^2+2^r by^2+2^s cz^2$ where $0<r\leq s$; a, b, c are odd and no two of a, b, c have a factor in common.

f will be irregular unless derived from a form g in table I by processes 1 and 2 applied in some order or succession.

A. If $g=ax^2+by^2+cz^2$ (a, b, c odd), only process 1 applies. By symmetry take $b\leq c$.

$4g=ax^2+4by^2+4cz^2=f$

If a=1 and b=1; c=1,3,9 see table II.

c=5, use k=77 and theorem 4 to prove f irregular.

c=21 use k=21 for $28z^2-7=35\cdot 3$ for z=2 and $12\cdot 3^2-3=15\cdot 7$ and theorem 6 applies.

If a=3, b=1 and c=1 see table II.

If a=5,9 or 21 and b=1=c use k=1 to prove f irregular.

$16g=ax^2+16by^2+16cz^2=f'$ (applying process 1 to forms f not proven regular, i.e. the forms underlined above).

If a=1, b=1 and c=1 or 3 see table II.

If a=1, b=1, c=9 use k=473, to prove f' irregular.

If a=3, b=1=c use k=11 to prove f' irregular.

$64g=ax^2+64by^2+64cz^2=f''$

If a=1, b=1, c=1 or 3 use k=17 to prove f'' irregular.

B. If $g=ax^2+by^2+2cz^2$ either process 1 or process 2 may be here applied. (Interchange y and z for exact correspondence with the theory).

(i) Process 2.

$2g = 2ax^2 + 2by^2 + cz^2 = f$.

If a=1; <u>b=1, c=1 or 3 see table II</u>.

<u>b=3, c=1 or 5 see table II.</u>

<u>b=5, c=1 see table II.</u>

We also have forms obtained from the above by interchanging <u>a</u> and <u>b</u>.

$4g = ax^2 + 4by^2 + 2cz^2 = f'$.

If a=1; b=1, <u>c=1 or 3 see table II.</u>

b=3, c=1 or 5 use k=7 to prove f' irregular.

b=5, c=1 use k=7 to prove f' irregular.

If a=3, b=1, c=1 or 5 use k=1 to prove f' irregular.

If a=5, b=1=c use k=1.

$8g = 2ax^2 + 8by^2 + cz^2 = f''$.

<u>If a=1, b=1, c=1 or 3 see table II.</u>

$16g = ax^2 + 16by^2 + 2cz^2 = h$.

<u>If a=1, b=1, c=1 or 3 see table II.</u>

$32g = 2ax^2 + 32by^2 + cz^2 = h'$

If a=1, b=1; <u>c=1 see table II.</u>

c=3 take k=13 to prove h' irregular.

$64g = ax^2 + 64by^2 + 2cz^2 = h''$.

If a=1=b=c use k=35 to prove h" irregular.

It remains to apply process 1 to the underlined forms above and to regular forms g.

(ii) Process 1.

a) $4g = ax^2 + 4by^2 + 8cz^2 = F$.

If a=1; <u>b=1, c=1 or 3 see table II.</u>

b=3, c=1 or 5 use k=5 or 17 respectively.

b=5, c=1 use k=13.

If a=3, b=1, c=1 or 5 use k=23.

If a=5, b=1, c=1 use k=1.

$16g = ax^2 + 16by^2 + 32cz^2$.

If a=1, b=1, c=1 or 3 use k=161 or 33 respectively (for k=33 use theorem 6).

b) $4f = 8ax^2 + 8by^2 + cz^2 = F$.

If a=1; <u>b=1, c=1 or 3 see table II.</u>

<u>b=3, c=1 or 5 see table II.</u>

<u>b=5, c=1 see table II</u>.

$16f = 32ax^2 + 32by^2 + cz^2$.

If a=1; b=1, c=1 or 3 use k=17 or 11 respectively.

b=3, c=1 or 5 use k=17 or 13 respectively.

b=5, c=1 use k=17.

c) $4f' = ax^2 + 16by^2 + 8cz^2$.

<u>If a=1, b=1, c=1 or 3 see table II</u>.

$16f' = ax^2 + 64by^2 + 32cz^2$.

If a=1, b=1, c=1 or 3 use k=17 to prove 16f' irregular.

d) $4f'' = 8ax^2 + 32by^2 + cz^2$.

If a=1, b=1; <u>c=1 see table II.</u>

c=3 use k=19.

$16f'' = 32ax^2 + 128by^2 + cz^2$.

If a=1=b=c use k=17.

e) $4h = ax^2 + 64by^2 + 8cz^2$.

If $a=1$, $b=1$; $c=1$ see table II.

$c=3$ use $k=17$.

$16h=ax^2+256by^2+32cz^2$.

If $a=1=b=c$ use $k=17$.

f) $4h'=8ax^2+128by^2+cz^2$.

If $a=1=b=c$ use $k=65$.

C. If $g=ax^2+by^2+4cz^2$ only process 1 can be applied.

$4g=ax^2+4by^2+16cz^2=f$.

If $a=1, b=1$; $c=1$ see table II.

$c=3$ use $k=21$ and theorem 6 to prove f irregular.

If $a=1$, $b=3$, $c=1$ use $k=5$.

If $a=3$, $b=1$, $c=1$ use $k=11$.

$16g=ax^2+16by^2+64cz^2$.

If $a=1=b=c$ take $k=33$.

D. If $g=ax^2+by^2+8cz^2$ only process 1 applies.

$4g=ax^2+4by^2+32cz^2$.

If $a=1; b=1, c=1$ or 3 use $k=21$ or 77 respectively.

$b=5$, $c=1$ use $k=13$.

If $a=5, b=1=c$ use $k=1$.

E. If $g=ax^2+by^2+16cz^2$ only process 1 applies.

$4g=ax^2+4by^2+64cz^2$.

If $a=1=b=c$ use $k=21$.

F. Since no regular form g has as a factor of one of its coefficients the integer 32, all forms $f=ax^2+2^rby^2+2^scz^2$ where a, b, c are odd and no two of a, b, c, have a factor in common, $0<r\leq s$, are irregular except those included in table II.

VI. $f=ax^2+3^r by^2+3^s cz^2$ where $0<r\leq s$; a, b, c are prime to 3 and no two of a, b, c have an odd prime factor in common.

We apply theorem 7 with the specified k to prove f irregular unless the contrary is specifically stated. f will be irregular unless derived from a form g in table I or II by processes 1 or 2 applied in some order or succession.

A. If $g=ax^2+by^2+cz^2$ (a, b, c prime to 3) only process 1 applies. From symmetry take $b\leq c$.

$$9g=ax^2+9by^2+9cz^2=f.$$

If a=1, b=1; c=1 see table III.

c=2 or 5, use k=7.

c=4 use k=22 for, since $f\equiv 6 \pmod 8$ is solvable with x and y odd, theorems 5 and 6 apply with theorem 7.

c=8, f=4h where $h=x^2+9y^2+18z^2$ which is proved irregular immediately above.

c=16 use k=133.

If a=1, b=2; c=2, 4, 5 or 10 use k=13.

c=8, f=4h' where $h'=x^2+18y^2+18z^2$ just proved irregular.

c=16, f=4h" where $h''=x^2+18y^2+36z^2$ just proved irregular.

c=32, f=16h'.

If a=1, b=4 use k=13 for no c in table I or II has a factor 13.

If a=1, b=5, c=8 use k=13.

If a=1, b=8, c=8m (reference to table II shows that g is irregular unless $c\equiv 0 \pmod 8$)) $f=4H$ where $H=x^2+18y^2+18mz^2$ which is irregular from the above theory.

If $a=1$, $b=16=c$ use $k=73$.

If $a=2$, $b=1$; $c=1,2$ or 4 use $k=5$.

$c=5$ or 10 use $k=29$.

$c=8$, 16 or 32 use $k=35$.

If $a=4$, $b=1$ use $k=1$.

If $a=5$, $b=1$, $c=2$, 8 or 1 use $k=17$.

If $a=8$, $b=1$; $c=1$ or 4 use $k=5$.

$c=5$ use $k=14$ and theorem 5($p=2$) with theorem 7 to prove f irregular.

$c=2$ use $k=11$.

$c=8,16,32,40,64$ use $k=65$ (this holds for $c=40$ by the application of theorem 6 with theorem 7).

Note that all forms of minimum ≥ 2 in table II have a factor 3 in one of the coefficients and are thus barred from present consideration. There thus remains $a>8$, $b=1$

$a\equiv 1 \pmod 3$, $b=1$, $a>8$, use $k=1$.

$a\equiv 2 \pmod 3$, $b=1$, $a>8$, use $k=17$ since no coefficient in table I or II has 17 as a factor.

$81g=ax^2+81by^2+81cz^2=f$.

If $a=1=b=c$ use $k=73$ to prove f irregular.

B. If $g=ax^2+3by^2+cz^2$ either process 1 or process 2 may be here applied.

(i) Process 2.

$3g=3ax^2+by^2+3cz^2=f$.

If $b=1$; $a=1, c=1, 2, 4$ or 10 see table III.

$a=2, c=2$ or 8 see table III.

$a=4,\ c=4$ see table III.

$a=8,\ c=8$ see table III.

If $b=2$; $a=1, c=1, 2, 4$ or 16 see table III.

$a=2, c=5$ see table III.

If $b=4, a=1, c=1$ or 4 see table III.

If $b=7, a=1=c$ see table III.

If $b=8$; $a=1, c=1, 4, 8, 16$ see table III.

If $b=8, a=5, c=8$ see table III.

If $b=16, a=1,\ c=16$ see table III.

We have also forms obtained from the above by interchanging a and c. This has to be taken into account below though the form f is symmetrical in a and c.

$$9g=ax^2+3by^2+9cz^2=f'.$$

If $b=1$; $a=1, c=1, 2$ or 4 see table III.

$a=1, c=10$ use $k=22$ and theorem 5 with theorem 7.

$a=2,\ c=1$ or 2 see table III.

$a=2, c=8$, then $f'=2F$ where $F=x^2+6y^2+36z^2$ is proved irregular by taking $k=13$.

$a=4$; $c=1$ use $k=49$.

$c=4$ see table III.

$a=8$; $c=2$ use $k=5$.

$c=8$ see table III.

$a=10,\ c=1$ use $k=7$.

If $b=2$; $a=1, c=1$ or 2 see table III.

$a=1,\ c=4$ use $k=13$.

a=1, c=16, then $f'=4F$ where F is above proved irregular.

a=2; <u>c=1 see table III.</u>

c=5 use k=11.

a=4 or 16 and c=1 use k=1.

a=5, c=2, then $f'=2F'$ where $F'=10x^2+3y^2+9z^2$ is irregular from above theory.

If b=4; <u>a=1, c=1 or 4 see table III.</u>

a=4, c=1 use k=1.

If <u>b=7, a=1=c see table III</u>.

If b=8; <u>a=1, c=1 or 8 see table III.</u>

a=1, c=4 use k=13.

If b=8; a=1, c=16, then $f'=4F$ where $F=x^2+6y^2+36z^2$ is irregular from the above theory.

a=4 or 16 and c=1 use k=1.

a=5, c=8 use k=53.

a=8; <u>c=1 see table III.</u>

c=5 use k=29.

If b=16; <u>a=1, c=16 see table III</u>.

a=16, c=1 use k=1.

For all other coefficients either g or 3g is irregular. Making use of the remark after the discussion of process 2 we consider only

$$27g=3ax^2+by^2+27cz^2=f''$$ for which g is in table I.

If b=1; a=1, c=1 or 4 use k=85.

a=1, c=2 or 10 use k=34 and apply theorems 5 and 7.

a=2, c=1 use k=13.

a=4 or 10 and c=1 use k=7.

If b=2, a=1, c=1 use k=17.

If b=4, a=1, c=1 use k=10 and apply theorems 5 and 7.

If b=7, a=1=c use k=13.

If b=8, a=1, c=1 use k=14 applying theorems 5 and 7.

It remains to apply process 1 to the underlined forms above and to regular forms g.

(ii) Process 1.

$9g = ax^2 + 27by^2 + 9cz^2$. Making use of the remark after the discussion of process 2 we consider only those 9g for which g is in table I. (Interchange b and c for strict conformity with (1) in the description of process 1).

If b=1; a=1, c=1,2,4 or 10 use k=7.

a=2, c=1 use k=5.

a=4 or 10, c=1 use k=1.

If b=2, a=1=c use k=7.

If b=4, a=1=c use k=22 applying theorems 5 and 7.

If b=7, a=1=c use k=133 and theorems 6 and 7.

If b=8, a=1=c use k=133.

$9f = 27ax^2 + by^2 + 27cz^2$. (Make use of the symmetry in a and c and of the remark above referred to).

If b=1, a=1, c=1,2, 4 or 10 use k=7.

If b=2,4,7 or 8 and a=1=c use k=5,7,13 or 11 respectively.

$9f' = ax^2 + 27by^2 + 81cz^2$. Consider only those values of a, b, c underlined for f'.

If $b=1$; $a=1$, $c=1$, 2 or 4 use $k=13$.

$a=2$, $c=1$ or 2 use $k=11$.

$a=4$, $c=4$ use $k=7$.

$a=8$, $c=8$ use $k=11$.

If $b=2$; $a=1$, $c=1$ or 2 use $k=73$.

$a=2$, $c=1$ use $k=17$.

If $b=4$, $a=1$, $c=1$ or 4 use $k=13$.

If $b=7$, $a=1=c$ use $k=73$.

If $b=8$; $a=1$, $c=1$ or 8 use $k=73$.

$a=8$, $c=1$ use $k=17$.

If $b=16$, $a=1$, $c=16$ use $k=73$.

C. If $g=ax^2+by^2+9cz^2$ only process 1 applies.

$9g=ax^2+9by^2+81cz^2$.

If $a=b=c=1$ use $k=19$ and thus by the remark after process 2 all $9g$ are irregular.

D. Since no regular form g has 27 as a factor of one of its coefficients all the forms $f=ax^2+3^r by^2+3^s cz^2$ where a, b and c are prime to 3 and no two of a, b, c have an odd prime factor in common and $0<r\leq s$, are irregular except those in table III.

VII. $f = ax^2 + p^2by^2 + p^2cz^2$ where p is a prime ≥ 5 not dividing a, and no factor is common to a, b and c.

Take $b \leq c$.

Lemma 6. All forms f for which $g = ax^2 + by^2 + cz^2 = f/p^2$ is one of the forms below are irregular:

(1) (1,1,1) (1,1,5) (1,1,21) (1,1,3) (1,2,3) (1,2,5) (1,3,10).

Proof: Noting theorem 7 we see that a form f for which g is one of (1) is irregular if we can find a positive integer k such that:

a) k is prime to abc.

b) $f \equiv k \pmod 8$ is solvable if k is odd.

b') $f \equiv k \pmod 8$ is solvable with two of x, y, z odd if k is even.

c) $f \equiv k \pmod p$ is solvable.

d) $f \neq k$.

We may replace c) by c') $\left(\frac{k}{p}\right) = \left(\frac{a}{p}\right)$

and d by d') $k < p^2b$

and d") $k \neq ax^2$. This follows from a) unless a=1.

A. If $a = 1 = b = c$, take $k = 4pbc + a \equiv 5 \pmod 8$. Obviously conditions a), b), c') and d") are satisfied since $x^2 \not\equiv 5 \pmod 8$. d') holds if $a < pb(p-4c)$ which is true since $1 < p(p-4)$ for all $p \geq 5$.

B. If g is one of the forms (1, 1, 5), (1, 1, 21) note that $f \equiv 1, 5$ or $7 \pmod 8$ is solvable and that $f \equiv 2$ or $6 \pmod 8$ is solvable with x and y odd.

(i) If $p \equiv 1 (\text{mod } 4)$ take $k = pbc + a \equiv 2 (\text{mod } 4)$ satisfying conditions a), b'), c'), d"). d') holds if $a < pb(p-c)$ which holds with the following exceptions:

a) If $p=5$; $c=5$ use $k=21$.

$c=21$ use $k=21$ with theorems 6 and 7.

$a=21$ use $k=1$.

b) If $p=13$, $c=21$ use $k=10$ with theorem 5 and theorem 7.

c) If $p=17$, $c=21$ use $k=2$ with theorem 5 and theorem 7.

(ii) If $p \equiv 3 (\text{mod } 8)$ take $k = pbc - 2a \equiv 5$ or $1 (\text{mod } 8)$ according as $a=1$ or $a \neq 1$. k satisfies conditions a) b) c') c") obviously. k will be positive and satisfy d') if $pbc > 2a > pb(c-p)$. Such is the case with the following exceptions:

a) If $p=11$, c or $a=21$ use $k=5$ or 13 respectively.

b) If $p=19$, c or $a=21$ use $k=5$ or 13 respectively.

c) If $p=43$, $a=21$ use $k=1$.

(iii) If $p \equiv 7 (\text{mod } 8)$ take $k = pbc + 2a \equiv 5$ or $1 (\text{mod } 8)$ according as $a=1$ or $a \neq 1$. k satisfies conditions a), b), c'), d") obviously. k will satisfy d') if $2a < pb(p-c)$. Such is the case with the following exceptions.

a) If $p=7$, then $a \neq 21$. If $c=21$ use $k=2$ with theorems 5 and 7.

(If $p=23$, $c=21$ and $2 < 23 \cdot 2$; if $p=23$, $a=21$, $42 < 23 \cdot 22$).

C. If $g=(1,1,3)$, then $f \equiv 1, 3, 5, 7 (\text{mod } 8)$ is solvable and $f \equiv 2, 6 (\text{mod } 8)$ is solvable with x and y odd.

(i) If $a=1$, $b=1$, $c=3$ and

a) If $p \equiv 1 \pmod 4$ take $k = 2pbc + a \equiv 3 \pmod 4$. Conditions a), b), c'), d") are obviously satisfied. d') holds if $a < pb(p-2c)$ which holds unless $p = 5$ when we use $k = 11$.

b) If $p \equiv 3 \pmod 4$ take $k = pbc + a \equiv 2 \pmod 4$. Conditions a), b'), c'), d") are obviously satisfied and d') holds since $a < pb(p-c)$, that is $1 < p(p-3)$.

(ii) If $a = 3$, $b = 1 = c$ use $k = 4pbc + a \equiv 3 \pmod 4$ which satisfies all the conditions on k since $3 < p(p-4)$ for $p \geq 5$.

D. If g is one of the forms (1,2,3), (1,2,5), (1,3,10) note that $f \equiv 1,3,5,7 \pmod 8$ is solvable.

(i) If a is odd use $k = pbc + a \equiv 3$ or $1 \pmod 4$ according as $a \equiv 1$ or $3 \pmod 4$ and thus conditions a), b), c'), d") are satisfied. d') holds if $a < pb(p-c)$ which is true with the following exceptions:

a) If $p = 5$, then $a \neq 5$. If $a = 1$ and $c = 5$ or 10 use $k = 19$.
If $a = 3$, $c = 10$ use $k = 17$.

b) If $p = 7$; $a = 1$ or 3 and $c = 10$ use $k = 11$.

(ii) If $a \equiv 2 \pmod 4$, then $bc \leq 5$ and use $k = pbc + a \equiv$ an odd (mod 8) and conditions a), b), c), d") are obviously satisfied. Also d') holds since $a < pb(p-c)$ unless $p = 5 = c$ in which case we use $k = 17$.

This completes the proof of the lemma.

<u>Theorem 8.</u> All forms $f = ax^2 + p^2by^2 + p^2cz^2$ (where p is a prime ≥ 5 not dividing a and no factor is common to a, b and c) and those derived from such forms by processes 1 or 2, are irregular.

Proof:

1. We first prove that for every form F in table I there exists a positive integer m such that $F=mg$ where g is one of the forms (1).

a). Consider $h=x^2+y^2+2rz^2$ where $r=1$ or 3. $h\equiv 0 \pmod 2$ implies $x-y=2X$, $x+y=2Y$ are solvable for X and Y and thus $h=2g$ where $g=X^2+Y^2+rz^2$ and thus $m=2$.

b). $h=x^2+y^2+9z^2=9(X^2+Y^2+z^2)$ since $h\equiv 0 \pmod 3$ implies $x=3X$, $y=3Y$.

c). $h=x^2+3y^2+4z^2= 4h'$ where $h'=x'^2+3y'^2+z^2$ for $h\equiv 0 \pmod 4$ implies $x\equiv y \pmod 2$. If $x\equiv y\equiv 0 \pmod 2$ the above is obvious. If $x\equiv y\equiv 1 \pmod 2$, $x-y=2X$, $x+y=2Y$ are solvable with one of X, Y even.[1] $h=(2X-Y)^2+3Y^2+4z^2= (2Y-X)^2+3X^2+4z^2$. If $Y=2y'$ is solvable for y' then take $x'=X-y'$. If $X=2y'$ is solvable take $x'=Y-y'$ and in either case we have the desired result.

d). $h=x^2+y^2+4rz^2 = 4h'$ where $r=1$, or 3 and $h'=X^2+Y^2+rz^2$.

e). $h=x^2+y^2+8rz^2= 8h'$ where $h' = X^2+Y^2+rz^2$ where $r=1$ or 3 is obtained by applying a method similar to that for a).

f). $h=x^2+5y^2+8z^2= 4(X^2+5Y^2+2z^2)$.

h). $H=x^2+y^2+16z^2 =16(X^2+Y^2+z^2)$.

2. Suppose F is a regular form $ax^2+by^2+cz^2$ having no factor common to all of a, b, c. If p' is a prime factor of two of

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1 Cf. J. G. A. Arndt, Dissertation, Göttingen, 1925, p. 25; also the corollary to lemma b in part B of this thesis.

a, b, c we see by the discussion preceeding processes 1 and 2 that there is a positive integer t such that F/p'^t is a form in which no two coefficients have a factor p' in common. If two have a factor $p'' \neq p'$ in common we know by the same reasoning on F/p'^t that there exists a positive integer t' such that $F/p'^t p''^{t'}$ is a form in which no two coefficients have a prime factor p' or p" in common. Thus proceeding we see that there exists an m' such that F=m'g' where g' is a form such that no two coefficients have a factor in common. g' must be regular since F is and therefore g' is one of the forms of table I. Thus, by the above there exists an m such that F=mg where g is one of the forms (1).

3. Now f is irregular unless f/p^2 is regular, i.e. unless $f/p^2 = F = mg$ in which case $f = mf_g$ where f_g is a form proved in Lemma 6 to be irregular. Thus, in any case, f is irregular and it follows that any forms derived from it by process 1 or 2 is irregular.

Corollary: All forms obtained by applying process 1 when $p \geq 5$ to any form are irregular. This is evident since process 1 increases by 2 two of the exponents of p and thus the resulting f is of the form in theorem 8.

VIII. $f=ax^2+5^r by^2+5^s cz^2$ where $0<r\leq s$; a, b, c are prime to 5 and no two of a, b, c have an odd prime factor >5 in common.

We apply theorem 7 with the specified k to prove f irregular unless the contrary is specifically stated. f will be irregular unless derived from a form g in table I, II or III by processes 1 and 2 applied in some order or succession.

A. $g=ax^2+by^2+cz^2$ (a, b, c prime to 5). Then only process 1 applies and by the corollary to theorem 8, f is irregular.

B. If $g=ax^2+5by^2+cz^2$ either process 1 or process 2 may be here applied.

(i) Process 2.

$5g=5ax^2+by^2+5cz^2=f.$

If a=1; b=1; c=1, 2 or 8 see table IV.

b=2, c=2 or 3 see table IV.

b=6, c=3 see table IV.

b=8, c=8 see table IV.

If a=2; b=1 or 3 and c=6 see table IV.

If a=8, b=1 or 3 and c=24 see table IV.

We have also forms f obtained from the above by interchanging a and c above. This produces no new forms above but must be taken into account below.

$5f=ax^2+5by^2+25cz^2=f'.$

If a=1; b=1, c=1 or 8 see table IV.

b=1, c=2 use k=31.

b=2, c=2 or 3 use k=29.

$b=6$, $c=3$ use $k=19$.

$b=8$, $c=8$ use $k=129$.

If $a=2$, $b=1$ or 2 and $c=1$ use $k=17$.

$b=1$ or 3 and $c=6$ use $k=43$ or 53 respectively.

If $a=3$, $b=2$ or 6 and $c=1$ use $k=17$ or 7 respectively.

If $a=6$, $b=1$ or 3 and $c=2$ use $k=19$ or 29 respectively.

If $a=8$; $b=1$ or 8 and $c=1$ use $k=17$.

$b=1$, $c=24$, then $f'=4h$ where $h=2x^2+5y^2+150z^2$ is above proved irregular.

$b=3$, $c=24$, then $f'=4h'$ where $h'=2x^2+15y^2+150z^2$ is above proved irregular.

If $a=24$, $b=1$ or 3 and $c=8$ use $k=61$ or 71 respectively.

We consider only the underlined forms f' for $125g=5f'=5ax^2+by^2+125cz^2=f''$.

If $a=1=b=c$ use $k=19$.

If $a=1=b$, $c=8$ then $f''=4h''$ where $h''=5x^2+y^2+250cz^2$ is proved irregular by taking $k=19$.

(ii) It remains to apply process 1 to the underlined forms above and to seemingly regular forms g but by the corollary to theorem 8 all forms so obtained are irregular.

C. Since no regular form g has as a factor of one of its coefficients the integer 25, all the forms $f=ax^2+5^r by^2+5^s cz^2$ where a, b and c are prime to 5 and no two of a, b, c have a prime factor >5 in common and $0<r\leq s$ are irregular except those in table IV.

IX. $f = ax^2 + 7^r by^2 + 7^s cz^2$ where $0 < r \leq s$; a, b, c are prime to 7 and no two of a, b, c have a prime factor > 7 in common.

We apply theorem 7 with the specified k to prove f irregular unless the contrary is specifically stated. f will be irregular unless derived from a form in table I, II, III or IV (exclusive of forms 94, 98, 99) by process 1 and applied in some order or succession.

A. $g = ax^2 + by^2 + cz^2$ (a, b, c prime to 7). Then only process 1 applies and by the corollary to theorem 8, f is irregular.

B. If $g = ax^2 + 7by^2 + cz^2$ either process 1 or process 2 may be here applied.

(i) Process 2.

$7g = 7ax^2 + by^2 + 7cz^2 = f.$

If $a=1$; $b=3$, $c=1$ or 9 see table IV.

If $a=3$, $b=1$, $c=3$ see table IV.

$7f = ax^2 + 7by^2 + 49cz^2 = f'.$

If $a=1, c=1, b=3$ use $k=23$ to prove $f' = H$ irregular.

If $a=3, c=3, b=1$ then $f' = h' = 3H$ which is therefore irregular.

If $a=1, c=9, b=3$, then $f' = 3h'$ which is therefore irregular.

If $a=9, c=1, b=3$ use $k=1$.

It remains to apply process 1 to the underlined forms above and to seemingly regular forms g but by the corollary to theorem 8, all forms so obtained are irregular.

C. Since no regular form g has as a factor of one of its coefficients the integer 49, all the forms $f = ax^2 + 7^r by^2 + 7^s cz^2$ where a, b and c are prime to 5 and no two of a, b, c have a prime factor > 7 in common and $0 < r \leq s$, are irregular except those in table IV.

X. $f = ax^2 + p^r by^2 + p^s cz^2$ where $0 < r \leq s$, p is a prime > 7 not dividing abc and no two of a, b, c have a prime factor $> p$ in common.

If $r \geq 2$, f is irregular from theorem 8.

If $p = 11$, since no form in tables I to IV has a prime > 7 as a factor of one of its coefficients, the only possibly regular forms f with $p = 11$ would be derived by process 1 from seemingly regular forms $g = ax^2 + by^2 + cz^2$ in tables I to IV and would thus by the corollary to theorem 8 be irregular.

We may proceed similarly going from one prime to the next proving that for every p, f is irregular and

all forms $f = ax^2 + by^2 + cz^2$ not listed in tables I to IV, where a, b, c have no factor common to all three, are irregular.

XI. Reduced positive ternary quadratic forms with cross products and Hessian ≤ 20.

We prove that all such forms above are irregular except those appearing in table V.

1. $f=(1,2,4,-2,0,0)=x^2+2y^2+4z^2-2yz$ is irregular. (H=7)

Proof: $2f=2x^2+(2y-z)^2+7z^2$. Now $g=2x^2+Y^2+7z^2\equiv 0 \pmod 2$ implies $Y+z\equiv 0 \pmod 2$ and thus that $Y=2y-z$ is solvable for y and thus we have $g/2=f$, i.e. the evens represented by g coincide with double the integers represented by f. Now $g\equiv 6 \pmod{2^n}$ is solvable for n arbitrary since $g\equiv 6 \pmod 8$ implies $Y=2y'$, $z=2z'$ and $g'=x^2+2y'^2+14z'^2\equiv 3 \pmod 8$ and therefore $\equiv 3 \pmod{2^{n-1}}$ is solvable for n arbitrary. Furthermore $g\equiv 14 \pmod{7^n}$ is solvable by lemma 4b. And thus, by theorems 4b and 4c, $g\equiv 14 \pmod{2N}$ is solvable for N arbitrary. But $g\neq 14$. Thus $f\neq 7$ and $f\equiv 7 \pmod N$ is solvable for N arbitrary and thus by theorem 4a (which applies equally well for forms with cross products) f is irregular.

2. $f=(2,2,3,0,-2,0)$ (Hessian 10) is irregular.

Proof: As above $g/2=f$ where $g=X^2+4y^2+5z^2\equiv 2 \pmod 8$ is solvable with X and z odd, $g\neq 2$ and thus f is irregular proceeding as above.

3. $f=(1,3,4,-2,0,0)$ (Hessian 11) is irregular.

Proof: $3f=3x^2+(3y-z)^2+11z^2$. Now $g=3x^2+Y^2+11z^2\equiv 0 \pmod 3$ implies $Y^2+11z^2\equiv 0 \pmod 3$ and $Y\equiv \pm z \pmod 3$ where one of the signs holds. Thus $z-3y=\pm Y$ is solvable for y and $g/3=f$. Now $g\equiv 6 \pmod 8$ is solvable with two of x, Y, z odd and

$g \equiv 6 \pmod 3$ is solvable with Y and z prime to 3 and thus $g \equiv 6 \pmod{3N}$ is solvable for N arbitrary and $g \neq 6$. Thus $f \equiv 2 \pmod N$ is solvable for N arbitrary, $f \neq 2$ and therefore by theorem 4a is irregular.

4. $f=(1,2,7,-2,0,0)$ $(H=13)$ is irregular.

Proof: as for 1. $g/2=f$ where $g=2x^2+Y^2+13z^2 \equiv 2 \pmod 8$ implies $Y=2y'$, $z=2z'$ and $g'=x^2+2y'^2+26z'^2 \equiv 5 \pmod 8$ is solvable implying that $g \equiv 10 \pmod{2N}$ is solvable for N arbitrary, $g \neq 10$ and thus as in 1 f is irregular.

5. $f=(2,2,5,2,2,2)$ $(H=13)$ is irregular.

Proof: $6f=3(2x+y+z)^2+(3y+z)^2+26z^2$. Consider $g=3X^2+Y^2+26z^2$. Now $g \equiv 0 \pmod 3$ implies $Y \equiv \pm z \pmod 3$ where one of the signs holds as in 3 and thus $3y+z=\pm Y$ is solvable for y. Furthermore $g \equiv 0 \pmod 2$ implies $X \equiv Y \equiv y+z \pmod 2$ and thus that $2x+y+z=X$ is solvable for x. Thus $g/6=f$. Now $g \equiv 6 \pmod 3$ is solvable with Y and z prime to 3, $g \equiv 6 \pmod 8$ is solvable with X and Y odd, $g \neq 6$ and thus as for 1 f is irregular.

6. $f=(2,3,3,-2,0,-2)$ $(H=13)$ is irregular.

Proof: As for the form above $g/6=f$ where $g=3(2x-y)^2+2(3z-y)^2+13y^2=3X^2+2Y^2+13Z^2 \equiv 6 \pmod{2^n}$ is solvable since $g \equiv 6 \pmod 8$ implies $X=2x'$, $Z=2z'$ and $g'=6x'^2+Y^2+26z'^2 \equiv 3 \pmod 8$ is solvable. $g \equiv 6 \pmod 3$ is solvable with Y and z prime to 3. Thus $g \equiv 6 \pmod N$ is solvable, $g \neq 6$ proves f irregular.

7. $f=(1,2,8,-2,0,0)$ $(H=15)$ is irregular.

Proof: as in 1 $g/2=f$ where $g=2x^2+Y^2+15z^2 \equiv 2 \pmod 8$ is solvable with Y and z odd, $g \equiv 10 \pmod{5^n}$ is solvable by lemma 1b and thus by Lemma 3, and theorems 4b and 4c we have $g \equiv 10 \pmod{2N}$ is solvable for N arbitrary, $g \neq 10$ and thus f is irregular.

8. $f = (1,4,4,-2,0,0)$ (H=15) is irregular.

Proof: $4f = 4x^2 + (4y-z)^2 + 15z^2$. Now $g = 4x^2 + Y^2 + 15z^2 = 4f \equiv 0 \pmod 8$ for suppose $g \equiv 0 \pmod 8$ with x odd; then Y and z are both odd. If $g \equiv 0 \pmod 8$ with x even we know $Y^2 \equiv z^2 \pmod 8$ and thus in any case $Y \equiv \pm z \pmod 4$ where one of the signs holds. Thus $4y - z = \pm Y$ is solvable for y. Now $g \equiv 8 \pmod 8$ is solvable with Y and z odd, $g \neq 8$ and thus, since $f \neq 2$ and $f \equiv 2 \pmod N$ is solvable for all N and f is irregular.

9. $f = (2,2,5,-2,-2,0)$ (H=16) is irregular.

Proof: $f = (x+y-z)^2 + (x-y)^2 + 4z^2$. Now $g = X^2 + Y^2 + 4z^2 \equiv 1 \pmod 4$ implies $X \not\equiv Y \pmod 2$ and thus that $x+y-z = X$ and $x-y = Y$ i.e. $2x = X+Y+z$ and $2y = X-Y+z$ are solvable for x and y, if z is odd. Now $g \equiv 1 \pmod 8$ is solvable with z odd and $X \not\equiv Y \pmod 2$ and thus from corollary 1 of lemma 3, $g \equiv 1 \pmod{2^n}$ is solvable with z odd and $X \not\equiv Y \pmod 2$. Furthermore $g \equiv 1 \pmod{N'}$ is solvable for N' odd by theorem 4b and thus with z odd and $X \not\equiv Y \pmod 2$ for suppose a solution $x', y', 2z'$ exists; then $x', y', 2z'+N'$ is also a solution and z is odd. And if $x' \equiv y' \pmod 2$ we know that $x', y'+N', 2z'+N'$ is a solution with $x' \not\equiv y'' \pmod 2$. Thus $g \equiv 1 \pmod N$ is solvable with $X \not\equiv Y \pmod 2$ and z odd for N an arbitrary integer (for the solutions of $g \equiv 1 \pmod N$ are congruent (mod 2) to the solutions of $g \equiv 1 \pmod 8$ if N is even). Thus $f \equiv 1 \pmod N$ is solvable for N arbitrary and since $f \neq 1$, f is irregular.

10. $f=(1,2,9,-2,0,0)$ $(H=17)$ is irregular.

Proof: As for 1, $g/2=f$ where $g=2x^2+Y^2+17z^2 \equiv 2 \pmod 8$ is solvable with Y and z odd, $g\neq 10$ and thus f is irregular as above.

11. $f=(1,3,6,-2,0,0)$ $(H=17)$ is irregular.

Proof: As for 3 $g/3=f$ where $g=3x^2+Y^2+17z^2$ and since $g \equiv 15 \pmod 3$ is solvable with Y and z prime to 3, $g\neq 15$ we know that f is irregular as above.

12. $f=(2,3,4,2,2,2)$ $(H=17)$ is irregular.

Proof: $10f=5(2x+y+z)^2+(5y+z)^2+34z^2$. Consider $g=5X^2+Y^2+34z^2$. Now $g \equiv 0 \pmod 5$ implies $Y \equiv \pm z \pmod 5$ where one of the signs holds and thus $5y+z=\pm Y$ is solvable for y. Furthermore $g \equiv 0 \pmod 2$ implies $X \equiv Y \equiv y+z \pmod 2$ and thus that $2x+y+z=X$ is solvable for x. Thus $g/10=f$. Now $g \equiv 60 \pmod{2^n}$ is solvable for n an arbitrary positive integer for $g \equiv 4 \pmod 8$ implies $X=2x'$, $Y=2y'$, $z=2z'$ and $g/4=5x'^2+y'^2+34z'^2 \equiv 15 \pmod 8$ is solvable. Also $g \equiv 60 \pmod 5$ is solvable with Y and z prime to 5 and thus by lemma 3, theorems 4b and 4c, $g \equiv 60 \pmod N$ is solvable, $g\neq 60$ and thus f is irregular.

13. $f=(2,2,5,0,-2,0)$ $(H=18)$ is irregular.

Proof: As for 1 $g/2=f$ where $g=X^2+4y^2+9z^2=2 \pmod 8$ is solvable with X and z odd, $g\neq 2$ and thus f is irregular as above.

14. $f=(2,3,4,-2,0,-2)$ $(H=18)$ is irregular.

Proof: $4f=2(2x-y)^2+(4z-y)^2+9y^2$. Consider $g=2X^2+Z^2+9y^2 \equiv 4 \pmod 8$ is solvable with X, Z and y all odd. Thus as for 9 $g \equiv 4 \pmod N$ is solvable with X, Z and y all odd for N an arbitrary integer. Under these conditions $Z \equiv \pm y \pmod 4$ and $y-4z=\pm Z$ is solvable for z and $2x-y=X$ is solvable for x. Thus $4f \equiv 4 \pmod N$ is solvable for N arbitrary, $f \equiv 1 \pmod{N'}$ is solvable for N' arbitrary; $f \neq 1$ shows by theorem 4a that f is irregular.

15. $f=(1,2,10,-2,0,0)$ $(H=19)$ is irregular.

Proof: As for 1 $g/2=f$ where $g=2x^2+Y^2+19z^2 \equiv 6 \pmod 8$ is solvable with Y and z odd, $g \neq 14$ and thus f is irregular as above.

16. $f=(1,4,5,-2,0,0)$ $(H=19)$ is irregular.

Proof: As for 3 $g/5=f$ where $g=5x^2+Z^2+19y^2 \equiv 7 \pmod 8$ is solvable, $g \equiv 15 \pmod 5$ is solvable with y and Z prime to 5, $g \neq 15$ and thus f is irregular as above.

17. $f=(2,2,7,2,2,2)$ $(H=19)$ is irregular.

Proof: As for 5 $g/6=f$ where $g=3(2x+y+z)^2+(3y+z)^2+38z^2=3X^2+Y^2+38z^2 \equiv 2 \pmod 8$ is solvable with X and Y odd, $g \equiv 18 \pmod 3$ is solvable with Y and z prime to 3, $g \neq 18$ and thus f is irregular as above.

18. $f=(2,3,4,-2,-2,0)$ $(H=19)$ is irregular.

Proof: As for 5 $g/6=f$ where $g=3(2x-z)^2+2(3y-z)^2+19z^2=3X^2+2Y^2+19z^2 \equiv 6 \pmod 8$ is solvable with X and z odd, $g \equiv 6 \pmod 3$ is solvable with Y and z prime to 3, $g \neq 6$ and thus f is irregular as above.

19. $f=(1,4,6,-4,0,0)$ $(H=20)$ is irregular.

Proof: $f \equiv 0 \pmod 2$ implies $x=2X$ and thus $f/2=2X^2+2y^2+3z^2-2yz$ which, noting the symmetry in X and y, is proved irregular above in 2. Thus f is irregular.

PART B

REGULAR FORMS

I. General methods and theorems.

In addition to the methods of Dirichlet,[1] Dickson's modification[2] and a further modification (see proof for form 11) the following elementary methods used are numbered for convenience.

Method 1: see forms 5 and 13.

Method 2: is applied to a form $f=ax^2+by^2+2cz^2$ $(a\leq b)$ where a and b are odd. Now $f\equiv 0 \pmod 2$ implies $x+y\equiv 0 \pmod 2$ and $x+y=2X$, $x-y=2Y$ is solvable for X and Y. Thus $f/2=aX^2+bY^2+(b-a)(X-Y)^2/2+cz^2=(a+b)X^2/2+(a+b)Y^2/2+cz^2-(b-a)XY$. An alternative equivalent substitution is $x=2X-y$, $y=y$ giving $f/2=2aX^2+(a+b)y^2/2+cz^2-2aXy$. This method can be applied if the integers represented by f are known to find those represented by f/2, or conversely (see the proof for form 6, for example).

Method 3: see the proof for form 49.

Method 4 uses the corollary to lemma b in the following pages (see the proof for forms 35 and 44 for example).

The following theorems and lemmas apply chiefly to proofs for semi-regular forms in part C, but since the

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1 Journal fur Mathematik, vol.40 (1850), pp.228-32.
2 Bull. Amer. Math. Soc., 33 (1927), p.65.

corollary to lemma b and theorem 9 apply to certain regular forms we include the complete theory below.

<u>Lemma a:</u>[1] If x^2+by^2 represents two integers m and n, where b is a positive integer, it represents mn.

Proof: Suppose $x^2+by^2=m$ and $x'^2+by'^2=n$. Multiplying we get

(1) $$X^2+bY^2 = X'^2+bY'^2 = mn \text{ where}$$

(2) $X=xx'+byy'$, $Y=xy'-x'y$ and $X'=xx'-byy'$, $Y'=xy'+x'y$ are thus integers.

<u>Theorem 10a</u>. If $f=dx^2+dby^2+cz^2$ represents an integer a, then $g=dx^2+dby^2+cmz^2$ represents ma where m is an integer represented by x^2+by^2, where b is a positive integer as well as d.

Proof: "f represents a" means that there exists a z such that $(a-cz^2)/d$ is represented by x^2+by^2. Thus by lemma a we know $(ma-mcz^2)/d$ is represented by x^2+by^2 and thus ma is represented by g.

<u>Lemma b.</u> If x^2+by^2 represents p^n and mp^n with x and y prime to p in each case where n is a positive integer and p a prime which is odd in case $n>2$, then x^2+by^2 represents m.

Proof: Suppose $X^2+bY^2=mp^n$ and $x'^2+by'^2=p^n$ where X, Y, x', y' are prime to p. Then, solving (2) we obtain

$$x = \frac{Xx'+bYy'}{x'^2+by'^2} = \frac{Xx'+bYy'}{p^n} \text{ and } -y = \frac{x'Y-y'X}{p^n}$$

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1 Used by Brahmegupta and L. Euler, see "History of the Theory of Numbers", L. E. Dickson: vol. 2 p. 355 and vol.3 p. 60.

and from the derivation of (2), $x^2+by^2=m$ and thus it remains to prove that x and y are integers. We know $X^2\equiv -bY^2 \pmod{p^n}$ and $x'^2\equiv -by'^2 \pmod{p^n}$ and, multiplying, $(Xx')^2\equiv (bYy')^2 \pmod{p^n}$. Thus $Xx'\equiv \pm bYy' \pmod{p^n}$ where one of the signs holds for, if p is an odd prime $Xx'\equiv bYy'\equiv -bYy' \pmod{p}$ implies $b\equiv 0 \pmod{p}$ since Y and y' are both prime to p, which is false since $b\equiv 0 \pmod{p}$ and $X^2+bY^2\equiv mp^n$ would imply $X\equiv 0 \pmod{p}$; if p is even the statement also holds since $n\leq 2$, and the terms on each side of the congruence sign are odd. If the plus sign holds we may substitute $-y'$ for y' since the original equations are not affected by such a change and thus have in any case that x is an integer. Then since $x^2+by^2=m$ and b is prime to p we know that y is an integer.

<u>Corollary:</u> If $f=dx^2+bdy^2+cz^2$ represents an integer m with $x^2+by^2\equiv 0 \pmod{p^2}$ where p is a prime not dividing b (b and d are positive integers), then $g=dp^2x^2+bdp^2y^2+cz^2$ represents m,[1] if x^2+by^2 represents p^2 with x and y prime to p.

Proof: By hypothesis there exists a z such that $(m-cz^2)/d\equiv 0 \pmod{p^2}$ is represented by x^2+by^2. Thus $x\equiv y\equiv 0 \pmod{p}$ or x and y are prime to p. In the latter case we know from lemma b that $(m-cz^2)/dp^2$ is represented by x^2+by^2. Thus, in any case, $m-cz^2$ is represented by $dp^2x^2+bdp^2y^2$.

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1 Cf. J.G.A. Arndt, Göttingen Thesis, 1925, p. 25, for case $p=2$.

Theorem 9. If $f=x^2+by^2$ represents an odd prime p, where b is a positive integer prime to p, then every rp represented by f (r a positive integer) is represented by f with x and y prime to p. (This theorem is used in the proof for form 31).

Suppose $r=p^u$ (where u is an integer ≥ 1). Then, by lemma a, with $m=n=p$ there exist X, Y, X', Y' satisfying equations (1) and (2) where $x=x'$, $y=y'$ are prime to p since $x^2+by^2 \equiv 0 \pmod p$ and $x \equiv 0 \pmod p$ implies $y \equiv 0 \pmod p$ implies $x^2+by^2 \equiv 0 \pmod{p^2}$. If $Y \equiv 0 \pmod p$ we know $Y' \not\equiv 0 \pmod p$ for $xy'-x'y \equiv xy'+x'y \equiv 0 \pmod p$ implies $x'y \equiv 0 \pmod p$ which is impossible since x' and y are prime to p. Y' prime to p implies that X' is prime to p and thus in any case there exists an X and Y prime to p such that $X^2+bY^2=p^2$. Apply the same reasoning to lemma a with $m=p$ and $n=p^2$ with the above X, Y replaced by x', y' and find there exist an X and Y prime to p such that $X^2+bY^2=p^3$. Thus proceeding this case may be established.

Suppose $r=p^{s-1}t$ where t is an integer prime to p and s is integral and ≥ 1. If f represents p^st with x or $y \equiv 0 \pmod p$ then $x \equiv y \equiv 0 \pmod p$ and setting $x=px'$, $y=py'$ we see that $x'^2+by'^2=p^{s-2}t$. If x' or y' is divisible by p, both are unless s=2. Postponing the case s=2, set $x'=px''$, $y=py''$ and find that $x''^2+by''^2=p^{s-4}t$. Thus we continue until we find an integer v, $0 \leq v \leq s/2$ such that x^2+by^2 represents $p^{s-2v}t$, with not both x and y divisible by p. This must eventually come to pass since $x^2+by^2=pt$ or t implies that not both x and y are divisible by p. (This includes the case above postponed: s=2). Thus we have an x_1 and a y_1 not both divisible by p such that $x_1^2+by_1^2=p^{s-2v}t$. If $v=0$, x_1 and y_1 are both prime to p since $s\geq 1$ and our theorem is proved. If $v>0$ from the preceeding paragraph above there exist an x and y both prime to p such that $x^2+by^2=p^{2v}$. Thus, by lemma a, there exist X, Y, X', Y' defined by (2) with x_1 substituted for x' and y_1 for y'. Now $Y\equiv 0 \pmod p$ implies $Y'\not\equiv 0 \pmod p$ for $xy_1-x_1y\equiv xy_1+x_1y\equiv 0 \pmod p$ implies $x_1y\equiv 0 \pmod p$ implies $x_1\equiv 0 \pmod p$ implies $xy_1\equiv 0 \pmod p$ implies $y_1\equiv 0 \pmod p$ which contradicts the statement that not both x_1 and y_1 are divisible by p. Since $Y'\not\equiv 0 \pmod p$ and $X'^2+bY'^2=p^st$ implies that X' is prime to p, we know that in any case there exists an X and Y both prime to p such that $X^2+bY^2=p^st=rp$.

<u>Corollary 1</u>. If $f=x^2+by^2$ represents a prime p and if it represents mp^n where n is an integer ≥ 1, then f represents m.

Proof: 1. If p is odd the hypothesis of the corollary together with theorem 9 combine to show that f represents p^n and mp^n with x and y prime to p in each case and therefore from lemma b, f represents m.

2. If p=2, then b=1 and $x^2+y^2=2^nm$ implies $x+y=2X$, $x-y=2Y$ are solvable for X and Y and $X^2+Y^2=2^{n-1}m$. If $n>1$, then $X+Y=2x'$, $X-Y=2y'$ are solvable for x' and y' and

$x'^2+y'^2=2^{n-2}m$. Thus we may continue until we find an x and y such that $x^2+y^2=m$.

Corollary 2. If $f=dp^2x^2+dbp^2y^2+cz^2$ represents m and x^2+by^2 represents p, an odd prime, where b, d and c are positive integers, then $dx^2+bdy^2+cz^2$ represents m with x and y prime to p. (Cf. the corollary to lemma b).

Proof: There exists a z such that $(m-cz^2)/d$ is represented by $x^2+by^2\equiv 0 \pmod{p^2}$ and thus by theorem 9, $(m-cz^2)/d$ is represented with x and y prime to p.

Corollary 3. If $f=dx^2+bdy^2+pcz^2$ represents pm, where p is a prime represented by x^2+by^2 and m a positive integer and b and d positive integers prime to p, then $g=dx^2+bdy^2+cz^2$ represents m.

Proof: There exists a z such that $p(m-cz^2)/d$ is represented by x^2+by^2 and thus from corollary 1. $(m-cz^2)/d$ is represented by x^2+by^2.

Theorem 10b. If $f=dx^2+bdy^2+ncz^2$ represents nm where x^2+by^2 represents all the (prime) factors of n (d and b are positive integers prime to n), then $g=dx^2+bdy^2+cz^2$ represents m (m and n are positive integers).

Proof: Suppose the prime factors of n are $p_1, p_2, \ldots, p_r$ where any prime appearing to the t-th power in n is repeated t times in the display. Then from corollary 3 above:

$$g_1=dx^2+bdy^2+p_2p_3\ldots p_rcz^2 \text{ represents } p_2p_3\ldots p_rm.$$

$$g_2=dx^2+bdy^2+p_3p_4\ldots p_rcz^2 \text{ represents } p_3\ldots p_rm.$$

....

$$g_r=dx^2+bdy^2+cz^2 \text{ represents } m.$$

Theorem 10. If $f=dx^2+dby^2+cz^2$ and $g=dx^2+dby^2+cmz^2$ where all the (prime) factors of the positive integer m are represented by x^2+by^2, then g represents ma if and only if f represents a, an integer (b and d positive integers prime to m).

This results directly from theorems 10a and 10b if we note that if x^2+by^2 represents all the prime factors of an integer it represents that integer from lemma a.

Note 1: That it is not sufficient to say merely that m shall be represented by x^2+by^2 is illustrated by the fact that x^2+14y^2 represents 15 (but not 3 or 5) and while $g=x^2+14y^2+4\cdot 15z^2$ represents 30, $f=x^2+14y^2+4z^2$ does not represent 2. However we have

Note 2: Examination of the proof of theorem 10 and corollaries shows that theorem 9 may be altered to read: Given f and g where all the prime factors occuring to an odd power in m are represented by x^2+by^2 and the squares of all prime factors occuring to an even power in m are represented by x^2+by^2 with x and y prime to p, then g represents ma if and only if f represents a.

Lemma 7. $f=ax^2+by^2+c'pz^2+2pryz+2psxz+2ptxy$ represents no integer $\equiv pk \pmod{p^2}$ if a and b are prime to p and $\left(\frac{a}{p}\right)=-\left(\frac{-1}{p}\right)\left(\frac{b}{p}\right)$, $\left(\frac{c'}{p}\right)=-\left(\frac{k}{p}\right)$ and p is an odd prime not dividing k. (This lemma is used, for example, in the modification of Dirichlet's method in the proof for form 11).

1. $ax^2+by^2\equiv 0 \pmod p$ implies $x\equiv y\equiv 0 \pmod p$ since

$\left(\frac{a}{p}\right)=-\left(\frac{-1}{p}\right)\left(\frac{b}{p}\right)$. For, suppose $ax^2+by^2 \equiv 0 \pmod p$ with x or y prime to p. Then, since a and b are prime to p, both x and y are prime to p and there exists a z such that $xz \equiv 1 \pmod p$ and $a+b(yz)^2 \equiv 0 \pmod p$, $ab+(byz)^2 \equiv 0 \pmod p$ and $\left(\frac{-ab}{p}\right) = 1 = \left(\frac{-1}{p}\right)\left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$. Thus $\left(\frac{a}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{b}{p}\right)$ which contradicts the hypothesis.

2. $g = aX^2+bY^2+pc'z^2 \not\equiv pk \pmod{p^2}$ for $g \equiv 0 \pmod p$ implies $X=px'$, $Y=py'$ and $g/p = apx'^2+bpy'^2+c'z^2 \not\equiv k \pmod p$ since $\left(\frac{c'}{p}\right) = -\left(\frac{k}{p}\right)$. Set $X=x+pvy+pv'z$, $Y=y+pv''z$ and g becomes $g' = a(x+pvy+pv'z)^2+b(y+pv''z)^2+pc'z^2 \equiv ax^2+by^2+pc'z^2+2apvxy+2apv'xz+2bpv''yz \pmod{p^2}$. For no choice of v, v', v'' (integral) is $g' \equiv pk \pmod{p^2}$ since all integers represented by g' are represented by g. Also, since a and b are prime to p we may choose v, v', v'' so that $av \equiv t \pmod p$, $av' \equiv s \pmod p$, $bv'' \equiv r \pmod p$. Then $g' \equiv f \not\equiv pk \pmod{p^2}$.

II. Regular forms $f = ax^2 + by^2 + cz^2$.

(Regular forms completely dealt with in the references given in the tables are considered below only when a simpler proof has been found). The forms are numbered as in the tables.

4. $f = (1,1,4)$[*] $\neq 4^k(8n+7)$, $8n+3$. (This proof is contained essentially in some notes of L. E. Dickson).

f represents all $4n+1$ for $g = x^2+y^2+Z^2 \equiv 1 \pmod 4$ implies that one of x, y, Z is even. Permute if necessary and take $Z = 2z$ to prove $f = g \equiv 1 \pmod 4$ (see notations), and g represents all $4n+1$.

f represents no $4n+3$.

f represents all evens $\neq 4^k(8n+7)$ for, using method 2, we find $f/2 = x^2+y^2+2z^2$ which represents exclusively all $\neq 4^k(16n+14)$.

5. $f = (1,1,5)$[*] $\neq 4^k(8n+3)$.

For every $5a \neq 4^k(8n+7)$ reference to table I shows that there exists an x, y, z such that $f = x^2+y^2+z^2 = 5a$. Now $f = 5a$ implies that x, y or $z \equiv 0 \pmod 5$. Thus, from symmetry, there exists an x, y, $z = 5Z$ such that $x^2+y^2+25Z^2 = 5a$ which implies $x^2+y^2 \equiv 0 \pmod 5$ and $x \equiv \pm 2y \pmod 5$ where one of the signs holds. Now $\pm x = 5X+2y$ is solvable for X and thus 5a is represented by $(5X+2y)^2+y^2+25Z^2 = 25X^2+5y^2+25Z^2+20Xy$. Thus

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[*] $f = (1,1,4) \neq 4^k(8n+7)$, $8n+3$ is an abbreviation under such circumstances for "f represents exclusively all positive integers not of the forms given".

a is represented by $5X^2+y^2+5Z^2+4Xy = X^2+(2X+y)^2+5Z^2 \sim g$. Conversely if g represents a, f represents 5a and thus g represents exclusively all $\neq 4^k(8n'+7)/5 = 4^k(8n+3)$.[1]

6. $f=(1,1,6) \neq 9^k(9n+3)$.

f represents all evens $\neq 9^k(9n+3)$ for, using method 2, we have $f/2 = x^2+y^2+3z^2$ which from table I represents exclusively all positive integers $\neq 9^k(9n'+6)$.[2]

f represents all odds $\neq 9^k(9n+3)$ for $g = x^2+y^2+3z^2 \equiv 2 \pmod 4$ implies $x+y \equiv 0 \pmod 2$ and $z=2Z$. Thus set $x+y=2Y$, $x-y=2X$ and have $X^2+Y^2+6Z^2$ represents all odds $\neq 9^k(9n'+6)/2 = 9^k(9n+3)$, and none of that form.

10. $f=(1,1,16) \neq 4n+3,\ 8n+6,\ 32n+12,\ 4^k(8n+7)$.

[3]f represents all evens exclusively not of the last three forms above since $f/2 = X^2+Y^2+8z^2$ using method 2 and results for (1,1,8).

[3]f represents all 8n+5 since $g=x^2+4y^2+4z^2=f \equiv 5 \pmod 8$ for $g \equiv 5 \pmod 8$ implies y or z is even and by symmetry we may take $y=2y'$.

f represents all 8n+1. This has been proved by Arnold Chaimovitch applying results obtained by P. S. Nazimov "On The Application of the Theory of Elliptic Functions to the Theory of Numbers" (Dissertation, 1884) now being translated from the Russian by Mr. Chaimovitch.

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[1] This method gives a relationship between the number of representations of a by g and 5a by f here. At some future date the writer intends to work out the details for several forms proven by this method.

[2] See Amer. Jour. of Math., 49 (1927), p. 43.

[3] See Annals of Math. (2), 28 (1927), p. 339.

f represents no $4n+3$.

11. $f=(1,1,21)\neq 9^k(9n+6),\ 4^k(8n+3),\ 49^k(49n+7e)$ where $e=1$, 2 or 4. We apply a modification of Dirichlet's method and lemma 7. The only other reduced positive ternary quadratic forms of Hessian 21 are[1]: $g_1=(1,2,11,-2,0,0)$, $g_2=(1,5,5,-4,0,0)$ and $g_3=(3,3,3,0,-2,-2)$ all represent 6. $g_4=(1,3,7)$, $g_5=(2,2,7,0,0,-2)$ and $g_6=(2,3,4,0,-2,0)$ all represent 7.

Thus a form of Hessian 21 representing no $9n+6$ nor $49n+7$ cannot be equivalent to g_i $(i=1,\ldots,6)$ and thus must be equivalent to f.

I. For every integer $a\not\equiv 3 \pmod 8$ and not divisible by 4, 3 or 7 there exists a form $h=ax^2+3by^2+7cz^2+42ryz+42szx$ equivalent to f. By lemma 7, h represents no $9n+6$ nor $49n+7e$ where $\left(\frac{e}{7}\right)=1$ if

(1) $\left(\frac{b}{3}\right)=1,\ \left(\frac{a}{7}\right)=-\left(\frac{b}{7}\right),\ \left(\frac{a}{3}\right)=\left(\frac{c}{3}\right)$ and $\left(\frac{e}{7}\right)=-1$.

Thus we will have proved the statement above if we can find integers b, c, r and s satisfying (1) such that

$$H=21=a(21bc-21^2r^2)-21^2\cdot 3bs^2 \quad \text{i.e.}$$

(2) $$63s^2b=at-1 \quad \text{where } t=bc-21r^2.$$

(3) Now $\left(\frac{a}{7}\right)=\left(\frac{t}{7}\right)$ and $\left(\frac{a}{3}\right)=\left(\frac{t}{3}\right)$ since $at-1\equiv 0 \pmod{63}$ and thus if conditions (1) on b are satisfied, those on c will follow for:

$\left(\frac{a}{7}\right)=\left(\frac{t}{7}\right)=\left(\frac{b}{7}\right)\left(\frac{c}{7}\right)=-\left(\frac{a}{7}\right)\left(\frac{c}{7}\right)$ thus $\left(\frac{c}{7}\right)=-1$ and $\left(\frac{a}{3}\right)=\left(\frac{t}{3}\right)=\left(\frac{b}{3}\right)\left(\frac{c}{3}\right)=\left(\frac{c}{3}\right)$.

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[1] Eisenstein, Journal fur Mathematatik, vol. 41 (1851), p.169.

A. If a is odd take s=1, $t=8\cdot 63\cdot 21k+v$ and $b=2b'$.

Then $126b' = at-1$ and $b'=4\cdot 21k+(av-1)/126$. For each $\underline{a}$ we choose a w such that $\left(\frac{w}{7}\right)=-\left(\frac{a}{7}\right)$ and $\left(\frac{w}{3}\right)=-1$. Then there exists a $v \equiv a+2 \pmod 4$ and $\equiv v' \pmod 8$ such that $av-1 \equiv 126w \pmod{126\cdot 21}$. Then $(av-1)/126 = w' \equiv w \pmod{21}$ and w' and thus b' satisfies the conditions on w and thus b satisfies conditions (1). Furthermore w' is odd from the choice of v and is prime to a, 3 and 7. Thus we may and do choose k so that b' is a prime > 7.[1]

Then if $\left(\frac{a}{7}\right)=\pm 1$, $\left(\frac{b'}{7}\right)=\mp 1$ and $\left(\frac{t}{7}\right)=\pm 1$ also $\left(\frac{b'}{3}\right)=-1$.

Then $\left(\frac{-21t}{b'}\right)=\left(\frac{-3}{b'}\right)\left(\frac{-7}{b'}\right)\left(\frac{-t}{b'}\right)=\pm\left(\frac{-t}{b'}\right)$ and $\left(\frac{-126}{t}\right)=\left(\frac{-14}{t}\right)=\left(\frac{2}{t}\right)\left(\frac{-7}{t}\right)=\left(\frac{2}{t}\right)$.

1) If $a \equiv 1$ or $7 \pmod 8$ take $v'=7$ or 1 respectively and have $b' \equiv 1 \pmod 4$

$$\left(\frac{-126}{t}\right)=\pm\left(\frac{2}{t}\right)=\pm 1 \quad \text{and} \quad \left(\frac{-21t}{b'}\right)=\pm\left(\frac{-t}{b'}\right)=\pm\left(\frac{b'}{t}\right)=\left(\frac{-126b'}{t}\right)=1$$

2) If $a \equiv 5 \pmod 8$ take $v'=7$ and have $b' \equiv 3 \pmod 4$

$$\text{and} \left(\frac{-21t}{b'}\right)=\pm\left(\frac{-t}{b'}\right)=\pm\left(\frac{b'}{t}\right)=\left(\frac{-126b'}{t}\right)=\left(\frac{1}{t}\right)=1$$

B. If $a=2a'$ where a' is odd take s=1, $t=4\cdot 63\cdot 21k+v$.

Then $63b=2a't-1$ and choose v as above (omitting the restriction $v \equiv a+2 \pmod 4$ and k so that b is a prime except that this time we choose w and thus v so that $\left(\frac{w}{3}\right)=1$, i.e. $\left(\frac{b}{3}\right)=1$.

If $\left(\frac{a'}{7}\right)=\pm 1$, then $\left(\frac{b}{7}\right)=+1$ and $\left(\frac{t}{7}\right)=\pm 1$. Take $v'=1$ and have $b \equiv 3 \pmod 4$ and thus $\left(\frac{-63}{t}\right)=\left(\frac{-7}{t}\right)=\left(\frac{t}{7}\right) = \pm 1$ and $\left(\frac{-21t}{b}\right)=\left(\frac{-3}{b}\right)\left(\frac{-7}{b}\right)\left(\frac{-t}{b}\right)= \mp\left(\frac{-t}{b}\right)=\left(\frac{-63b}{t}\right)=1$

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1 "Verteilung der Primzahlen", Landau, vol.1,1909,p.422.

Thus in both cases A and B there exists an r' such that $2(21t+r'^2) \equiv 0 \pmod{b}$ and we can find an r (odd in case A) such that $21r \equiv r' \pmod{b}$ and thus $(t+21r^2)/b = c$ an integer.

II. For every integer $3a$ where $a \equiv 1 \pmod{3}$, $a \not\equiv 1 \pmod{8}$ and a is not divisible by 4 or 7 there exists a form $h = 3ax^2 + by^2 + 7cz^2 + 42ryz + 42xz$ equivalent to f. To prove this we seek as above b, r and c such that

(4) $\left(\frac{a}{7}\right) = -\left(\frac{b}{7}\right)$, $\left(\frac{b}{3}\right) = \left(\frac{c}{3}\right)$ and $\left(\frac{c}{7}\right) = -1$ and

$H = 21 = 3a(7bc - 21^2r^2) - 21^2b$ that is

(5) $21b = at - 1$ where $t = bc - 63r^2$.

(6) Now $\left(\frac{a}{7}\right) = \left(\frac{t}{7}\right)$ and $\left(\frac{a}{3}\right) = 1 = \left(\frac{t}{3}\right)$ since $at - 1 \equiv 0 \pmod{21}$ and thus if $\left(\frac{a}{7}\right) = -\left(\frac{b}{7}\right)$ the rest of (4) holds since $1 = \left(\frac{t}{3}\right) = \left(\frac{b}{3}\right)\left(\frac{c}{3}\right)$ and therefore $\left(\frac{b}{3}\right) = \left(\frac{c}{3}\right)$. Also $\left(\frac{a}{7}\right) = \left(\frac{t}{7}\right) = \left(\frac{b}{7}\right)\left(\frac{c}{7}\right) = -\left(\frac{a}{7}\right)\left(\frac{c}{7}\right)$ and thus $\left(\frac{c}{7}\right) = -1$.

A. If a is odd, choose $t = 8\cdot 63\cdot 21k + v$, $b = 2b'$.

Then $42b' = at - 1$ where as in IA for any given a, v may be so chosen that $\left(\frac{a}{7}\right) = -\left(\frac{b}{7}\right)$, b' an odd integer and k such that b' is a prime > 7. Then if $\left(\frac{a}{7}\right) = \pm 1$ we know $\left(\frac{b'}{7}\right) = \mp 1$ and $\left(\frac{t}{7}\right) = \pm 1$.

Then $\left(\frac{-7t}{b'}\right) = \left(\frac{b'}{7}\right)\left(\frac{t}{b'}\right) = \mp\left(\frac{t}{b'}\right)$ and $\left(\frac{42}{t}\right) = \left(\frac{2}{t}\right)\left(\frac{-3}{t}\right)\left(\frac{-7}{t}\right) = \left(\frac{2}{t}\right)\left(\frac{t}{7}\right) = \pm\left(\frac{2}{t}\right)$

1) If $a \equiv 3$ or $5 \pmod{8}$ take $v' = 5$ or 3 respectively and have $b' \equiv 3 \pmod{4}$ and $\left(\frac{42}{t}\right) = \pm 1$ and thus $\left(\frac{-7t}{b'}\right) = \mp\left(\frac{-b'}{t}\right) = \left(\frac{-42b'}{t}\right) = 1$.

2) If $a \equiv 7 \pmod{8}$ take $v' = 5$ and have $b' \equiv 1 \pmod{4}$ and $\left(\frac{42}{t}\right) = \mp 1$ and $\left(\frac{-7t}{b'}\right) = \mp\left(\frac{b'}{t}\right) = \left(\frac{42b'}{t}\right) = \left(\frac{-1}{t}\right) = 1$

B. If $a = 2a'$ where a' is odd, take $t = 4\cdot 21^2k + v$.

Then $21b = 2a't - 1$ where v is chosen as in the preceeding case and k so that b is a prime > 7.

If $\left(\frac{a}{7}\right) = \pm 1$, then $\left(\frac{b}{7}\right) = \mp 1$ and $\left(\frac{t}{7}\right) = \pm 1$ and, taking $v' = 3$, $b \equiv 1 \pmod 4$. Then $\left(\frac{21}{t}\right) = \left(\frac{-3}{t}\right)\left(\frac{-7}{t}\right) = \left(\frac{t}{7}\right) = \pm 1$ and $\left(\frac{-7t}{b}\right) = \left(\frac{b}{7}\right)\left(\frac{t}{b}\right) = \mp\left(\frac{t}{b}\right) = \mp\left(\frac{b}{t}\right) = \left(\frac{21b}{t}\right) = 1.$

Thus in cases A and B there exists a r' such that $7t + r'^2 \equiv 0 \pmod{b'}$ and $\pmod{b}$ respectively. We may choose an r (odd for A) such that $21r \equiv r' \pmod{b'}$ and $\pmod b$ respectively. Thus $(t + 63r^2)/b = c$ is an integer.

III. For every integer $7a$ where $a \not\equiv 5 \pmod 8$, $\left(\frac{a}{7}\right) = -1$ and a is not divisible by 3, 4 or 7 there exists a form $h = 7ax^2 + by^2 + 3cz^2 + 42ryz + 42xz$ equivalent to f. To prove this we seek as previously b, r and c such that

(7) $\left(\frac{c}{3}\right) = 1$, $\left(\frac{a}{3}\right) = \left(\frac{b}{3}\right)$, $\left(\frac{b}{7}\right) = -\left(\frac{c}{7}\right)$ and

$H = 21 = 7a(3bc - 21^2r^2) - 21^2b$, that is

(8) $21b = at - 1$ where $t = bc - 49 \cdot 3r^2$.

(9) Now $\left(\frac{a}{7}\right) = \left(\frac{t}{7}\right) = -1$ and $\left(\frac{a}{3}\right) = \left(\frac{t}{3}\right)$ since $at - 1 \equiv 0 \pmod{21}$. And thus if $\left(\frac{a}{3}\right) = \left(\frac{b}{3}\right)$ the rest of (7) holds since $-1 = \left(\frac{t}{7}\right) = \left(\frac{b}{7}\right)\left(\frac{c}{7}\right)$ and thus $\left(\frac{b}{7}\right) = -\left(\frac{c}{7}\right)$ and $\left(\frac{a}{3}\right) = \left(\frac{t}{3}\right) = \left(\frac{b}{3}\right)\left(\frac{c}{3}\right) = \left(\frac{a}{3}\right)\left(\frac{c}{3}\right)$ and thus $\left(\frac{c}{3}\right) = 1$.

A. If a is odd, choose $t = 8 \cdot 21^2k + v$ and $b = 2b'$.

Then $42b' = at - 1$ where as in IA for any given a, $v \equiv v' \pmod 8$ may be so chosen that $\left(\frac{a}{3}\right) = \left(\frac{b}{3}\right)$ and b' odd and k so that b' is a prime > 7. ($v' \equiv a + 2 \pmod 4$).

If $\left(\frac{a}{3}\right) = \pm 1$ then $\left(\frac{b'}{3}\right) = \mp 1$ and $\left(\frac{t}{3}\right) = \pm 1$.

$\left(\frac{-3t}{b'}\right) = \left(\frac{b'}{3}\right)\left(\frac{t}{b'}\right) = \mp\left(\frac{t}{b'}\right)$ and $\left(\frac{42}{t}\right) = \left(\frac{2}{t}\right)\left(\frac{-3}{t}\right)\left(\frac{-7}{t}\right) = \mp\left(\frac{2}{t}\right)$

1) If $a \equiv 1$ or $7 \pmod 8$ take $v' = 7$ or 1 respectively. Then $b' \equiv 3 \pmod 4$ and $\left(\frac{42}{t}\right) = \mp 1$ and $\left(\frac{-3t}{b'}\right) = \mp\left(\frac{b'}{t}\right) = \left(\frac{-42b'}{t}\right) = 1$.

2) If $a \equiv 3 \pmod 8$ take $v' = 1$. Then $b' \equiv 1 \pmod 4$ and $\left(\frac{42}{t}\right) = \mp 1$ and $\left(\frac{-3t}{b'}\right) = \mp\left(\frac{b'}{t}\right) = \left(\frac{42b'}{t}\right) = 1$.

B. If $a=2a'$ where a' is odd, take $t=8\cdot 21^2k+v$.

Then $21b=2a't-1$ where as in IA for any given a, $v\equiv v'(\text{mod } 8)$ may be so chosen that $\left(\frac{a}{3}\right)=\left(\frac{b}{3}\right)$ and k so that b is a prime >7. Take $v'=3$, then $b\equiv 1(\text{mod } 4)$ and if $\left(\frac{a'}{3}\right)=\pm 1$, we know $\left(\frac{b}{3}\right)=\mp 1$ and $\left(\frac{t}{3}\right)=\mp 1$ and $\left(\frac{21}{t}\right)=\left(\frac{-3}{t}\right)\left(\frac{-7}{t}\right)=\pm 1$ and thus $\left(\frac{-3t}{b}\right)=\mp\left(\frac{t}{b}\right)=\mp\left(\frac{b}{t}\right)=-\left(\frac{21b}{t}\right)=-\left(\frac{-1}{t}\right)=1$. Thus in cases A and B there exists an r' such that $3t+r'^2\equiv 0(\text{mod } b)$ (for if r' is even in case A replace it by $r''=r'+b'$ and have $3t+r''\equiv 0(\text{mod } 2)$). And choose an r (odd in case A) such that $21r\equiv r'(\text{mod } b')$ or (mod b) respectively for cases A and B, and have $(t+49\cdot 3r^2)/b=c$ an integer.

IV. For every integer 21a where $\left(\frac{a}{7}\right)=1=\left(\frac{a}{3}\right)$, $a\not\equiv 7(\text{mod } 8)$ and a prime to 3 and 7 and not divisible by 4, there exists a form $h=21ax^2+by^2+cz^2+42ryz+42xz$ equivalent to f. To prove this we seek, as previously, b, r and c such that

(10) $\left(\frac{b}{7}\right)=\left(\frac{c}{7}\right)$ and $\left(\frac{b}{3}\right)=\left(\frac{c}{3}\right)$ and

$H=21=21a(bc-21^2r^2)-21^2b$, that is

(11) $21b=at-1$ where $t=bc-(21r)^2$.

Now $\left(\frac{a}{7}\right)=1=\left(\frac{t}{7}\right)$ and $1=\left(\frac{a}{3}\right)=\left(\frac{t}{3}\right)$ since $at-1\equiv 0(\text{mod } 21)$ and thus (10) follows from (11) since $\left(\frac{t}{3}\right)=1=\left(\frac{b}{3}\right)\left(\frac{c}{3}\right)$. Thus $\left(\frac{b}{3}\right)=\left(\frac{c}{3}\right)$; $\left(\frac{t}{7}\right)=1=\left(\frac{b}{7}\right)\left(\frac{c}{7}\right)$. Thus $\left(\frac{b}{7}\right)=\left(\frac{c}{7}\right)$.

A. If a is odd, choose $t=8\cdot 21^2k+v$ and $b=2b'$.

Then $42b'=at-1$ and as in IA for any given a, v may be so chosen $\equiv v'(\text{mod } 8)$ where $v'\equiv a+2(\text{mod } 4)$ such that $av-1\equiv 0(\text{mod } 21)$ and k so that b' is a prime >7.

Then $\left(\frac{42}{t}\right)=\left(\frac{2}{t}\right)\left(\frac{-3}{t}\right)\left(\frac{-7}{t}\right)=\left(\frac{2}{t}\right)$.

1) If $a \equiv 3$ or $5 \pmod 8$ take $v' = 5$ or 3 respectively. Then $b' \equiv 3 \pmod 4$ and $\left(\frac{42}{t}\right) = -1$ and $\left(\frac{-t}{b'}\right) = -\left(\frac{t}{b'}\right) = -\left(\frac{-b'}{t}\right) = \left(\frac{-42b'}{t}\right) = 1$.

2) If $a \equiv 1 \pmod 8$ take $v' = 3$. Then $b' \equiv 1 \pmod 4$ and $\left(\frac{-t}{b'}\right) = \left(\frac{b'}{t}\right) = -\left(\frac{42b'}{t}\right) = 1$.

B. If $a = 2a'$ where a' is odd, take $t = 4 \cdot 21^2 k + v$.

Then $21b = 2a't - 1$ where as in IA, $v \equiv 1 \pmod 8$ is chosen so that $av - 1 \equiv 0 \pmod{21}$ and k so that b is a prime > 7. Then $b \equiv 1 \pmod 4$ and $\left(\frac{21}{t}\right) = \left(\frac{-3}{t}\right)\left(\frac{-7}{t}\right) = 1$ and thus $\left(\frac{-t}{b}\right) = \left(\frac{b}{t}\right) = \left(\frac{21b}{t}\right) = \left(\frac{-1}{t}\right) = 1$. Thus in cases A and B there exists an r' (odd in case A) such that $t + r'^2 \equiv 0 \pmod b$. And choose an r (odd in case A) such that $21r \equiv r' \pmod b$ and have $(t + 21^2 r^2)/b = c$ an integer.

V. Thus we have proved that for any $a \neq 8n+3$, $9n+6$, $49n+7e$ (where $e = 1, 2$ or 4) not divisible by 9, 49 or 4 there is a form h with leading coefficient <u>a</u> equivalent to f. Thus f represents all such a. Furthermore it is apparent that f represents no a excluded. Now, since $f \equiv 0 \pmod{m^2}$ implies $x \equiv y \equiv z \equiv 0 \pmod m$ where $m = 2, 3$ or 7 we see that $f = m^2 f$ and the proof is completed.

13. $f = (1, 2, 3) \neq 4^k(16n+10)$.

Reference to table I shows that for every $3a \neq 4^k(16n+14)$ there exists an x, y, z such that $g = x^2 + y^2 + 2z^2 = 3a$. Now $g = 3a$ implies that x or $y \equiv 0 \pmod 3$ and thus there exists an $x = 3X$ for which $g = 3a$ which implies $y \equiv \pm z \pmod 3$ where one of the signs holds. Then $\pm y = 3Y + z$ is solvable for Y, $3a$ is represented by $9X^2 + (3Y+z)^2 + 2z^2$ and a is represented by $3X^2 + 2Y^2 + (z+Y)^2 \sim f$. Conversely if f represents a, g represents 3a and thus f represents exclusively all $\neq 4^k(16n'+14)/3 = 4^k(16n+10)$.

16. $f=(1,3,10) \neq 9^k(9n+6),\ 25^k(25n\pm5),\ 4^k(16n+2)$.

We apply a modification of Dirichlet's method and lemma 7. The only other reduced positive ternary quadratic forms of Hessian 30 are:[1]

$g_1=(1,1,30),\quad g_2=(2,3,5)$ which represent 5;

$g_3=(1,2,15),\quad g_4=(1,5,6),\quad g_5=(2,3,6,0,0,-2),$
$g_6=(3,3,4,-2,-2,0),\quad g_7=(2,2,10,0,0,-2),\quad g_8=(2,4,4,-2,0,0)$
which represent 6;

and $g_9=(2,2,8,0,-2,0)$ which represents 20.

Thus a form of Hessian 30 representing no $9n+6$ nor $25n\pm5$ cannot be equivalent to g_i $(i=1,..,9)$ and thus must be equivalent to f.

I. For every integer $a \neq 4^k(16n+2)$ and not divisible by 3, 5 or 4 there exists a form $h=ax^2+3by^2+5cz^2+30ryz+30sxz$ equivalent to f. By lemma 7, h represents no $9n+6$ nor $25n\pm5$ if

(1) $\left(\frac{a}{3}\right)=-\left(\frac{c}{3}\right),\ \left(\frac{a}{5}\right)=\left(\frac{b}{5}\right)$ and $\left(\frac{b}{3}\right)=1,\ \left(\frac{c}{5}\right)=-1.$

Thus we will have proved the statement above if we can find integers b, c, r and s satisfying (1) such that

$$H=30=15a(bc-15r^2)-3\cdot15^2bs^2, \quad \text{i.e.}$$

(2) $45bs^2=at-2$ where $t=bc-15r^2$.

(3) Now $\left(\frac{a}{3}\right)=+\left(\frac{t}{3}\right)$ and $\left(\frac{a}{5}\right)=+\left(\frac{t}{5}\right)$ since $at-2\equiv 0 \pmod{15}$, and thus

1 Eisenstein, Journal für Mathematatik, vol. 41 (1851), p.169.

if conditions (1) on b are satisfied, those on c will follow for: $-\left(\frac{a}{3}\right)=\left(\frac{t}{3}\right)=\left(\frac{b}{3}\right)\left(\frac{c}{3}\right)=\left(\frac{c}{3}\right)$ and $-\left(\frac{a}{5}\right)=\left(\frac{t}{5}\right)=\left(\frac{b}{5}\right)\left(\frac{c}{5}\right)=\left(\frac{a}{5}\right)\left(\frac{c}{5}\right)$ and thus $\left(\frac{c}{5}\right)=-1$.

A. If a is odd take $t=4T$, $b=2B$, $T=4\cdot45\cdot15k+v$ and $s=1$.

Then $45B=2aT-1$ and $B=8\cdot15ak+(2av-1)/45$. Then for any given a we can choose a w (prime to 3 and 5) such that $\left(\frac{w}{5}\right)=-\left(\frac{a}{5}\right)$ and $\left(\frac{w}{3}\right)=-1$. Then for any odd v' there exists a $v\equiv v'(\text{mod } 8)$ such that $2av-1\equiv45w(\text{mod } 45\cdot15)$. Then $(2av-1)/45$ $=w'\equiv w(\text{mod } 15)$ and w' and thus B satisfies the conditions on w and thus b satisfies conditions (1) on b. Furthermore w' is odd and is prime to a, 3 and 5. Thus we may choose k so that B is a prime >5.

Take $v'=3$ and have $\left(\frac{-15t}{B}\right)=\left(\frac{-15T}{B}\right)=\left(\frac{5}{B}\right)\left(\frac{-3}{B}\right)\left(\frac{T}{B}\right)=\left(\frac{B}{5}\right)\left(\frac{B}{3}\right)\left(\frac{T}{B}\right)=-\left(\frac{B}{5}\right)\left(\frac{B}{T}\right)$ for since T is odd $B\equiv1(\text{mod } 4)$ and since $\left(\frac{b}{3}\right)=1$, $\left(\frac{B}{3}\right)=-1$. Also $\left(\frac{45}{T}\right)=\left(\frac{5}{T}\right)=\left(\frac{T}{5}\right)$.

Now if $\left(\frac{a}{5}\right)=\pm1$, then from (3) and $t=4T$ and (1) we have $\left(\frac{T}{5}\right)=\mp1$, $\left(\frac{b}{5}\right)=\pm1$ and therefore $\left(\frac{B}{5}\right)=\mp1$. Thus $\left(\frac{-15t}{B}\right)=\pm\left(\frac{B}{T}\right)=-\left(\frac{45B}{T}\right)=-\left(\frac{-1}{T}\right)=1$ and therefore there exists an r' such that $15t+r'^2\equiv0(\text{mod } B)$ and we can find an even r such that $15r\equiv r'(\text{mod } B)$ which gives $(t+15r^2)/2B=c$ is integral.

B. If $a=2a'$ where $a'\equiv3,5$ or $7(\text{mod } 8)$, let $s=2$ and $t=8\cdot45\cdot15k+v$.

Then $90b=a't-1$ and as above we can choose $v\equiv v'(\text{mod } 8)$ so that b is an odd integer satisfying the conditions (1) on b, and k so that b is a prime >5.

Now $\left(\frac{-15t}{b}\right)=\left(\frac{5}{b}\right)\left(\frac{-3}{b}\right)\left(\frac{t}{b}\right)=\left(\frac{b}{5}\right)\left(\frac{b}{3}\right)\left(\frac{t}{b}\right)=\left(\frac{b}{5}\right)\left(\frac{t}{b}\right)$.

1). If $a' \equiv 3$ or $5 \pmod 8$ take $v' = 5$ or 3 respectively. Then $b \equiv 3 \pmod 4$, and when $\left(\frac{a'}{5}\right) = \pm 1$ we have $\left(\frac{t}{5}\right) = \pm 1$ and $\left(\frac{b}{5}\right) = \mp 1$ from (1) and (3).

Then $\left(\frac{90}{t}\right) = \left(\frac{10}{t}\right) = \left(\frac{2}{t}\right)\left(\frac{5}{t}\right) = \mp 1$ and $\left(\frac{-15t}{b}\right) = \mp\left(\frac{t}{b}\right) = \mp\left(\frac{-b}{t}\right) = \left(\frac{-90b}{t}\right) = \left(\frac{1}{t}\right) = 1$

2). If $a' \equiv 7 \pmod 8$ take $v' = 5$. Then $b \equiv 1 \pmod 4$ and $\left(\frac{90}{t}\right) = \mp 1$ and $\left(\frac{-15t}{b}\right) = \mp\left(\frac{b}{t}\right) = \left(\frac{90b}{t}\right) = \left(\frac{-1}{t}\right) = 1$.

Thus, as above there exists an r such that $t + 15r^2 \equiv 0 \pmod b$ and c is integral $= (t + 15r^2)/b$.

II. For every integer $3a$ where $a \equiv 1 \pmod 3$ and $a \neq 4^k(16n+6)$ and not divisible by 5 or 4 there exists a form $h = 3ax^2 + by^2 + 5cz^2 + 30ryz + 30sxz$ equivalent to f. To prove this we seek as above b, r, c and s such that

(4) $\left(\frac{b}{3}\right) = -\left(\frac{c}{3}\right)$, $\left(\frac{a}{5}\right) = \left(\frac{b}{5}\right)$ and $\left(\frac{c}{5}\right) = -1$ and

$H = 30 = 3a(5bc - 15^2r^2) - (15s)^2 b$ that is

(5) $15bs^2 = at - 2$ where $t = bc - 45r^2$.

Now $\left(\frac{a}{3}\right) = 1 = -\left(\frac{t}{3}\right)$ and $\left(\frac{a}{5}\right) = -\left(\frac{t}{5}\right)$ since $at - 2 \equiv 0 \pmod{15}$ and thus if $\left(\frac{a}{5}\right) = \left(\frac{b}{5}\right)$ the rest of (4) follows for $\left(\frac{t}{5}\right) = \left(\frac{bc}{5}\right) = \left(\frac{a}{5}\right)\left(\frac{c}{5}\right) = -\left(\frac{a}{5}\right)$ gives $\left(\frac{c}{5}\right) = -1$ and $-1 = \left(\frac{t}{3}\right) = \left(\frac{bc}{3}\right)$ gives $\left(\frac{c}{3}\right) = -\left(\frac{b}{3}\right)$.

A. If a is odd let $t = 4T$, $b = 2B$ and $T = 8 \cdot 15^2 k + v$, $s = 1$.

Then $15B = 2aT - 1$ and as above we can choose $v \equiv v' \pmod 8$ so that B is an integer satisfying the condition $\left(\frac{2B}{5}\right) = \left(\frac{a}{5}\right)$ and k so that B is a prime > 5.

Now $\left(\frac{-5T}{B}\right) = \left(\frac{5}{B}\right)\left(\frac{-T}{B}\right) = \left(\frac{B}{5}\right)\left(\frac{-T}{B}\right)$ and taking $v' = 1$ we have $B \equiv 3 \pmod 4$.

If $\left(\frac{a}{5}\right) = \pm 1$ then $\left(\frac{B}{5}\right) = \mp 1$ and $\left(\frac{t}{5}\right) = \mp 1$ and thus $\left(\frac{T}{5}\right) = \mp 1$.

Then $\left(\frac{15}{T}\right) = \left(\frac{3}{T}\right)\left(\frac{5}{T}\right) = \left(\frac{T}{3}\right)\left(\frac{T}{5}\right) = \pm 1$ and $\left(\frac{-5t}{B}\right) = \pm\left(\frac{T}{B}\right) = \pm\left(\frac{B}{T}\right) = \left(\frac{15B}{T}\right) = 1$ and there exists an r' such that $r'^2 + 5t \equiv 0 \pmod B$ and choose

r, even, such that $15r \equiv r'(\text{mod } B)$ and have $(45r^2+t)/2B = c$ is an integer.

B. If $a=2a'$ where $a' \equiv 1,5$ or $7(\text{mod } 8)$ take $s=2$ and $t=8\cdot 15^2k+v$.

Then $30b = a't-1$ and as above we can choose $v \equiv v'(\text{mod } 8)$ so that b is an odd integer satisfying the condition $\left(\frac{b}{5}\right)=\left(\frac{a}{5}\right)$ and k so that b is a prime > 5.

1). If $a' \equiv 1$ or $7(\text{mod } 8)$ take $v'=7$ or 1 respectively. Then $b \equiv 1(\text{mod } 4)$. When $\left(\frac{a'}{5}\right) = \pm 1$ we have $\left(\frac{b}{5}\right)=\mp 1, \left(\frac{t}{5}\right)=\pm 1$. Thus $\left(\frac{-30}{t}\right)=\left(\frac{2}{t}\right)\left(\frac{5}{t}\right)\left(\frac{-3}{t}\right) = \pm\left(\frac{t}{3}\right)=\mp 1$ and $\left(\frac{-5t}{b}\right)=\left(\frac{5}{b}\right)\left(\frac{-t}{b}\right) = \mp\left(\frac{b}{t}\right)=\left(\frac{-30b}{t}\right)=1.$

2). If $a' \equiv 5(\text{mod } 8)$ take $v'=7$ and have $b \equiv 3(\text{mod } 4)$. Then $\left(\frac{-30}{t}\right)= \mp 1$ and $\left(\frac{-5t}{b}\right)=\left(\frac{5}{b}\right)\left(\frac{-t}{b}\right)=\mp\left(\frac{-t}{b}\right) = \mp\left(\frac{b}{t}\right)=\left(\frac{-30b}{t}\right)=1.$

Thus in both cases there exists an r' such that $r'^2+5t \equiv 0(\text{mod } b)$ and we choose r such that $15r \equiv r'(\text{mod } b)$ and have $(45r^2+t)/b = c$.

III. For every integer $5a$ where $\left(\frac{a}{5}\right)=-1$, $a \not\equiv 10(\text{mod } 16)$ and not divisible by 3 or 4 there exists a form $h=5ax^2+by^2+3cz^2+30ryz+30sxz$ equivalent to f. To prove this we seek as above b, r, c and s such that

(6) $\left(\frac{b}{5}\right)=\left(\frac{c}{5}\right)$, $\left(\frac{a}{3}\right) = -\left(\frac{b}{3}\right)$ and $\left(\frac{c}{3}\right) = 1$ and

$H=30=5a(3bc-15^2r^2)-(15s)^2b$, that is

(7) $15bs^2=at-2$ where $t=bc-75r^2$.

Now $\left(\frac{a}{3}\right)=-\left(\frac{t}{3}\right)$ and $\left(\frac{a}{5}\right)=-\left(\frac{t}{5}\right)$ since $at-2 \equiv 0(\text{mod } 15)$ and thus $\left(\frac{t}{5}\right)=1$ and if $\left(\frac{a}{3}\right)=-\left(\frac{b}{3}\right)$ the remaining conditions (6) hold for $\left(\frac{t}{3}\right)=\left(\frac{bc}{3}\right)=\left(\frac{b}{3}\right)\left(\frac{c}{3}\right)=-\left(\frac{a}{3}\right)\left(\frac{c}{3}\right) = -\left(\frac{a}{3}\right)$ and thus $\left(\frac{c}{3}\right) = 1$ and $\left(\frac{t}{5}\right) = 1 = \left(\frac{b}{5}\right)\left(\frac{c}{5}\right)$ giving $\left(\frac{b}{5}\right)=\left(\frac{c}{5}\right)$.

A. If a is odd let $t=4T$, $b=2B$, $T=8\cdot 15^2k+v$ and $s=1$.

Then $15B=2aT-1$ and as above we can choose $v\equiv 3 \pmod 8$ so that B is an integer satisfying the condition $\left(\frac{a}{3}\right)=-\left(\frac{-2B}{3}\right)$ and k so that B is a prime >5.

If $\left(\frac{a}{3}\right)=\pm 1$, then $\left(\frac{B}{3}\right)=\pm 1$, $\left(\frac{t}{3}\right)=\mp 1$ and thus $\left(\frac{15}{T}\right)=\left(\frac{3}{T}\right)\left(\frac{5}{T}\right)=-\left(\frac{T}{3}\right)=\pm 1$.

Also $\left(\frac{-3t}{B}\right)=\left(\frac{-3}{B}\right)\left(\frac{T}{B}\right)=\left(\frac{B}{3}\right)\left(\frac{-B}{T}\right)=\pm\left(\frac{-B}{T}\right)=\left(\frac{-15B}{T}\right)=\left(\frac{1}{T}\right)=1$.

Thus there exists an r' such that $r'^2+3t\equiv 0 \pmod B$ and choose r, even, such that $15r\equiv r' \pmod B$ and have $(75r^2+t)/b=c$ is integral.

B. If $a=2a'$ where $a'\equiv 1, 3$ or $7 \pmod 8$ take $s=2$, $t=8\cdot 15^2k+v$.

Then $30b=a't-1$ and as above we can choose $v\equiv v' \pmod 8$ so that b is an odd integer satisfying the condition $\left(\frac{a}{3}\right)=-\left(\frac{b}{3}\right)$ and k so that b is a prime >5.

If $\left(\frac{a'}{3}\right)=\pm 1$, then $\left(\frac{b}{3}\right)=\pm 1=\left(\frac{t}{3}\right)$ and

1) If $a'\equiv 1$ or $7 \pmod 8$ take $v'=7$ or 1 respectively. Then $b\equiv 1 \pmod 4$ and $\left(\frac{-30}{t}\right)=\left(\frac{2}{t}\right)\left(\frac{5}{t}\right)\left(\frac{-3}{t}\right)=\left(\frac{t}{3}\right)=\pm 1$ and thus

$\left(\frac{-3t}{b}\right)=\left(\frac{b}{3}\right)\left(\frac{t}{b}\right)=\pm\left(\frac{b}{t}\right)=\left(\frac{-30b}{t}\right)=1$.

2) If $a'\equiv 3 \pmod 8$ take $v'=1$. Then $b\equiv 3 \pmod 4$ and $\left(\frac{-30}{t}\right)=\pm 1$ gives $\left(\frac{-3t}{b}\right)=\pm\left(\frac{b}{t}\right)=1$ and thus in both cases as above there exists an r such that $(75r^2+t)/b=c$ is an integer.

IV. For every integer 15a where $\left(\frac{a}{3}\right)=-1$, $\left(\frac{a}{5}\right)=1$, $a\not\equiv 14 \pmod{16}$ and not divisible by 4, there exists a form $h=15ax^2+by^2+cz^2+30ryz+30sxz$ equivalent to f. To prove this we seek as above b, r, c and s such that

(8) $\left(\frac{b}{3}\right)=\left(\frac{c}{3}\right)$ and $\left(\frac{b}{5}\right)=-\left(\frac{c}{5}\right)$ and

$H=30=15a(bc-15^2r^2)-(15s)^2b$, that is

(9) $15bs^2=at-2$ where $t=bc-(15r)^2$.

Now $-\left(\frac{a}{5}\right)=\left(\frac{t}{5}\right)=-1$ and $1=-\left(\frac{a}{3}\right)=\left(\frac{t}{3}\right)$ since $at-2\equiv 0 \pmod{15}$ and thus (8) holds if b is an integer for $\left(\frac{t}{3}\right)=1=\left(\frac{b}{3}\right)\left(\frac{c}{3}\right)$ implies $\left(\frac{b}{3}\right)=\left(\frac{c}{3}\right)$ and $\left(\frac{t}{5}\right)=-1=\left(\frac{b}{5}\right)\left(\frac{c}{5}\right)$ implies $-\left(\frac{b}{5}\right)=\left(\frac{c}{5}\right)$.

A. If a is odd let $t=4T$, $b=2B$, $T=8\cdot 15^2k+v$ and $s=1$.

Then $15B=2aT-1$ and as above we can choose $v\equiv 1 \pmod 8$ so that B is an integer and k so that B is a prime > 5. Then $\left(\frac{15}{T}\right)=\left(\frac{3}{T}\right)\left(\frac{5}{T}\right)=\left(\frac{T}{3}\right)\left(\frac{T}{5}\right)=-1$ and $\left(\frac{-T}{B}\right)=-\left(\frac{T}{B}\right)=-\left(\frac{-B}{T}\right)=\left(\frac{-15B}{T}\right)=1$ and an r' exists such that $t+r'^2\equiv 0 \pmod B$. Choose r even so that $15r\equiv r' \pmod B$ and have $(t+15^2r^2)/2B=c$ is integral.

B. If $a=2a'$ where $a'\equiv 1, 3$ or $5 \pmod 8$ take $s=2$ and $t=8\cdot 15^2k+v$.

Then $30b=a't-1$ and as above we can choose $v\equiv v' \pmod 8$ so that b is an odd integer, and k so that b is a prime >5.

1) If $a'\equiv 3$ or $5 \pmod 8$ take $v'=5$ or 3 respectively. Then $b\equiv 1 \pmod 4$ and $\left(\frac{-30}{t}\right)=\left(\frac{2}{t}\right)\left(\frac{-3}{t}\right)\left(\frac{5}{t}\right)=1$ and $\left(\frac{-t}{b}\right)=\left(\frac{b}{t}\right)=\left(\frac{-30b}{t}\right)=1$.

2) If $a'\equiv 1 \pmod 8$ take $v'=3$. Then $b\equiv 3 \pmod 4$, $\left(\frac{-30}{t}\right)=1$ and $\left(\frac{-t}{b}\right)=1$. Thus in both cases there exists as above an r such that $(t+15^2r^2)/b=c$ is integral.

V. Thus we have proved that for every $a \neq 9n+6$, $25n\pm 5$ nor $16n+2$ there is a form h with leading coefficient <u>a</u> equivalent to f. Thus f represents all such a. Furthermore it is apparent that f represents no <u>a</u> excluded. Now

$f \equiv 0 \pmod{m^2}$ implies $x \equiv y \equiv z \equiv 0 \pmod{m}$ for $m=3$ or 5. Also $f \equiv 0 \pmod 4$ implies $z \equiv 0 \pmod 2$ and thus all $x + 3y^2 \equiv 0 \pmod 4$ and thus by the corollary to **lemma** b, every multiple of 4 represented by f is represented with x and y even. Thus $f = m^2 f$ where $m=2, 3$ or 5 and the proof is complete.

17. $f = (1,5,8) \neq 4n+3,\ 8n+2,\ 25^k(25n \pm 10)$.

[2] f represents all $4n+1 \neq 25^k(25n' \pm 10)$ as is shown by reference to table I and $g = x^2 + 5y^2 + 2Z^2 \equiv 1 \pmod 4$ implies $Z = 2z$, i.e. $g = f \equiv 1 \pmod 4$.

f represents no integers of the forms excluded.

f represents all $\equiv 6 \pmod 8$ not of the form $25^k(25n \pm 10)$.

Proof: $f \equiv 6 \pmod 8$ implies $x+y \equiv 0 \pmod 2$ and thus $x+y = 2X$, is solvable for X and $f/2 = g = 2X^2 + 3y^2 + 4z^2 - 2Xy$. The only other reduced positive ternary quadratic forms of Hessian 20[1] are: forms of minimum 1 and $g_1 = (2,2,5)$ which represents 5, two forms representing no odds and $g_2 = (3,3,3,2,2,2)$ which represents no $4n+2$ for $3x^2 + 3y^2 + 3z^2 + 2yz + 2xz + 2yx \equiv 0 \pmod 2$ implies that one of x, y, z is even and the other two both odd or both even. From symmetry take $x = 2X$, $y+z = 2Y$, $y-z = 2Z$ which are solvable for X, Y and Z and g_2 becomes $12X^2 + 8Y^2 + 4Z^2 + 8XY \not\equiv 2 \pmod 4$. Since we wish to prove that g represents all $a \equiv 3 \pmod 4$ not of the form $25^k(25n \pm 5)$, for such a**s** a **we** form

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[1] Eisenstein, Journal fur Mathematik, vol.41 (1851), p.169.
[2] See Annals of Math. (2), 28 (1927), p. 340.

$h = ax^2 + by^2 + 4cz^2 + 4ryz + 4sxz$ where $b \equiv 3 \pmod 4$.

Now h represents no $4n+1$ and thus is not equivalent to a form with minimum 1 or to g_1. h represents an odd and thus is not equivalent to either of the forms representing only evens. h represents $a+b \equiv 2 \pmod 4$ and thus is not equivalent to g_2. Thus h is equivalent to g if we can find integers $b \equiv 3 \pmod 4$, c, r and s such that

$$H = 20 = 4a(bc - r^2) - 4bs^2; \quad \text{that is}$$

$$bs^2 = at - 5 \text{ where } t = bc - r^2.$$

I. If a is prime to 5 let $s = 1$, $t = 4T$, $T = 5k+2$.

Then $b = 4aT - 5 \equiv 3 \pmod 4$, $b = 20ak + 8a - 5$ and since $20a$ and $8a-5$ are relatively prime we can and do choose k so that b is prime. Then $\left(\frac{-t}{b}\right) = \left(\frac{-T}{b}\right) = -\left(\frac{T}{b}\right) = -\left(\frac{-b}{T}\right) = -\left(\frac{5}{T}\right) = -\left(\frac{T}{5}\right) = 1$ and there exists an r such that $t + r^2 \equiv 0 \pmod b$ and $(t + r^2)/b = c$ is an integer.

II. If $a = 5a'$ where $a' = 5w \pm 2$.[1] Take $b = 4a'T - 5$, $s = 1 \pm 2a'$, $T = 5k+1$.

Then $5 = 5a'(bc - r^2) - b(1 \pm 4a' + 4a'^2)$, i.e. $5 + b = a'b(5c \mp 4 - 4a') - 5a'r^2$, $b = 20a'k + 4a' - 5$ and choose k so that b is a prime. Replace $5+b$ by $4a'T$ above, divide through by a' and have $4T + 5r^2 = bP$ where $P = 5c \mp 4 - 4a' \equiv \pm 3 \pmod 5$ since $b = 4a'T - 5 \equiv \pm 3 \pmod 5$ and $bP \equiv 4T \equiv 4 \pmod 5$ provided we can find integers r and P such that $4T + 5r^2 = bP$. Further-

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1 This is an example of Dickson's modification of Dirichlet's proof. See Bull. Amer. Math. Soc., 33 (1927), p. 65.

more since $\mp 4-4a' \equiv \pm 3 \pmod 5$ we know if P, integral, exists, $P = 5c \mp 4 - 4a'$ is solvable for c, an integer. Thus it remains to find an r such that $4T+5r^2 \equiv 0 \pmod b$.

Now $\left(\frac{-5T}{b}\right) = \left(\frac{5}{b}\right)\left(\frac{-T}{b}\right), \left(\frac{5}{b}\right) = \left(\frac{b}{5}\right) = \left(\frac{a'T}{5}\right) = -\left(\frac{T}{5}\right) = -1, \left(\frac{-T}{b}\right) = -\left(\frac{-b}{T}\right) = -\left(\frac{5}{T}\right) = -\left(\frac{T}{5}\right) = -1.$

Thus $\left(\frac{-5T}{b}\right) = 1$ and such an r exists.

III. Thus we have proved that for every $a \equiv 3 \pmod 4$ and not of the form $25n \pm 5$ nor divisible by 25 there is an h having leading coefficient a which is equivalent to g. Thus f/2 represents all such a. Since $f \equiv 0 \pmod{25}$ implies $x \equiv y \equiv z \equiv 0 \pmod 5$ we know $f = 25f$ and the proof is complete.

[1] f represents all $\equiv 0 \pmod 4$ not of the form $25^k(25n \pm 10)$ for $f \equiv 0 \pmod 4$ implies $x = 2X$, $y = 2Y$ and thus $f/4 = X^2 + 5Y^2 + 2z^2$ which, from table I represents exclusively all positive integers $\neq 25^k(25n \pm 10)$.

20. $f = (1,2,6) \neq 4^k(8n+5)$.

Apply method 1 (see proof for form 13) to prove that for every $3a \neq 4^k(8n+7)$, $g = x^2+2y^2+2z^2$ represents 3a with $z = 3Z$, $x \equiv \pm y \pmod 3$ where one of the signs holds. Thus $(3X+y)^2+2y^2+18Z^2$ represents 3a and a is represented by $2X^2+(X+y)^2+6Z^2 \sim f$. Also if f represents a, g represents 3a and thus f represents exclusively all $\neq 4^k(8n+7)/3 = 4^k(8n'+5)$.

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[1] See Annals of Math. (2), 28 (1927), p. 340.

22. $f=(1,2,10)\neq 8n+7,\ 25^k(25n\pm5)$.

$f=a\equiv 0 \pmod 2$ implies $x=2X$ and $f/2=g=2X^2+y^2+5z^2$ which reference to table I shows represents all and only those positive integers not of the form $25^k(25n'\pm10)$.

f represents no $8n+7$.

f represents all $4n+1\neq 25^k(25n\pm5)$ for if $g\equiv 2 \pmod 8$ y and z are even and thus $2x^2+4Y^2+20Z^2$ represents all $\equiv 2 \pmod 8$ not of the form $25^k(25n'\pm10)$.

f represents all $8n+3$.

Proof: The only other reduced positive ternary quadratic forms of Hessian 20 are: $g_1=(1,1,20)$, $g_2=(1,4,5)$, $g_3=(1,4,6,-4,0,0)$, $g_4=(2,2,5)$ which represent no $8n+3$ $[g_3=x^2+(2y-z)^2+5z^2]$; $g_5=(1,3,7,-2,0,0)$ which represents no $4n+2$ since $3g_5=3x^2+(3y-z)^2+20z^2\not\equiv 2 \pmod 4$; $g_6=(2,3,4,0,0,-2)$ which represents no $4n+1$ since $2g_6=(2x-y)^2+5y^2+8z^2\not\equiv 2 \pmod 8$; two forms which represent no odds; and $g_7=(3,3,3,2,2,2)$ which represents no $4n+2$ (see proof for form 17). Since we wish to prove that f represents all $a\equiv 3 \pmod 8$ where a is not of the form $25^k(25n\pm5)$ for such an a we form

$$h=ax^2+by^2+cz^2+2ryz+2sxz \quad \text{where } b\equiv 2 \pmod 4.$$

Now h represents an $8n+3$ and thus is not equivalent to g_i $(i=1,.,4)$ nor to the forms which represent no odds. h represents $b\equiv 2 \pmod 4$ and thus is not equivalent to g_5 nor g_7. h represents $a+b\equiv 1 \pmod 4$ and thus is not equivalent to g_6. Thus h is equivalent to f if we can find integers

$b \equiv 2 \pmod 4$, c, r and s such that

$$H=20=a(bc-r^2)-bs^2.$$

I. If a is prime to 5 take $t=2T$, $b=2B$, $T=40k+11$, $s=1$ where $t=bc-r^2$.

Then $B=aT-10=40ak+11a-10 \equiv 7 \pmod 8$ and since 40a and 11a-10 are relatively prime we choose k so that B is an odd prime. Now $\left(\frac{-t}{B}\right)=\left(\frac{-2T}{B}\right)=-\left(\frac{T}{B}\right)=\left(\frac{B}{T}\right)=\left(\frac{-10}{T}\right)=\left(\frac{5}{T}\right)=\left(\frac{T}{5}\right)=$ 1 and thus there exists an r' such that $t+r'^2 \equiv 0 \pmod B$. Choose $r \equiv r' \pmod B$ and even and have $(t+r^2)/b=c$ is an integer.

II. If $a=5a'$ where $a'=5w \pm 2 \equiv 7 \pmod 8$ let $b=2B$, $B=a'T-10$, $T=40k+5\mp 4$, $s=1\pm 2a'$ and $r=2r'$.

Then $20=5a'(bc-r^2)-b(1\pm 2a')^2$.

$10=5a'(Bc-2r'^2)-B(1\pm 2a')^2$.

Then $B=a'T-10=a'(40k+5\mp 4)-10=40a'k+(5\mp 4)a'-10 \equiv 5 \pmod 8$ and since 40a' and $(5\mp 4)a'-10$ are relatively prime we choose k so that B is a prime > 5.

We then have from the above: $10r'^2a'+10+B=a'B(5c\mp 4-4a')$ $=a'BP$ where $P=5c\mp 4-4a'$. Substitute $a'T$ for $10+B$ on the left, divide through by a' and have $10r'^2+T=BP$ where $P \equiv \mp 2 \pmod 5$ since $B \equiv a'T \equiv 2 \pmod 5$ and $BP \equiv T \equiv \pm 1 \pmod 5$ provided we can find integers r' and P such that $10r'^2+T=BP$. Furthermore since $\mp 4-4a' \equiv \mp 2 \pmod 5$ we know that if P, integral, exists, $P=5c\mp 4-4a'$ is solvable for c, an integer. Thus it remains to find an r' such that $10r'^2+T \equiv 0 \pmod B$. Now $\left(\frac{-10T}{B}\right)=\left(\frac{2}{B}\right)\left(\frac{5}{B}\right)\left(\frac{-T}{B}\right)$; $\left(\frac{2}{B}\right)=-1$; $\left(\frac{-T}{B}\right)=\left(\frac{B}{T}\right)=\left(\frac{-10}{T}\right)=\left(\frac{5}{T}\right)=\left(\frac{T}{5}\right)=1$; $\left(\frac{5}{B}\right)=\left(\frac{B}{5}\right)=\left(\frac{a'T}{5}\right)=-\left(\frac{T}{5}\right)=-1$

Thus $\left(\frac{-10T}{B}\right)=1$ and there exists an r'' such that $10T+r''^2 \equiv 0 \pmod B$, choose $10r' \equiv r'' \pmod B$ and have $(T+10r'^2)/B=P$ an integer.

III. Thus we have proved that for every $a \equiv 3 \pmod 8$ and not of the form $25^k(25n+5)$ nor divisible by 25 there is an h having leading coefficient a which is equivalent to f. Thus f represents all such a. Since $f \equiv 0 \pmod{25}$ implies $x \equiv y \equiv z \equiv 0 \pmod 5$ we know $f = 25f$ and the proof is complete.

23. $f = (1,2,16) \neq 8n+5,\ 8n+7,\ 16n+10,\ 4^k(16n+14)$. (This proof is contained essentially in some notes of L.E.Dickson).

f represents all $8n+1$, $8n+3$ for $g = x^2+2y^2+4Z^2 \equiv 1$ or $3 \pmod 8$ implies $Z = 2z$ and thus $g = f \equiv 1$ or $3 \pmod 8$ and reference to table II shows that g represents all $8n+1$ and $8n+3$.

$f = 2a$ implies x is even and $f/2 = 2X^2+y^2+8z^2$ which reference to table II shows represents all $\neq 8n+5, 4^k(8n+7)$ thus completing the proof, since f represents no $8n+5$, $8n+7$.

25. $f = (1,4,4) \neq 4n+2,\ 4n+3,\ 4^k(8n+7)$. (This proof is contained essentially in some notes of L. E. Dickson).

The only odds represented by f are of the form $4n+1$ and $g = x^2+y^2+4z^2 = f \equiv 1 \pmod 4$ since $g \equiv 1 \pmod 4$ implies x or y is even and thus f represents all $4n+1$ since g does from table I.

$f = 2a$ implies x is even and thus $f = 4F$ where $F = x^2+y^2+z^2$. Thus f represents no $4n+2$ and represents all multiples of $4 \neq 4^k(8n+7)$ and none of the form $4^k(8n+7)$.

26. $f=(1,4,6)\neq 16n+2,\ 9^k(9n+3)$.

$f=2a$ implies $x=2X$ and $f/2=2X^2+2y^2+3z^2$ which, from table II, represents exclusively all $\neq 8n+1,\ 9^k(9n+6)$.

$f=a$ an odd integer. Then consider $g=x^2+y^2+6z^2$ which represents exclusively all $\neq 9^k(9n+3)$ and $g\equiv 1 \pmod 2$ implies x or y even and thus $g=f\equiv 1 \pmod 2$.

27. $f=(1,4,8)\neq 4n+3,\ 4n+2,\ 4^k(16n+14)$. (This proof is contained essentially in some notes of L. E. Dickson).

$f=2a$ implies $x=2X$ and $f/4=X^2+y^2+2z^2$ which represents exclusively all $\neq 4^k(16n+14)$. Thus also $f\neq 4n+2$.

$f=a$ an odd integer implies $a\equiv 1 \pmod 4$ and $g=x^2+y^2+8z^2$ represents all $4n+1$, $g\equiv 1 \pmod 4$ implies x or y is even and thus $g=f\equiv 1 \pmod 4$ and f represents all $4n+1$.

28. $f=(1,4,12)\neq 4n+2,\ 4n+3,\ 9^k(9n+6)$.

$f=2a$ implies $x=2X$, $f/4=X^2+y^2+3z^2$ which represents exclusively all positive integers $\neq 9^k(9n+6)$. Thus also $f\neq 4n+2$.

$f=a$ an odd integer implies $a\equiv 1 \pmod 4$ and $g=x^2+4y^2+3z^2=a$ implies $z=2Z$, g represents all positive odd integers $\neq 9^k(9n+6)$ and thus $f=g\equiv 1 \pmod 4$ represents all $4n'+1\neq 9^k(9n+6)$.

29. $f=(1,4,16)\neq 4n+2,\ 4n+3,\ 16n+12,\ 4^k(8n+7)$.

$f=2a$ implies $x=2X$, $f/4=X^2+y^2+4z^2$ which represents exclusively all positive integers $\neq 4n+3$ nor $4^k(8n+7)$. Also then, $f\neq 4n+2$.

$f=a$ an odd integer implies $a\equiv 1 \pmod 4$. $g=x^2+y^2+16z^2$ represents all $4n+1$, $g\equiv 1 \pmod 4$ implies x or y even and thus

$f = g \equiv 1 \pmod 4$ and f represents all $4n+1$.

30. $f=(1,4,24) \neq 4n+2,\ 4n+3,\ 9^k(9n+3)$.

$f=2a$ implies $x=2X$, $f/4=X^2+y^2+6z^2$ which represents exclusively all positive integers $\neq 9^k(9n+3)$. Thus also $f \neq 4n+2$.

$f=a$ an odd integer implies $a \equiv 1 \pmod 4$. $g=x^2+4y^2+6z^2$ represents all a $(\equiv 1 \pmod 4))$ $\neq 9^k(9n+3)$. $g=a$ implies $z=2Z$ and thus $g=f\equiv 1 \pmod 4$ and f represents all $4n+1 \neq 9^k(9n'+3)$.

31. $f=(1,4,36) \neq 4n+2,\ 4n+3,\ 9n\pm 3,\ 4^k(8n+7)$.

$f=a\equiv 0 \pmod 3$ implies $x=3X$, $y=3Y$ and $f/9=X^2+4Y^2+4Z^2$ which represents exclusively all positive integers $\neq 4n+2$, $4n+3$, $4^k(8n+7)$. Thus also $f \neq 9n\pm 3$.

$f=a\equiv 0 \pmod 2$ implies $x=2X$ and $f/4=X^2+y^2+9z^2$ which represents exclusively all positive integers $\neq 9n\pm 3, 4^k(8n+7)$.

f represents no $4n+3$. It remains to prove

f represents all $a\equiv 1 \pmod 4$ prime to 3. Now f is equivalent to $x^2+4(y+3z)^2+36z^2 \equiv x^2+2y^2+2(6z+y)^2$. We prove that all $4n+1$ prime to 3 represented by $g=x^2+2y^2+2Z^2$, that is, all $12n+1$, $12n+5$ are represented by $x^2+2y^2+2(6z+y)^2$ and thus by f.

1) $a=12n+1$. $g\equiv 1 \pmod{12}$ implies $y\equiv Z \pmod 2$ and $y\equiv \pm Z \pmod 3$ where one of the signs holds and thus $6z+y=\pm Z$ is solvable for z.

2) $a=12n+5$. $g\equiv 5 \pmod{12}$ implies $y\equiv Z \pmod 2$ and, by interchanging y and Z if necessary, we may take $x\equiv \pm y \pmod 3$ where one of the signs holds and Z is prime to 3. Since g

represents all $12n+5$ we know that for any $a=12n+5$ there exists a Z such that $x^2+2y^2=a-2Z^2\equiv 0 \pmod 3$. Thus by theorem 9 (with $p=3$, $b=2$) we know that $a-2Z^2$ is represented by x^2+2y^2 with x and y prime to 3. Thus there exists a solution of $g=a$ for which $y\equiv \pm Z \pmod 3$. $6z+y=\pm Z$ is solvable for z, and the proof is complete.

43. $f=(2,3,8)\neq 8n\pm 1,\ 32n+4,\ 9^k(9n+6)$.

$f=a$ an odd integer implies $a\equiv \pm 3 \pmod 8$. $g=2x^2+2y^2+3z^2=a$ implies that either x or y is even and thus $g=f\equiv \pm 3 \pmod 8$ and f represents all such $\underline{a}\neq 9^k(9n+6)$ since g does.

$f=2a$ implies $y=2Y$, $f/2=x^2+6Y^2+4z^2$ which represents all and only those positive integers not of the form $16n+2$, $9^k(9n+3)$.

32. $f=(1,6,16)\neq 9^k(9n+3),\ 8n\pm 3,\ 16n\pm 2,\ 64n+8$.

$f=a$ an odd integer implies $a\equiv \pm 1 \pmod 8$. $g=x^2+6y^2+4z^2=a$ then implies $z=2Z$, $g=f\equiv \pm 1 \pmod 8$ and thus f represents all $8n\pm 1$ exclusively not of the form $9^k(9n+3)$.

$f=2a$ implies $x=2X$, $f/2=2X^2+3y^2+8z^2$ which from the proof preceeding represents exclusively all positive integers $\neq 8n\pm 1,\ 32n+4,\ 9^k(9n+6)$ thus giving the desired result for f.

33. $f=(1,8,8)\neq 8n+5,\ 4n+2,\ 4n+3,\ 4^k(8n+7)$. (This proof is contained essentially in some notes of L.E.Dickson).

$f=2a$ implies $x=2X$, $f/4=X^2+2y^2+2z^2$ which represents exclusively all positive integers $\neq 4^k(8n+7)$. Thus also $f\neq 4n+2$.

$f=a$ an odd integer implies $a\equiv 1 \pmod 8$. $g=x^2+2Y^2+2Z^2=a$ implies $Y=2y$, $Z=2z$ and thus $g=f\equiv 1 \pmod 8$ and f represents all $8n+1$, since g does.

34. $f=(1,8,16) \neq 8n+5,\ 4n+2,\ 4n+3,\ 4^k(16n+14)$. (This proof is contained essentially in some notes of L. E. Dickson).

$f=2a$ implies $x=2X$, $f/4=X^2+2y^2+4z^2$ which represents exclusively all positive integers $\neq 4^k(16n+14)$. Thus also $f \neq 4n+2$.

$f=a$ an odd integer implies $a\equiv 1 \pmod 8$. $g=x^2+8y^2+4z^2=a$ implies $z=2Z$ and thus $g=f\equiv 1 \pmod 8$ and f represents all $8n+1$ since g does.

35. $f=(1,8,24) \neq 4n+2,\ 4n+3,\ 4^k(8n+5)$.

f represents all $8n+1$ for consider $g=x^2+2y^2+6z^2=8n+1$ implies $y\equiv z \pmod 2$. Thus $y^2+3z^2\equiv 0 \pmod 4$ for $g=8n+1$ and the corollary to lemma b, ($d=2$, $b=3$, $p=2$) applies to prove that g represents $8n+1$ with y and z even (since g represents all $8n+1$) and thus f represents all $8n+1$.

f represents no $8n+5$, $4n+3$, $4n+2$.

$f=4a$ implies $x=2X$, $f/4=X^2+2y^2+6z^2$ which represents exclusively all positive integers not of the form $4^k(8n+5)$.

37. $f=(1,8,40) \neq 4n+3,\ 4n+2,\ 8n+5,\ 32n+28,\ 25^k(25n\pm 5)$.

$f=a$ an odd integer implies $a\equiv 1 \pmod 8$. $g=x^2+2y^2+10z^2=a$ implies $y=2Y$, $z=2Z$ and thus $f=g\equiv 1 \pmod 8$ represents all $8n+1$ not of the form $25^k(25n\pm 5)$.

$f=2a$ implies $x=2X$, $f/4=X^2+2y^2+10z^2$ which represents all positive integers exclusively not of the forms $8n+7$, $25^k(25n\pm 5)$.

39. $f=(1,16,16)\neq 4n+2,\ 4n+3,\ 8n+5,\ 16n+8,\ 16n+12,\ 4^k(8n+7)$.

$f=a$ an odd integer implies $a\equiv 1 \pmod 8$. $g=x^2+4y^2+16z^2=a$ implies $y=2Y$ and thus $f=g\equiv 1 \pmod 8$ represents all $8n+1$ since g does.

$f=2a$ implies $x=2X$, $f/4=X^2+4y^2+4z^2$ which represents exclusively all positive integers not of the forms $4n+2$, $4n+3$, $4^k(8n+7)$.

40. $f=(1,16,24)\neq 4n+2,\ 4n+3,\ 8n+5,\ 64n+8,\ 9^k(9n+3)$.

$f=a$ an odd integer implies $a\equiv 1 \pmod 8$. $g=x^2+4y^2+6z^2=a$ implies $y=2Y$, $z=2Z$ and thus $f=g\equiv 1 \pmod 8$ represents all $8n+1$ not of the form $9^k(9n+3)$ since g does.

$f=2a$ implies $x=2X$, $f/4=X^2+4y^2+6z^2$ which represents exclusively all positive integers not of the form $16n+2$, $9^k(9n+3)$.

41. $f=(1,16,48)\neq 4n+2,\ 4n+3,\ 8n+5,\ 16n+8,\ 16n+12,\ 9^k(9n+6)$.

$f=2a$ implies $x=2X$, $f/4=X^2+4y^2+12z^2$ which represents exclusively all positive integers not of the forms $4n+2$, $4n+3$, $9^k(9n+6)$.

f represents no $4n+2$, $4n+3$, $8n+5$ obviously. It remains to prove

f represents all $a=8n+1\neq 9^k(9n'+6)$. We know that for any such a there exists an x, y, z such that $g=x^2+4y^2+12z^2=a$. Now $g\equiv 1 \pmod 8$ implies $y\equiv z \pmod 2$. Thus $y^2+3z^2\equiv 0 \pmod 4$ for $g=8n+1$ and the corollary to lemma b, ($d=4$, $b=3$, $p=2$) applies to prove that g represents $8n+1$ with y and z even if $8n+1\neq 9^k(9n'+6)$ and thus f represents a.

43. See immediately preceeding the proof for form 32.

44. $f=(2,5,6)\neq 4^k(8n+1),\ 9^k(9n+3),\ 25^k(25n\pm 10)$.

Consider $g=x^2+3y^2+10z^2=2a\neq 9^k(9n+6),\ 25^k(25n\pm 20),\ 4^k(16n+2)$. Reference to table I shows that g represents all such 2a. Now $g=2a$ implies $x\equiv y \pmod 2$ and thus the corollary to lemma b applies ($d=1$, $b=3$, $p=2$) to prove that g represents 2a with $x=2X$, $y=2Y$ and thus $g/2=f$ represants all such <u>a</u>.

46. $f=(3,8,8)\neq 4n+1,\ 4n+2,\ 8n+7,\ 32n+4,\ 9^k(9n+6)$.

$f=2a$ implies $x=2X$, $f/4=3X^2+2y^2+2z^2$ which represents exclusively all positive integers $\neq 8n+1,\ 9^k(9n+6)$. Thus also $f\neq 4n+2$.

$f=a$ an odd integer implies $a\equiv 3 \pmod 8$. $g=3x^2+2y^2+2z^2\equiv 3 \pmod 8$ implies $y=2Y$, $z=2Z$ and thus $f=g\equiv 3 \pmod 8$ represents all such a not of the form $9^k(9n+6)$.

47. $f=(5,8,24)\neq 4^k(8n+1),\ 4n+2,\ 4n+3,\ 9^k(9n+3),\ 25^k(25n\pm 10)$.

f represents all $8n+5=a\neq 9^k(9n+3),\ 25^k(25n\pm 10)$ for $g=2x^2+6y^2+5z^2=a$ implies $x\equiv y \pmod 2$ and thus the corollary to lemma b applies to prove that, since g represents all such a, it represents a with x and y even (take $p=2$, $b=3$, $d=2$) and thus that f represents all such <u>a</u>.

f represents no $4n+2$, $4n+3$, $8n+1$ obviously.

$f=2a$ implies $x=2X$, $f/4=5X^2+2y^2+6z^2$ which represents exclusively all positive integers $\neq 4^k(8n+1),\ 9^k(9n+3),\ 25^k(25n\pm 10)$. Also $f\neq 4n+2$.

49. $f=(1,3,6)\neq 3n+2,\ 4^k(16n+14)$.

Every integer (positive) $a\neq 4^k(16n+14)$ nor $3n+2$ is represented by $g=x^2+y^2+2z^2$. For every such a, $f=a$ implies

$x^2 \equiv y^2 \equiv z^2 \pmod 3$ or $x^2 \not\equiv y^2 \pmod 3$. Thus, on account of the symmetry in x and y there exists a solution of f= a with $y \equiv \pm z \pmod 3$ where one of the signs holds. Then $3Y+z = \pm y$ is solvable and a is represented by $x^2+(3Y+z)^2+2z^2 = x^2+3(z+Y)^2+6Y^2$ which is equivalent to f and thus f represents all such a and none others.

50. $f=(1,3,9) \neq 9^k(9n+6)$, $3n+2$.

f= a prime to 3 implies $a \equiv 1 \pmod 3$ and $g=x^2+3y^2+z^2 \equiv 1 \pmod 3$ implies x or $z \equiv 0 \pmod 3$ and thus $f=g \equiv 1 \pmod 3$ represents all 3n+1 since g does.

f=3a implies x=3X, $f/3=3X^2+y^2+3z^2$ which represents exclusively all positive integers not of the form $9^k(3n+2)$.

52. $f=(1,3,18) \neq 3n+2$, $9n+6$, $4^k(16n+10)$.

f= a prime to 3 implies $a \equiv 1 \pmod 3$ and $g=x^2+3y^2+2z^2 \equiv 1 \pmod 3$ implies z=3Z and thus $f=g \equiv 1 \pmod 3$ represents all 3n+1 not of the form $4^k(16n+10)$ since g does.

f=3a implies x=3X, $f/3=3X^2+y^2+6z^2$ which represents all positive integers not of the forms 3n+2, $4^k(16n+14)$ and none others.

53. $f=(1,3,30) \neq 9^k(3n+2)$, $25^k(25n \pm 10)$, $4^k(16n+6)$.

Reference to table I shows that $g=x^2+3y^2+10z^2$ represents exclusively all $3a \neq 9^k(9n+6)$, $25^k(25n \pm 5)$, $4^k(16n+2)$. But g=3a implies x=3X, z=3Z and thus g/3= f represents all such a and none others.

55. $f=(1,6,6) \neq 8n+3$, $9^k(3n+2)$.

Reference to table II shows that $g=3x^2+2y^2+2z^2$

represents exclusively all $3a \neq 8n+1,\ 9^k(9n+6)$. But $g=3a$ implies $y=3Y$, $z=3Z$ and thus $g/3=f$ represents all such <u>a</u> and none others.

56.$f=(1,6,9) \neq 3n+2,\ 9^k(9n+3)$.

$f=a$ prime to 3 implies $a \equiv 1 \pmod 3$ and $g=x^2+6y^2+z^2=a$ implies x or $z \equiv 0 \pmod 3$ and thus $f=g \equiv 1 \pmod 3$ represents all $3n+1$ since g does.

$f=3a$ implies $x=3X$, $f/3=3X^2+2y^2+3z^2$ which represents exclusively all positive integers not of the form $9^k(3n+1)$.

69.$f=(2,3,6) \neq 3n+1,\ 4^k(8n+7)$.

Applying method 3 as for form 49 we see that every integer $a \neq 4^k(8n+7),\ 3n+1$ is represented by $g=x^2+2y^2+2z^2$ with $x \equiv \pm y \pmod 3$. Then $\pm x=3X+y$ is solvable for X and have a is represented by $6X^2+3(y+X)^2+2z^2$ which is equivalent to f.

(Note: this proof may also be made using the corollary to lemma b on the form $g'=x^2+3y^2+6z^2 \equiv 0 \pmod 2$).

57.$f=(1,6,18) \neq 3n+2,\ 9n+3,\ 4^k(8n+5)$.

$f=a$ prime to 3 implies $a \equiv 1 \pmod 3$ and $g=x^2+6y^2+2z^2 \equiv 1 \pmod 3$ implies $z=3Z$ and thus $f=g \equiv 1 \pmod 3$ represents all $3n+1$ not of the form $4^k(8n+5)$ since g does.

$f=3a$ implies $x=3X$, $f/3=3X^2+2y^2+6z^2$ which, from above, represents exclusively all positive integers not of the form $3n+1,\ 4^k(8n+7)$.

71.$f=(2,3,12) \neq 16n+6,\ 9^k(3n+1)$.

$f = a$ an odd integer. Consider $g = 2x^2 + 3y^2 + 3z^2 = a$. Then y or z is even and thus $f = g \equiv 1 \pmod 2$ and f represents all odds not of the form $9^k(3n+1)$ and none such since g does.

$f = 2a$ implies $y = 2Y$ and $f/2 = x^2 + 6Y^2 + 6z^2$ which represents exclusively all positive integers not of the forms $9^k(3n+2)$, $8n+3$.

58. $f = (1,6,24) \neq 8n \pm 3,\ 9^k(3n+2),\ 32n+12$.

$f = a$ an odd implies $a \equiv \pm 1 \pmod 8$. Now $g = x^2 + 6y^2 + 6z^2 \equiv 1$ or $7 \pmod 8$ implies y or z is even and thus $f = g \equiv \pm 1 \pmod 8$ represents all such a not of the form $9^k(3n+2)$ since g does.

$f = 2a$ implies $x = 2X$, $f/2 = 2X^2 + 3y^2 + 12z^2$ which, from above, represents exclusively all positive integers $\neq 16n+6$, $9^k(3n+1)$.

59. $f = (1,9,9) \neq 9n \pm 3,\ 3n+2,\ 4^k(8n+7)$.

$f = a$ prime to 3 implies $a \equiv 1 \pmod 3$ and $g = x^2 + y^2 + z^2 = a$ implies that two of x, y, z are $\equiv 0 \pmod 3$. Thus $f = g \equiv 1 \pmod 3$ represents all such a not of the form $4^k(8n+7)$ and none such.

$f = 3a$ implies $x = 3X$, $f/9 = X^2 + y^2 + z^2$ which represents exclusively all positive integers $\neq 4^k(8n+7)$. Thus also $f \neq 9n \pm 3$.

60. $f = (1,9,12) \neq 3n+2,\ 4n+3,\ 9^k(9n+6)$.

$f = a$ prime to 3 implies $a \equiv 1 \pmod 3$. $g = x^2 + y^2 + 12z^2$. $a \equiv 1 \pmod 3$ implies x or $y \equiv 0 \pmod 3$ and thus $f = g \equiv 1 \pmod 3$ and thus represents all such a not of the form $4n+3$.

f=3a implies x=3X, $f/3=3X^2+3y^2+4z^2$ which represents exclusively all positive integers not of the forms 4n+1, $9^k(3n+2)$.

77.f=(3,3,7)$\neq 4^k(8n+1)$, $9^k(3n+2)$, $49^k(49n+7e)$ where e=3,5 or 6.

Reference to table I shows that $g=x^2+y^2+21z^2$ represents all 3a$\neq 4^k(8n+3)$, $9^k(9n+6)$, $49^k(49n+21e)$ where e=3, 5 or 6. But g=3a implies x=3X, y=3Y. Thus g/3=f which therefore represents all such a and none others.

61.f=(1,9,21)$\neq$3n+2, $9^k(9n+6)$, $4^k(8n+3)$, $49^k(49n+7e)$ where e=1, 2 or 4.

f=a prime to 3 implies $a\equiv 1 \pmod 3$. Then $g=x^2+y^2+21z^2=a\equiv 1 \pmod 3$ implies x or $y\equiv 0 \pmod 3$ and thus $f=g\equiv 1 \pmod 3$ represents all 3n+1 not excluded above and none excluded.

f=3a implies x=3X, $f/3=3X^2+3y^2+7z^2$ which from above represents exclusively all positive integers not of the forms $9^k(3n+2)$, $4^k(8n+1)$, $49^k(49n+7r)$ where r=3, 5 or 6.

78.f=(3,3,8)$\neq$4n+1, 8n+2, $9^k(3n+1)$.

f=a an odd integer implies $a\equiv 3 \pmod 4$. Now $g=3x^2+3y^2+2z^2=a$ implies z=2Z and thus $f=g\equiv 3 \pmod 4$ represents all 4n+3 not of the form $9^k(3n+1)$.

f=2a implies $x\equiv y \pmod 2$ and applying method 2 we have $f/2=3X^2+3Y^2+4z^2$ which represents exclusively all positive integers not of the forms 4n+1, $9^k(3n+2)$.

62.f=(1,9,24)$\neq$3n+2, 4n+3, 8n+6, $9^k(9n+3)$.

f=a prime to 3 implies $a\equiv 1 \pmod 3$. Now $g=x^2+y^2+24z^2=a$ implies x or $y\equiv 0 \pmod 3$ and thus $f=g\equiv 1 \pmod 3$

represents all $3n+1$ not of the forms $4n+3$, $8n+6$ since g does.

$f=3a$ implies $x=3X$, $f/3=3X^2+3y^2+8z^2$ which, from the preceeding proof, represents exclusively all positive integers not of the forms $4n+1$, $8n+2$, $9^k(3n+1)$.

65. $f=(1,24,24)\neq 4n+3,\ 8n+5,\ 4n+2,\ 32n+12,\ 9^k(3n+2)$.

$f=a$ an odd implies $a\equiv 1 \pmod 8$. $g=x^2+6y^2+6z^2=a$ implies $y=2Y$, $z=2Z$ and thus $f=g\equiv 1 \pmod 8$ represents all <u>a</u> not of the form $9^k(3n+2)$.

$f=2a$ implies $x=2X$, $f/4=X^2+6y^2+6z^2$ which represents exclusively all positive integers not of the forms $8n+3$, $9^k(3n+2)$. Thus also f represents no $4n+2$.

82. $f=(3,8,24)\neq 3n+1,\ 4n+1,\ 4n+2,\ 4^k(8n+7)$.

$f=2a$ implies $x=2X$, $f/4=3X^2+2y^2+6z^2$ which represents exclusively all positive integers not of the forms $3n+1$, $4^k(8n+7)$.

$f\neq 4n+1,\ 4n+2,\ 3n+1,\ 8n+7$ obviously. It remains to prove

f represents all $a\equiv 3 \pmod 8$ not of the form $3n+1$. $g=3x^2+2y^2+6z^2=a$ implies $y\equiv z \pmod 2$ and thus the corollary to lemma b applies to prove that, since g represents all such <u>a</u>, it represents <u>a</u> with y and z even ($p=2$, $b=c$, $d=2$) and thus that f represents all such <u>a</u>.

66. $f=(1,24,72)\neq 3n+2,\ 9n+3,\ 4n+3,\ 4n+2,\ 4^k(8n+5)$.

$f=a$ prime to 3 implies $a\equiv 1 \pmod 3$. $g=x^2+24y^2+8z^2=a\equiv 1 \pmod 3$ implies $z=3Z$, and thus $f=g\equiv 1 \pmod 3$ represents all $3n+1$ not of the forms $4n+3$, $4n+2$, $4^k(8n+5)$ since g does.

$f=3a$ implies $x=3X$, $f/3=3X^2+8y^2+24z^2$ which, from the preceeding proof represents exclusively all positive integers not of the forms $3n+1$, $4n+1$, $4n+2$, $4^k(8n+7)$.

69. See immediately following the proof for form 56.

70. $f=(2,3,9)\neq 3n+1,\ 9n+6,\ 4^k(16n+10)$.

$f=a$ prime to 3 implies $a\equiv 2 \pmod 3$. $g=2x^2+3y^2+z^2=a$ then implies $z=3Z$ and thus $f=g\equiv 2 \pmod 3$ represents all $3n+2$ not of the form $4^k(16n+10)$ since g does.

$f=3a$ implies $x=3X$, $f/3=6X^2+y^2+3z^2$ which represents exclusively all positive integers not of the forms $3n+2$, $4^k(16n+14)$.

71. See immediately following form 57.

72. $f=(2,3,18)\neq 9^k(9n+6),\ 3n+1,\ 8n+1$.

$f=a$ prime to 3 implies $a\equiv 2 \pmod 3$. $g=2x^2+3y^2+2z^2=a$ then implies x or $z\equiv 0 \pmod 3$ and thus $f=g\equiv 1 \pmod 3$ represents all such <u>a</u> not of the form $8n+1$.

$f=3a$ implies $x=3X$, $f/3=6X^2+y^2+6z^2$ which represents exclusively all positive integers $\neq 9^k(3n+2)$, $8n+3$.

73. $f=(2,3,48)\neq 16n\pm 6,\ 8n\pm 1,\ 64n+24,\ 9^k(3n+1)$.

$f=a$ an odd integer implies $a\equiv \pm 3 \pmod 8$. $g=2x^2+3y^2+12z^2=a$ then implies $z=2Z$ and thus $f=g\equiv \pm 3 \pmod 8$ represents all such <u>a</u> not of the form $9^k(3n+1)$ since g does.

$f=2a$ implies $y=2Y$, $f/2=x^2+6Y^2+24z^2$ which represents exclusively all positive integers not of the form $8n\pm 3$,

$32n+12$, $9^k(3n+2)$.

74. $f=(2,6,9) \neq 3n+1$, $9n+3$, $4^k(8n+5)$.

$f=a$ prime to 3 implies $a \equiv 2 \pmod 3$. $g=2x^2+6y^2+z^2=a$ then implies $z=3Z$ and $f=g\equiv 2 \pmod 3$ thus represents all $3n+2$ not of the form $4^k(8n+5)$.

$f=3a$ implies $x=3X$, $f/3=6X^2+2y^2+3z^2$ which represents exclusively all positive integers not of the forms $3n+1$, $4^k(8n+7)$.

75. $f=(2,6,15) \neq 9^k(3n+1)$, $25^k(25n\pm5)$, $4^k(8n+3)$.

Reference to table II shows that $g=2x^2+5y^2+6z^2$ represents all $3a \neq 9^k(9n+3)$, $25^k(25n\pm15)$, $4^k(8n+9)$. But $g=3a$ implies $x=3X$, $y=3Y$ and thus $g/3=f$ which thus represents all such a and none others.

77. See immediately following proof for form 60.

78. See immediately following proof for form 61.

80. $f=(3,4,36) \neq 3n+2$, $9^k(9n+6)$, $4n+1$, $4n+2$.

$f=a$ prime to 3 implies $a \equiv 1 \pmod 3$. $g=3x^2+4y^2+4z^2=a$ implies y or $z \equiv 0 \pmod 3$ and thus $f=g\equiv 1 \pmod 3$ and f represents all $3n+1$ not of the form $4n+1$, $4n+2$.

$f=3a$ implies $y=3Y$, $f/3=x^2+12Y^2+12z^2$ which represents exclusively all positive integers not of the forms $4n+3$, $4n+2$, $9^k(3n+2)$.

81. $f=(3,8,12) \neq 4n+1$, $4n+2$, $9^k(3n+1)$.

$f=a$ an odd integer implies $a \equiv 3 \pmod 4$. $g=3x^2+8y^2+3z^2=a$ then implies x or $z \equiv 0 \pmod 2$ and thus $f=g\equiv 3 \pmod 4$ represents all such a not of the form $9^k(3n+1)$.

$f=2a$ implies $x=2X$, $f/4=3X^2+2y^2+3z^2$ which represents exclusively all positive integers not of the form $9^k(3n+1)$. Thus also $f\neq 4n+2$.

82. See immediately following the proof for form 65.

83. $f=(3,8,48)\neq 4n+1,\ 4n+2,\ 64n+24,\ 8n+7,\ 9^k(3n+1)$.

$f=a$ an odd integer implies $a\equiv 3 \pmod 8$. $g=3x^2+8y^2+12z^2=a$ then implies $z=2Z$ and $f=g\equiv 3 \pmod 8$ represents all such a not of the form $9^k(3n+1)$ since g does.

$f=2a$ implies $x=2X$, $f/4=3X^2+2y^2+12z^2$ which represents exclusively all positive integers $\neq 16n+6,\ 9^k(3n+1)$. Thus also $f\neq 4n+2$.

84. $f=(3,8,72)\neq 3n+1,\ 8n+7,\ 4n+1,\ 4n+2,\ 32n+4,\ 9^k(9n+6)$.

$f=a$ prime to 3 implies $a\equiv 2 \pmod 3$. $g=3x^2+8y^2+8z^2=a$ implies y or $z\equiv 0 \pmod 3$ and thus $f=g\equiv 2 \pmod 3$ represents all such a not of the forms excluded since g does.

$f=3a$ implies $y=3Y$, $f/3=x^2+24Y^2+24z^2$ which represents exclusively all positive integers $\neq 4n+2,\ 4n+3,\ 8n+5,\ 32n+12,\ 9^k(3n+2)$.

85. $f=(3,16,48)\neq 4n+1,\ 4n+2,\ 8n+7,\ 16n+4,\ 16n+8,\ 9^k(3n+2)$.

Reference to table II shows that $g=x^2+16y^2+48z^2$ represents exclusively all $3a\neq 4n+6,\ 4n+3,\ 9^k(9n+6),16n+24,\ 16n+12,\ 8n+21$. But $g=3a$ implies $x=3X$, $y=3Y$ and thus $g/3=f$ represents all such a and none others.

86. $f=(8,9,24)\neq 3n+1,\ 4n+3,\ 9n+3,\ 4n+2,\ 4^k(8n+5)$.

$f=a$ prime to 3 implies $a\equiv 2 \pmod 3$. $g=8x^2+y^2+24z^2=a$ then implies $y=3Y$, $f=g\equiv 2 \pmod 3$ thus represents all such a not of the forms excluded and none excluded.

$f=3a$ implies $x=3X$, $f/3=24X^2+3y^2+8z^2$ which represents exclusively all positive integers $\neq 4n+1$, $3n+1$, $4n+2$, $4^k(8n+7)$.

87. $f=(8,15,24)\neq 4n+1$, $4n+2$, $4^k(8n+3)$, $9^k(3n+1)$, $25^k(25n\pm 5)$.

Reference to table II shows that $g=24x^2+5y^2+8z^2$ represents exclusively all $3a\neq 4n+3$, $4n+6$, $4^k(8n+9)$, $9^k(9n+3)$, $25^k(25n\pm 15)$. But $g=3a$ implies $y=3Y$, $z=3Z$ and thus $g/3=f$ represents all such a and none others.

88. $f=(1,5,5)\neq 5n\pm 2$, $4^k(8n+7)$.

Applying method 3 as for form 49 we see that every integer $a\neq 4^k(8n+7)$, $5n\pm 2$ is represented by $g=x^2+y^2+z^2$ with $y^2+z^2\equiv 0 \pmod 5$ i.e. with $y\equiv \pm 2z \pmod 5$ where one of the signs holds. Then $5Y+2z=\pm y$ is solvable for Y and a is represented by $x^2+(5Y+2z)^2+z^2=x^2+5Y^2+5(z+2Y)^2$ which is equivalent to f.

89. $f=(1,5,10)\neq 25^k(5n\pm 2)$.

Reference to table I shows that $g=x^2+2y^2+5z^2$ represents all $5a\neq 25^k(25n\pm 10)$. But $g=5a$ implies $x=5X$, $y=5Y$ and thus $g/5=f$ represents all $a\neq 25^k(5n\pm 2)$ and none others.

90. $f=(1,5,25)\neq 5n\pm 2$, $25n\pm 10$, $4^k(8n+3)$.

$f=a$ prime to 5 implies $a\equiv \pm 1 \pmod 5$. $g=x^2+5y^2+z^2=a$ then implies x or $z\equiv 0 \pmod 5$ and thus $f=g\equiv \pm 1 \pmod 5$ represents all such a not of the form $4^k(8n+3)$ since g does.

$f=5a$ implies $x=5X$, $f/5=5X^2+y^2+5z^2$ which represents exclusively all positive integers $\neq 5n\pm 2$, $4^k(8n+7)$.

91. $f=(1,5,40)\neq 4n+3,\ 8n+2,\ 25^k(5n\pm 2)$.

Reference to table I shows that $g=x^2+5y^2+8z^2$ represents exclusively all $5a\neq 8n+10,\ 4n+15,\ 25^k(25n\pm 10)$. But $g=5a$ implies $x=5X$, $z=5Z$ and thus $g/5=f$ represents all such <u>a</u> and none others.

92. $f=(1,5,200)\neq 5n\pm 2,\ 4n+3,\ 8n+2,\ 25^k(25n\pm 10)$.

Some mistake here

$f=a$ prime to 5 implies $a\equiv \pm 1 \pmod 5$. $g=x^2+5y^2+8z^2=a$ then implies $z=5Z$ and $f=g\equiv \pm 1 \pmod 5$ represents all such <u>a</u> not of the forms $4n+3$, $8n+2$ since g does.

$f=5a$ implies $x=5X$ and $f/5=5X^2+y^2+40z^2$ which represents exclusively all positive integers not of the forms $4n+3,\ 8n+2,\ 25^k(5n\pm 2)$.

93. $f=(1,10,30)\neq 9^k(9n+6),\ 25^k(5n\pm 2),\ 4^k(8n+5)$.

Reference to table II shows that $g=2x^2+5y^2+6z^2$ represents exclusively all $5a\neq 9^k(9n+30),\ 25^k(25n\pm 10),\ 4^k(8n+25)$. But $g=5a$ implies $x=5X$, $z=5Z$ and $g/5=f$ represents all such <u>a</u> and none others.

94. $f=(1,21,21)\neq 9^k(3n+2),\ 4^k(8n+7),\ 49^k(7n+e)$ where $e=3,5$, or 6.

Reference to table I shows that $g=x^2+y^2+21z^2$ represents exclusively all $21a\neq 4^k(8n+147),\ 9^k(9n+42), 49^k(49n+21e)$ where $e=3$, 5 or 6. But $g=21a$ implies $x=21X$, $y=21Y$ and thus $g/21=f$ represents all such <u>a</u> and none others.

95. $f=(1,40,120)\neq 4n+2,\ 4n+3,\ 4^k(8n+5),\ 9^k(9n+6),\ 25^k(5n\pm 2)$.

Reference to table II shows that $g=5x^2+8y^2+24z^2$

represents exclusively all $5a \neq 4n+10$, $4n+15$, $4^k(8n+25)$, $9^k(9n+30)$, $25^k(25n\pm10)$. But $g=5a$ implies $y=5Y$, $z=5Z$ and $g/5=f$ represents all such $\underline{a}$ and none others.

96.$f=(2,5,10) \neq 8n+3$, $25^k(5n\pm1)$.

Reference to table II shows that $g=x^2+2y^2+10z^2$ represents exclusively all $5a \neq 8n+15$, $25^k(25n\pm5)$. But $g=5a$ implies $x=5X$, $y=5Y$ and $g/5=f$ represents all such $\underline{a}$ and none others.

97.$f=(2,5,15) \neq 9^k(9n+3)$, $25^k(5n\pm1)$, $4^k(16n+10)$.

Reference to table I shows that $g=x^2+3y^2+10z^2$ represents exclusively all $5a \neq 9^k(9n+15)$, $25^k(25n\pm5)$, $4^k(16n+50)$. But $g=5a$ implies $x=5X$, $y=5Y$ and thus $g/5=f$ represents all such $\underline{a}$ and none others.

98.$f=(3,7,7) \neq 9^k(9n+6)$, $49^k(7n+e)$, $4^k(8n+5)$ where $e=1,2$ or 4.

Reference to table I shows that $g=x^2+y^2+21z^2$ represents exclusively all $7a \neq 4^k(8n+35)$, $9^k(9n+42)$, $49^k(49n+7e)$ where $e=1$, 2 or 4. But $g=7a$ implies $x=7X$, $y=7Y$ and $g/7=f$ represents all such $\underline{a}$ and none others.

99.$f=(3,7,63) \neq 3n+2$, $9^k(9n+6)$, $4^k(8n+5)$, $49^k(7n+e)$ where $e=1$, 2 or 4.

Reference to table III shows that $g=x^2+9y^2+21z^2$ represents exclusively all $7a \neq 3n+14$, $9^k(9n+42)$, $4^k(8n+35)$, $49^k(49n+7e)$ where $e=1,2$ or 4. But $g=7a$ implies $x=7X$, $y=7Y$ and $g/7=f$ represents all such $\underline{a}$ and none others.

100.$f=(3,10,30) \neq 9^k(3n+2)$, $25^k(5n\pm1)$, $4^k(8n+7)$.

Reference to table IV shows that $g=x^2+10y^2+30z^2$

represents exclusively all $3a \neq 9^k(9n+6)$, $25^k(5n\pm3)$, $4^k(8n+21)$. But $g=3a$ implies $x=3X$, $y=3Y$ and $g/3=f$ represents all such a and none others.

101. $f=(3,40,120) \neq 4n+2$, $4n+1$, $4^k(8n+7)$, $9^k(3n+2)$, $25^k(5n\pm1)$.

Reference to table IV shows that $g=x^2+40y^2+120z^2$ represents exclusively all $3a \neq 4n+6$, $4n+3$, $4^k(8n+21)$, $9^k(9n+6)$, $25^k(5n\pm3)$. But $g=3a$ implies $x=3X$, $y=3Y$ and $g/3=f$ represents all such a and none others.

102. $f=(5,6,15) \neq 9^k(3n+1)$, $25^k(5n\pm2)$, $4^k(16n+14)$.

Reference to table III shows that $g=x^2+3y^2+30z^2$ represents exclusively all $5a \neq 9^k(3n+5)$, $25^k(25n\pm10)$, $4^k(16n+70)$. But $g=5a$ implies $x=5X$, $y=5Y$ and $g/5=f$ represents all such a and none others.

103. $f=(5,8,40) \neq 4n+2$, $4n+3$, $8n+1$, $32n+12$, $25^k(5n\pm1)$.

Reference to table II shows that $g=x^2+8y^2+40z^2$ represents exclusively all $5a \neq 4n+15$, $4n+10$, $8n+5$, $32n+60$, $25^k(25n\pm5)$. But $g=5a$ implies $x=5X$, $y=5Y$ and $g/5=f$ represents all such a and none others.

III. Regular reduced positive forms $f=ax^2+by^2+cz^2+ryz+sxz+txy$ i.e. (a,b,c,r,s,t) of Hessian ≤ 20.

(Regular forms completely dealt with in the references given in Table V are considered below only when a simpler proof has been found. Also, since the proofs are similar to those in the preceding paragraph, only the essential details are given below).

104. $f=(1,2,2,-2,0,0)^1 \neq 4^k(8n+5)$. $(H=3)$.

Using method 1 we see that all and only the 3a represented by (1,1,1) are represented by $g=x^2+(3Y+x)^2+(3Z+x)^2$ and $g/3=(x+Y+Z)^2+2Y^2+2Z^2-2YZ$ is equivalent to f.

105. $f=(1,1,1,1,1,1)\neq 4^k(16n+14)$. $(H=4/8)$.

We know $g=3x^2+y^2+8z^2$ represents all multiples of $4\neq 4^k(16n+10)$, since $3x^2+y^2+2Z'^2\equiv 0 \pmod 4$ implies $Z'=2z$. $g=12a$ implies $3Y+z=\pm y$ is solvable for Y if the proper sign is taken, and $2X+Y+z=x$ is solvable for X since $x+y\equiv 0 \pmod 2$ and thus all $12a\neq 4^k(16n+10)$ are represented by $g'=3(2X+Y+z)^2+(3Y+z)^2+8z^2$ and $g'/12=f$. This is an application of Method 1.

106. $f=(1,2,3,-2,0,0)\neq 25^k(25n\pm 5)$. $(H=5)$.

Apply method 2 to prove $g=y^2+2x^2+5z^2=(2Y-z)^2+2x^2+5z^2\equiv 0 \pmod 2$ and $g/2=f$.

107. $f=(1,1,1,0,0,-1)\neq 9^k(9n+6)$. $(H=6/8)$.

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1 i.e. f represents exclusively all positive integers not of the form $4^k(8n+5)$.

Apply method 2 to prove $g=(1,3,4)=(2X-y)^2+3y^2+4z^2\equiv 0 \pmod 2$ and $g/4=f$.

108. $f=(2,2,3,2,2,2)\neq 4^k(8n+1)$. $(H=7)$.

Using method 1 we note that every $7a\neq 4^k(8n+7)$ is represented by $g=x^2+y^2+z^2$. Now $f=7a$ implies $x\equiv y\equiv z\equiv 0 \pmod 7$ or $x^2\not\equiv y^2\not\equiv z^2\not\equiv x^2 \pmod 7$ for suppose $y^2\equiv z^2 \pmod 7$; then $f=7a$ implies $x^2\equiv 5y^2 \pmod 7$ which is impossible unless $x\equiv y\equiv 0 \pmod 7$. Therefore there exists a solution x,y,z such that $x^2\equiv 2y^2\equiv 4z^2 \pmod 7$ and thus $x\equiv \pm 4y \pmod 7$ (where one of the signs holds) and $x\equiv 2bz \pmod 7$ where b is $+1$ or -1. Then $\pm y=7Y+2x, -bz=7Z+3x$ are solvable for Y and Z and $7a$ is represented by $g'=x^2+(7Y+2x)^2+(7Z+3x)^2$ and $g'/7$ of Hessian 7 represents exclusively all $a\neq 4^k(8n+1)$. But f is the only reduced form of Hessian 7 and minimum 2 and thus f is equivalent to $g'/7$.

109. $f=(1,3,3,-2,0,0)\neq 4n+2,\ 4^k(16n+14)$. $(H=8)$.

f represents all $4a\neq 4^k(16n+14)$ for, from the proof for form 105, we know $g=3x^2+y^2+8z^2$ represents all $12a\neq 4^k(16n+10)$ and for $g=12a$, $3Y-z=\pm y$ is solvable (for one of the signs) and thus all such $12a$ are represented by $3x^2+(3Y-z)^2+8z^2$ and thus $g/3=f$.

It remains to prove that f represents all odds. The other reduced forms of Hessian 8 are $h=(1,1,8)$, $h'=(1,2,4)$ and $g'=(2,2,3,-2,-2,0)$ all of which represent 2. For every odd $\underline{a}$ we prove the existence of a form $h''=ax^2+by^2+4cz^2+4ryz+4xz$ equivalent to f. If $a+b\equiv 0 \pmod 4$, h" represents no 4n+2 and thus is not equivalent to h, h' or g'. Thus h" will be equivalent to f if we can choose integers $b \equiv -a \pmod 4$, c, and r such that

$$H=8=a(4bc-4r^2)-4b.$$

That is, $b=at-2$ where $t=bc-r^2$.

Let $t=8k+v$ and have $b=8ak+av-2$ where, for any a and odd v, k may be chosen so that b is a prime.

1) If $a\equiv 3 \pmod 4$ take $v=1$. Then $b\equiv 1 \pmod 4$ and $\left(\frac{-t}{b}\right)=\left(\frac{b}{t}\right)=\left(\frac{-2}{t}\right)=1$.

2) If $a\equiv 1 \pmod 4$ take $v=5$. Then $b\equiv 3 \pmod 4$ and $\left(\frac{-t}{b}\right)=-\left(\frac{b}{t}\right)=-\left(\frac{-2}{t}\right)=1$.

Thus in both cases an r exists such that $r^2+t\equiv 0 \pmod b$ and thus $(r^2+t)/b=c$ is an integer and h" exists equivalent to f. Thus f represents all odd a. Furthermore since h" represents no 4n+2, f represents no 4n+2.

110. $f=(2,2,3,-2,-2,0)\neq 4n+1,\ 16n+6,\ 4^k(16n+14)$. $(H=8)$.

$2f=(2x-z)^2+(2y-z)^2+4z^2 \not\equiv 2 \pmod 8$ and thus f represents no 4n+1.

f represents all 4n+3 since $X^2+Y^2+4z^2\equiv 6 \pmod 8$ implies $2x-z=X$, $2y-z=Y$ are solvable for x and y.

f represents all $2a\neq 16n+6,\ 4^k(16n+14)$ for $f=2a$ implies $z=2Z$ and $f/2=(x-Z)^2+(y-Z)^2+4Z^2$ which is equivalent to (1,1,4).

111. $f=(1,2,5,-2,0,0)\neq 4^k(8n+7)$. $(H=9)$.

By Dirichlet's proof for the form $g=x^2+y^2+z^2$ for every 9a not of the form $4^k(8n+7)$ nor divisible by 4 there is a proper representation of 9a by g, i.e. 9a is represented by g with no factor common to all three of x, y and z.

Then since $g=4g$, for every $9a \neq 4^k(8n+7)$, there exist integers x, y and z having in common no prime factor greater than 2 such that $x^2+y^2+z^2=9a$. Now $g=9a$ and any one of x, y, $z \equiv 0 \pmod 3$ implies $x \equiv y \equiv z \equiv 0 \pmod 3$ and thus there exist x, y, z all prime to 3 such that $x^2+y^2+z^2=a$ and making use of the symmetry of g in x, y and z we have further that there exists a solution x, y, z of $g=9a$ for which x, y, z are prime to 3 and $4x^2 \equiv y^2 \equiv z^2 \pmod 9$, i.e. such that $2x \equiv \pm y \pmod 9$ (where one of the signs holds) and $2bx \equiv z \pmod 9$ where b is $+1$ or -1 since $2x \equiv -y \pmod 3$ and $2x \equiv y \pmod 3$ implies $y \equiv 0 \pmod 3$ and similarly for $2bx \equiv z \pmod 9$. Then $\pm y = 9Y+2x$ and $bz=9Z+2x$ are solvable for Y and Z and $9a$ is represented by $g'=x^2+(9Y+2x)^2+(9Z+2x)^2$ and thus $g'/9$ of Hessian 9 represents exclusively all positive integers $\neq 4^k(8n+7)$. The only reduced forms of Hessian 9 and minimum 1 are f and $h=(1,1,9)$ and $h'=(1,3,3)$. Now h does not represent 3, h' does not represent 2 both of which are represented by $g'/9$ which also represents 1 and thus $g'/9$ is equivalent to f.

112.$f=(2,2,3,0,0,-2) \neq 3n+1$, $4^k(8n+7)$. $(H=9)$.

Apply method 2 to prove $g=(1,3,6)=(2X-y)^2+3y^2+6z^2 \equiv 0 \pmod 2$ and $g/2=f$.

128.$f=(1,3,7,-2,0,0) \neq 4n+2$, $25^k(25n\pm5)$. $(H=20)$.

The only other reduced positive ternary quadratic forms of Hessian 20 representing an odd are: $g_1=(1,1,20)$, $g_2=(1,2,10)$, $g_3=(2,2,5)$, $g_4=(2,3,4,0,0,-2)$ all of which represent 2; $g_5=(1,4,5)$ and $g_6=(1,4,6,-4,0,0)$ which represent 6; $g_7=(3,3,3,2,2,2)$ represents no $4n+1$ for $g=$ an odd implies that one of x, y, z is odd and the other two both odd or both even. From symmetry take x odd, $y+z=2Y$, $y-z=2Z$ which are solvable for Y and Z and g_7 becomes $3x^2+8Y^2+4Z^2+4xY\not\equiv 1 \pmod 4$.

Thus a form of Hessian 20 representing no $4n+2$ and representing a positive integer $\equiv 1 \pmod 4$ cannot be equivalent to g_i $(i=1,..,7)$ and thus must be equivalent to f if it represents an odd.

I. For every odd $\underline{a} \neq 25^k(25n\pm 5)$ and not divisible by 25, there exists a form $h=ax^2+by^2+4cz^2+4ryz+4sxz$ equivalent to f. h represents no $4n+2$ if we choose b so that $a+b\equiv 0 \pmod 4$ and since by this choice either b or a is $\equiv 1 \pmod 4$ we will have proved our statement if we can prove the existence of integers $b\equiv -a \pmod 4$, c, r and s such that

$$H=20=a(4bc-4r^2)-4s^2b;\ \text{that is}$$

$$s^2b=at-5 \text{ where } t=bc-r^2.$$

A. If a is prime to 5 take $s=1$.

1) If $a\equiv 1 \pmod 4$ take $t=4T$, $T=20k-3$. Then $b=80ak-12a-5\equiv 3 \pmod 4$ and, choosing k so that b is a prime we have

$\left(\frac{-t}{b}\right)=\left(\frac{-T}{b}\right)=-\left(\frac{b}{T}\right)=-\left(\frac{-5}{T}\right)=-\left(\frac{T}{5}\right)=1.$

2) If $a\equiv 3 \pmod 4$ take $t=2T$, $T=20k+11$. Then $b=40ak+22a-5\equiv 5 \pmod 8$ and, choosing k so that b is a prime we have

$\left(\frac{-t}{b}\right)=\left(\frac{-2T}{b}\right)=-\left(\frac{T}{b}\right)=-\left(\frac{-b}{T}\right)=-\left(\frac{-5}{T}\right)\left(\frac{T}{5}\right)=1.$

B. If $a=5a'$ where $a'\equiv \pm 2 \pmod 5$. Take $s=5$ and have $5b=a't-1$.

1) If $a\equiv 1 \pmod 4$ take $t=4T$, $T=100k+v$ where v is chosen

$\equiv 1 \pmod 4$ and so that $4va' \equiv 21 \pmod{25}$. Then $b = 80a'k + (4a'v-1)/5 \equiv 3 \pmod 4$ is prime to 5. Choosing k so that b is a prime and noting that $4va' \equiv 21 \pmod{25}$ implies $\left(\frac{v}{5}\right) = \left(\frac{T}{5}\right) = -1$ we have $\left(\frac{-T}{b}\right) = \left(\frac{-T}{b}\right) = -\left(\frac{b}{T}\right) = \left(\frac{5b}{T}\right) = 1$.

2) If $a \equiv 3 \pmod 4$ take $t = 2T$, $T = 100k + v$ where v is chosen $\equiv 1 \pmod 4$ and so that $2va' \equiv 6 \pmod{25}$. Then $b = 40a'k + (2a'v-1)/5 \equiv 1 \pmod 8$ is an integer prime to 5. Choosing k so that b is a prime and noting that $2va' \equiv 6 \pmod{25}$ implies $\left(\frac{v}{5}\right) = \left(\frac{T}{5}\right) = 1$ we have $\left(\frac{-2T}{b}\right) = \left(\frac{T}{b}\right) = \left(\frac{b}{T}\right) = \left(\frac{5b}{T}\right) = 1$.

Thus in cases A and B an r exists such that $(t+r^2)/b$ is an integer c.

II. For every $4a \neq 25n \pm 5$ and not divisible by 25 there exists a form $h = 4ax^2 + 4by^2 + cz^2 + 4ryz + 2sxz$ equivalent to f. h represents no $4n+2$ if c is odd for $h \equiv 0 \pmod 2$ implies $z = 2Z$ and $h \equiv 0 \pmod 4$. Since for c odd either $c + 2s$ or c is $\equiv 1 \pmod 4$ if s odd and a $4n+1$ is thus represented by h we will have proved our statement if we can prove the existence of integers b, r, an odd c, and an odd s such that

$$H = 20 = 4a(4bc - 4r^2) - 4bs^2; \quad \text{that is}$$

$$bs^2 = 4at - 5 \text{ where } t = bc - r^2. \quad \text{Take } r = 2r' \text{ and}$$

have $t = bc - 4r'^2$ and t and b odd will insure us that c is odd if it exists.

A. If a is prime to 5 take $s = 1$, $t = 20k + 3$ and have $b = 80ak + 12a - 5$. Choosing k so that b is a prime we have $\left(\frac{-t}{b}\right) = \left(\frac{b}{t}\right) = \left(\frac{-5}{t}\right) = -\left(\frac{t}{5}\right) = 1$.

B. If $a=5a'$ where $a'\equiv\pm 2 \pmod 5$, take $s=5$, $t=100k+v$ where v is chosen $\equiv 3 \pmod 4$ so that $4a'v\equiv 16 \pmod{25}$. Then $b=80a'k+(4a'v-1)/5$ is prime to 5 and k may be chosen so that b is a prime. Noting that $4a'v\equiv 16 \pmod{25}$ implies $\left(\frac{v}{5}\right)=\left(\frac{t}{5}\right)=-1$ we have $\left(\frac{-t}{b}\right)=\left(\frac{b}{t}\right)=-\left(\frac{5b}{t}\right)=-\left(\frac{-1}{t}\right)=1$.

Thus in cases A and B there exists an r'' such that $t+r''^2\equiv 0 \pmod b$ and since either r'' or $r''+b$ is even we know there exists an r' such that $t+4r'^2\equiv 0 \pmod b$ and an integer $c=(t+4r'^2)/b$ exists.

III. Thus we have proved that for any $\underline{a}$ odd or $\equiv 0 \pmod 4$ and not of the form $25n\pm 5$ nor divisible by 25 there is a form h with leading coefficient $\underline{a}$ equivalent to f and representing no $4n+2$. Thus f represents all such $\underline{a}$ and no $4n+2$. Now $3f=3x^2+(3y-z)^2+20z^2\equiv 0 \pmod 5$ implies $x=5X$ and $3y-z=5Y$ and $3f/5=15X^2+5Y^2+4z^2\equiv 0 \pmod 5$ implies $z\equiv 0 \pmod 5$ and thus $y\equiv 0 \pmod 5$ we have $f\equiv 0 \pmod{25}$ implies $x\equiv y\equiv z\equiv 0 \pmod 5$ and $f=25f$ thus proving that f represents exclusively all positive integers not of the forms $4n+2$, $25^k(25n\pm 5)$.

113. $f=(1,1,2,1,1,1)\neq 25^k(25n\pm 5)$. $(H=10/8)$.

Consider g = form 128. Then from above $3g=3x^2+(3y-z)^2+20z^2\equiv 0 \pmod 4$ implies $x+3y-z\equiv x+y-z\equiv 0 \pmod 2$ and thus $x+y-z=-2X$, $z=-Z$ are solvable for X and Z. Thus $g=(2X+y+Z)^2+3y^2+7Z^2+2yZ=4f$.

114. $f=(1,2,6,-2,0,0)\neq 4^k(8n+5)$. $(H=11)$.

As in the discussion for form 111, $11a\neq 4^k(8n+7)$

is represented by $g=x^2+y^2+z^2$ with x, y, z prime to 11 for $g=11a$ and $z\equiv 0 \pmod{11}$ implies $x\equiv y\equiv 0 \pmod{11}$. For such a solution (x, y, z prime to 11) there exists a b such that $(bx)^2\equiv 1 \pmod{11}$. Then $f=11a$ implies $(by)^2+(bz)^2\equiv 10 \pmod{11}$ and thus $(by)^2\equiv 5\equiv(bz)^2 \pmod{11}$ or $(bz)^2\equiv 1 \pmod{11}$ and $(by)^2\equiv 9 \pmod{11}$ or $(by)^2\equiv 1 \pmod{11}$ and $(bz)^2\equiv 9 \pmod{11}$. Thus, making use of the symmetry of g in x, y and z there exists a solution x, y, z of $g=11a$ (x, y, z prime to 11) such that $5x^2\equiv y^2\equiv z^2 \pmod{11}$ or such that $x^2\equiv y^2 \pmod{11}$ and $z^2\equiv 9x^2 \pmod{11}$. But $z^2\equiv 9x^2 \pmod{11}$ implies $5z^2\equiv x^2 \pmod{11}$ and, thus on account of symmetry, the second case is included in the first and we have that there exists a solution of $g=11a$ with x, y, z prime to 11 such that $x^2\equiv y^2\equiv 5z^2 \pmod{11}$ i.e. $x\equiv \pm y \pmod{11}$ (where one of the signs holds) and $x\equiv 4cz \pmod{11}$ where c is +1 or -1. (Note that this is also true of $f=11a$ with $x\equiv y\equiv z\equiv 0 \pmod{11}$). Then $\pm y=11Y+x$, $cz=11Z+3x$ are solvable for Y and Z and 11a is represented by $x^2+(11Y+x)^2+(11Z+3x)^2=g'$ and thus $g'/11$ represents exclusively all positive integers not of the form $4^k(8n+5)$, and is of Hessian 11. The only reduced positive ternary quadratic forms of Hessian 11 are f, $g_1=(1,1,11)$ which represents no $11(11n+e)$ where $\left(\frac{e}{11}\right)=-1$ and $g_2=(1,3,4,-2,0,0)$ which represents no $11(11n+e)$ since $3g_2 = 3x^2+(3y-z)^2+11z^2$. Thus neither g_1 nor g_2 represents 22 which is represented by $g'/11$ thus proving that $g'/11$ is equivalent to f.

115. $f=(1,4,4,-4,0,0)\neq 4n+2,\ 4n+3,\ 9^k(9n+6)$. $(H=12)$.

$g=x^2+y^2+3z^2=a\equiv 0$ or $1 \pmod 4$ implies $x+z\equiv 0 \pmod 2$ or $y+z\equiv 0 \pmod 2$. From symmetry take $y+z\equiv 0 \pmod 2$ and have $2Y-z=y$ is solvable proving $g=f\equiv 0$ or $1 \pmod 4$.

116. $f=(2,3,3,2,2,2)\neq 8n+1,\ 4^k(8n+5)$. $(H=12)$.

Apply method 1 as for form 104 for $3a$ represented by $g=x^2+y^2+4z^2$ and find that $3a$ is represented by $4x^2+(3Y+x)^2+(3Z+x)^2=g'$ and $g'=3f'$ where f' represents exclusively all positive integers not of the forms $8n+1$, $4^k(8n+5)$. The only reduced forms of Hessian 12, minimum 2 and representing an odd are f and $g''=(2,2,3)$ which does not represent 6 which is represented by f'. Thus f' is equivalent to f.

117. $f=(1,1,2,-1,-1,0)\neq 9^k(9n+3)$. $(H=12/8)$.

$g=x^2+y^2+6z^2\equiv 0 \pmod 4$ implies $x\equiv y\equiv z \pmod 2$ and thus $x=2X-z$, $y=2Y-z$ are solvable for X and Y and $g/4=f$.

118. $f=(1,1,2,0,0,-1)\neq 4^k(16n+10)$. $(H=12/8)$.

From the proof for form 105, $g=x^2+3y^2+8z^2$ represents all multiples of $4\neq 4^k(16n+10)$. $g=4a$ implies $2X-y=x$ is solvable for X and $g/4=f$.

119. $f=(1,3,5,-2,0,0)\neq 4^k(16n+2)$. $(H=14)$.

Use method 1 as for form 108 with $7a\neq 4^k(16n+14)$ represented by $g=2x^2+y^2+z^2$ and find that there exists a solution $g=7a$ with $x\equiv \pm y \pmod 7$ (where one of the signs holds) and $z\equiv 2bx \pmod 7$ where b is $+1$ or -1. Thus $\pm y=7Y+x$ and $bz=7Z+2x$ are solvable for Y and Z, and $7a$ is represented by $g'=2x^2+(7Y+x)^2+(7Z+2x)^2$ and $g'/7$ represents exclusively all positive integers not of the form $4^k(16n+2)$. The reduced forms of Hessian 14 and minimum 1 are f, $h=(1,1,14)$ and $h'=(1,2,7)$. But h and h' represent 2 which $g'/7$ does not represent and thus f is equivalent to $g'/7$.

121. $f=(2,2,5,0,0,-2)\neq 9^k(9n+3),\ 25^k(25n\pm 10),\ 4^k(8n+1)$. (H=15).

$g=x^2+3y^2+10z^2=2a$ implies $x=2X-y$ is solvable and $g/2=f$.

122. $f=(2,3,3,0,0,-2)\neq 4^k(8n+1)$. (H=15).

Apply method 1 as for form 13 with $3a\neq 4^k(8n+3)$ represented by $g=5x^2+y^2+z^2$ and find there exists a solution for which $z=3Z$, $\pm y=3Y-x$ are solvable for Y and Z for one of the signs, and thus 3a is represented by $g'=5x^2+(3Y-x)^2+9z^2$ and $g'/3=f$.

123. $f=(1,4,5,-4,0,0)\neq 8n+2,\ 8n+3,\ 32n+12,\ 4^k(8n+7)$. (H=16).

$f=x^2+(2y-z)^2+4z^2$ obviously represents no $4n+3, 8n+2$.

f represents all $a\equiv 6 \pmod 8$ for $g=X^2+Y^2+4z^2=a$ implies $2y-z=Y$ is solvable for y and $g=f\equiv 6 \pmod 8$.

$f=4a$ implies $x=2X$, $z=2Z$ and $f/4=X^2+(y-Z)^2+4Z^2$. It remains to prove

f represents all $a\equiv 1 \pmod 4$. $g=a$ implies $X\not\equiv Y \pmod 2$. If $g=a$ with z odd permute X and Y if necessary and take Y odd having $2y-z=Y$ solvable for y. If $g=a$ with z even, permute X and Y if necessary and take Y even having $2y-z=Y$ solvable for y. Thus in any case $2y-z=Y$ is solvable for y and $g=f\equiv 1 \pmod 4$.

124. $f=(2,3,3,-2,0,0) \neq 8n+1,\ 4^k(8n+7)$. $(H=16)$.

The only other reduced positive ternary quadratic forms of Hessian 16 and minimum greater than 1 are $g=(2,2,5,-2,-2,0)$ which represents no $4n+3$ since $2g=(2x-z)^2+(2y-z)^2+8z^2 \not\equiv 6 \pmod 8$ and $g'=(3,3,3,-2,-2,-2)$ which represents no $4n+2$ by the same reasoning applied to g_2 in the proof for form 17, and a form g'' which represents no odds. We seek to prove first

f represents all $a=8n\pm3$. For such an $\underline{a}$ we construct $h=ax^2+2by^2+8cz^2+8ryz+8xz$ with $b\equiv\pm1 \pmod 4$. Now h represents no $8n\pm1$, it represents a $4n+3$ since either a or $2b+a$ is $\equiv 3 \pmod 4$ represents $\underline{a}$ an odd and thus h is equivalent to f if we can find, for every $a=8n\pm3$, a $b\equiv\pm1 \pmod 4$ and integers c and r such that

$$H=16=a(16bc-16r^2)-32b, \text{ that is}$$

$$2b=at-1 \text{ where } t=bc-r^2.$$

Take $t=8k+v$ where $va\equiv 3 \pmod 4$ and have $b=4ak+(va-1)/2$. For $a\equiv\pm3 \pmod 8$ take $v=1$ or 3 respectively and have $b\equiv\pm1 \pmod 4$ and, choosing k so that b is a prime, have $\left(\frac{-t}{b}\right)=\left(\frac{b}{t}\right)=\pm\left(\frac{2b}{t}\right)=1$. Thus an r exists such that $r^2+t\equiv 0 \pmod b$ and $(r^2+t)/b$ determines c as an integer. Also, since h represents no $8n+1$ nor $8n-1$, f represents no integers of that form.

$f=2a$ implies $y+z\equiv 0 \pmod 2$ and applying method 2 we get $f/2=(1,2,4)$.

125. $f=(3,3,3,-2,-2,-2) \neq 4n+1,\ 4n+2,\ 4^k(8n+7)$. $(H=16)$.

$f=a$ an odd or even integer implies one of x, y, z is odd or even respectively and the other two both odd or both even. By symmetry we may take $y+z\equiv 0 \pmod 2$ and have $y+z=2Y$, $y-z=2Z$ are solvable for Y and Z and all and only the integers represented by f are represented by $g=3x^2+4Y^2+8Z^2-4xY=2x^2+(2Y-x)^2+8z^2$. Thus the only odds represented by f are $\equiv 3 \pmod 8$. And we see that $g=(1,2,8)\equiv 3 \pmod 8$ or $\equiv 0 \pmod 4$.

126. $f=(1,1,3,1,1,1)\neq 4^k(64n+56)$, $4n+2$. $(H=16/8)$. (This proof is contained essentially in some notes of L. E. Dickson).

We first prove that f represents all odd integers <u>a</u>. 2f is the only reduced form of Hessian 16 with all coefficients even. Thus we seek a form $h=2ax^2+2by^2+2cz^2+2ryz+2xz$ of Hessian 16; that is, we wish to find integers b, c, r such that

$$H=16=2a(4bc-r^2)-2b; \text{ that is,}$$

$b=at-8$ where $t=4bc-r$. Take $t=32k$, $k=2T+1$, $b=8b'$ and have $b'=4ak-1\equiv 3 \pmod 8$ and T may be chosen so that b' is a prime. Then $\left(\frac{-T}{b'}\right)=\left(\frac{-2k}{b'}\right)=\left(\frac{k}{b'}\right)=\left(\frac{-b'}{k}\right)=\left(\frac{1}{k}\right)=1$ and an $r'=2r$ exists such that $t+4r^2\equiv 0 \pmod b$, and thus $(t+4r^2)/b=c$ an integer.

It remains to prove that f represents exclusively all evens not of the form $4^k(64n+56)$, $4n+2$. Now $f\equiv x^2+y^2+z^2+xy+xz+yz\equiv(1+x)(1+y)(1+z)+xyz+1\equiv 0 \pmod 2$. Then if any one of x, y, z is odd, $xyz\equiv 1 \pmod 2$ and all are odd. Since x, y, z are all odd or all even we may set

$x=-Z-2X$, $y=-Z-2Y$, $z=Z$ and get $f/4=g$ where g is form 105. Thus also $f\neq 4n+2$.

127. $f=(1,1,3,0,0,-1)\neq 9^k(3n+2)$. $(H=18/8)$.

Apply method 2 to $g=(1,3,12)$ letting $x=2X-y$ to prove $g/4=f$.

128. See immediately following the proof for 112.

129. $f=(2,3,4,0,0,-2)\neq 4n+1,\ 25^k(25n\pm 5)$. $(H=20)$.

Apply method 2 to $g=(1,5,8)$ letting $x=2X-y$ to prove $g/2=f$.

131. $f=(3,3,3,2,2,2)\neq 4n+1,\ 4n+2,\ 25^k(25n\pm 5)$. $(H=20)$.

As in the proof for form 125 all and only the integers represented by f are represented by $g=3x^2+8Y^2+4Z^2+4xY$. Thus f represents no $4n+1$, $4n+2$.

g and therefore f represents all $a\equiv 3 \pmod 4$ not of the form $25^k(25n\pm 5)$ for $2g=(4Y+x)^2+5x^2+8Z^2$ and $y'^2+5x^2+8Z^2\equiv 6 \pmod 8$ implies $\pm y'=4Y+x$ is solvable for Y where one of the signs holds.

$g=2a$ implies $-x=2X$ is solvable for X and $g/4=g'$ where g' is form 106 (with x and z interchanged).

131. $f=(1,1,3,-1,-1,0)\neq 4^k(16n+6)$. $(H=20/8)$.

$g=x^2+y^2+5z^2\equiv 0 \pmod 2$ implies $x+y+z\equiv 0 \pmod 2$ and thus that $x+y+z=2X$, $-x+y+z=2Y$ are solvable for X and Y and $g/2=f$.

132. $f=(1,2,2,2,1,1)\neq 25^k(25n\pm 10)$. $(H=20/8)$.

Form 106: $g=x^2+2y^2+3z^2-2yz\equiv 0 \pmod 2$ implies $z-x=2Y$, $z+x=2Z$, $y=-X$ are solvable for X, Y and Z and $g/2=f$.

IV. Certain regular reduced positive forms $f=ax^2+by^2+cz^2+ryz+sxz+txy$ i.e. (a,b,c,r,s,t) of Hessian >20.

(No attempt is made to prove the forms not included in this list are irregular though such is in general true for forms of Hessian <50. Also, since the methods used are the same as those in the preceeding sections, proofs are abbreviated to a minimum).

133. $f=(1,4,7,-4,0,0)\neq 4n+2,\ 9^k(9n+3)$. $(H=24)$.

$f=x^2+(2y-z)^2+6z^2=g\equiv 0,1,3 \pmod 4$ where $g=X^2+Y^2+6z^2$ for $g\equiv 1 \pmod 2$ implies $X\not\equiv Y \pmod 2$ and thus for any solution X, Y, z we can interchange X and Y if necessary so that $Y+z\equiv 0 \pmod 2$ and $2y-z=Y$ is solvable; and $g\equiv 0 \pmod 4$ implies $X\equiv Y\equiv z \pmod 2$ and $2y-z=Y$ is solvable.

f represents no $4n+2$.

134. $f=(2,2,7,-2,-2,0)\neq 4n+1,\ 8n+6,\ 9^k(9n+3)$. $(H=24)$.

$2f=(2x-z)^2+(2y-z)^2+12z^2=X^2+Y^2+12z^2\equiv 6 \pmod 8$.

$f\not\equiv 1 \pmod 4$ since $2f\not\equiv 2 \pmod 8$. $f/2=(1,1,12)$.

135. $f=(3,3,3,0,0,-2)\neq 4n+1,\ 16n+2,\ 4^k(16n+10)$. $(H=24)$.

$3f=(3x-y)^2+8y^2+9z^2=X^2+8y^2+9z^2\equiv 0 \pmod 3$ and $X^2+8y^2+9z^2=g=X^2+8y^2+Z^2\equiv 0 \pmod 3$ since $g\equiv 0 \pmod 3$ implies X or $Z\equiv 0 \pmod 3$.

136. $f=(1,1,4,0,0,-1)\neq 4n+2,\ 9^k(9n+6)$. $(H=24/8)$.

$f\equiv x^2+y^2-xy\equiv (x+y)^2-xy\equiv 0 \pmod 2$ implies $x=2X,\ y=2Y$ and $f/4=g$ where g is form 107. It remains to prove

f represents all odd integers $a>0$ not of the form

$9^k(9n+6)$. The only reduced forms of Hessian 24 and all coefficients even are 2f and $g=(2,4,4,2,2,2)$ which represents 4. We exhibit a form h with all coefficients even, of Hessian 24 representing no $8n+4$ and having 2a as the leading coefficient:

1) If a is prime to 3 (and odd) take

$$h=2ax^2+8by^2+2cz^2+16ryz+2xz$$ which represents no $8n+4$ if c is odd since $h\equiv 2ax^2+2cz^2+2xz \equiv (x+z)^2+(2a-1)x^2+(2c-1)z^2 \pmod 8$ and $h\equiv 0 \pmod 4$ implies $x\equiv z\equiv 0 \pmod 2$ and thus $h\equiv 0 \pmod 8$. Thus we seek integers b, c odd, r such that

$$H=24=2a(16bc-64r^2)-8b; \text{ that is}$$

$$b=4at-3\equiv 1 \pmod 4) \text{ where } t=bc-4r^2.$$ Take $t=12k+1$ where k is chosen so that b is a prime and thus, since t is odd, c is odd if it exists. Then $\left(\frac{-t}{b}\right)=\left(\frac{b}{t}\right)=\left(\frac{-3}{t}\right)=\left(\frac{t}{3}\right)=1$ and there exists an $r'=2r$ such that $t+4r^2\equiv 0 \pmod b$ and h is determined.

2). If $a=3a'$ where $a'\equiv 1 \pmod 6$ form

$$h=6a'x^2+8by^2+2cz^2+48ryz+6xz$$ which, as above, represents no $8n+4$ if c is odd and will be determined if we can find integers b, r and an odd c such that

$$H=24=6a'(16bc-144\cdot 4r^2)-72b; \text{ that is,}$$

$$3b=4a't-1 \text{ where } t=bc-36r^2.$$ Take $t=12k+1$ choosing k so that $b=4a'k+(4a'-1)/3\equiv 1 \pmod 4$ is a prime and thus, since t is odd, c is odd if it exists. Then $\left(\frac{-t}{b}\right)=\left(\frac{b}{t}\right)=\left(\frac{3b}{t}\right)=\left(\frac{-1}{t}\right)=1$ and $r'=6r$ exists so that $t+36r^2\equiv 0 \pmod b$ and h is

determined.

3) We have proved that f, then, represents all odd integers $a \neq 9n+6$ nor divisible by 9. Now $f \equiv (x+y)^2+z^2 \pmod 3$ and thus $f \equiv 0 \pmod 3$ implies $z=3Z$ and $x=3X-y$ are solvable and $f/3=3X^2+y^2+12Z^2-3Xy \equiv 0 \pmod 3$ implies $y \equiv 0 \pmod 3$ implies $x \equiv 0 \pmod 3$ and thus $f/9=f$ and the proof is complete.

137. $f=(2,3,5,0,0,-2) \neq 25^k(5n\pm 1)$. $(H=25)$.

$$2f=(2x-y)^2+5y^2+10z^2=(1,5,10) \equiv 0 \pmod 2 \text{ using method 2.}$$

138. $f=(1,2,2,-1,0,-1) \neq 169^k(169n+13e')$ where $e'=1,3,4,9,10$ or 12. $(H=26/8)$. (This proof is contained essentially in some notes of L. E. Dickson). Since 2f is the only reduced form of Hessian 26 with all coefficients even it is sufficient to exhibit, for any $a \neq 169^k(169n+13e')$, not divisible by 169, a form $h=2ax^2+2by^2+2cz^2+2ryz+2sxz$ of Hessian 26. That is, we seek integers b, c, r and s such that

$$H=26=2a(4bc-r^2)-2bs^2.$$

1) If a is prime to 13 let $s=1$ and have $b=at-13$ where $t=4bc-r^2$. Let $t=4T$ where $T=13k+2$ and, choosing k so that b is a prime, have $\left(\frac{-t}{b}\right)=\left(\frac{-T}{b}\right)=-\left(\frac{-6}{T}\right)=-\left(\frac{13}{T}\right)=-\left(\frac{T}{13}\right)=1$.

2) If $a=13a'$ where $a' \equiv 2,5,6,7,8,$ or $11 \pmod{13}$ i.e. $\left(\frac{a'}{13}\right)=-1$. Choose an <u>even</u> integer e so that $s=1-ea'$ is divisible by 13. Then $b+13+ar^2-a'bP=0$ where $P=4\cdot 13c+2e-e^2a'$. Take $b=8a'm-13$ where m is prime to 26. Replacing only the first b by its value and cancelling a' we get

$8m+13r^2-bP=0$. Hence $8m(1-a'P)\equiv 0$ and $a'P\equiv 1 \pmod{13}$. To show that $v=P-2e+e^2a'$ is divisible by 13, note that $a'v\equiv(1-ea')^2=s^2\equiv 0 \pmod{13}$. Also v will be divisible by 4 and hence will yield an integer c if r is even, so that P is divisible by 4. It remains only to show that $-8\cdot 13m\equiv x^2 \pmod b$ is solvable. For, since b is prime to 13 we can add a multiple of b to one root x and obtain a root x divisible by 13. If it be odd we add 13b and get a root $x=13r$, r even. We have $\left(\frac{-2}{b}\right)=1$, $\left(\frac{13}{b}\right)=\left(\frac{b}{13}\right)=\left(\frac{2a'm}{13}\right)=\left(\frac{m}{13}\right)$, $\left(\frac{m}{b}\right)=\left(\frac{-b}{m}\right)=\left(\frac{13}{m}\right)$, $\left(\frac{-8\cdot 13m}{b}\right)=1$. Thus we have proved that f represents all positive integers $a\neq 169^k(169n+13e')$ and not divisible by 169, where $\left(\frac{e'}{13}\right)=-1$.

$f/13=g=5X^2+5Y^2+2Z^2+3XY+XZ-YZ$ for $f\equiv(x+6y)^2+2(z+3y)^2 \pmod{13}$. But 2 and -2 are quadratic non-residues of 13. Hence $f\equiv 0 \pmod{13}$ if and only if $x+6y$ and $z+3y$ are. Then $x=2z+13X$, $y=-z+13Y$ and $f=13F$ where $F=2z^2-13zY+13X^2+26Y^2+13XY$ whence F represents no residue of 13 (and $f\neq 169n+13e'$). Replacing z by $Z+3Y$ and then Y by $Y-X$ we get g.

$f=169f$ for by its origin $g\equiv F\equiv 2(Z+3Y-3X)^2 \pmod{13}$. Hence g is divisible by 13 if and only if $Y=X+4Z+13y$. Then $g=13g'$ where $g'=X^2+4XZ+6Z^2+13Xy+39Zy+65y^2$. Replacing in turn X by $x-2Z$, x by $x-6y$, Z by $z-3y$, y by $-y$, we get f, thus completing the proof.

139. $f=(1,6,6,-6,0,0)\neq 3n+2,\ 9n+3,\ 4^k(8n+5)$. $(H=27)$.

$$2f=2x^2+3(2y-z)^2+9z^2=(2,3,9)\equiv 0 \pmod 2.$$

140. $f=(2,3,5,0,-2,0)\neq 3n+1,\ 9^k(9n+6)$. $(H=27)$.

$$2f=(2x-z)^2+6y^2+9z^2=(1,6,9)\equiv 0 \pmod 2.$$

141. $f=(1,4,8,-4,0,0)\neq 4n+2,\ 4n+3,\ 49^k(49n+7e)$ where $e=3$, 5 or 6. $(H=28)$.

$f=2a$ implies $x=2X$ and $f=4g$ where g is form 120.

f represents no $4n+2$, $4n+3$ obviously. It remains to prove

f represents all $a\equiv 1 \pmod 4$ not of the form $49^k(49n+7e)$. The properly reduced forms (forms representing odds) of Hessian 28 are f, $g_1=(1,1,28)$, $g_2=(1,2,14)$, $g_3=(1,4,7)$ and forms of minimum 2 or 3. Now g_1 and g_2 represent 2 and g_3 represents 11. We exhibit a form h of Hessian 28, with leading coefficient <u>a</u> and representing no $4n+2$ nor $4n+3$, which is therefore equivalent to f:

1) If a is prime to 7 (and $\equiv 1 \pmod 4$), take

$h=ax^2+4by^2+4cz^2+4ryz+4xz$ and seek integers b, c, r so that

$$H=28=a(16bc-4r^2)-16b; \quad \text{that is,}$$

$$4b=at-7 \text{ where } t=4bc-r^2.$$

Let $t=7\cdot 16k+v$ where v is chosen $\equiv 1 \pmod 7$ so that $av\equiv 3 \pmod 8$. Thus $t\equiv 3 \pmod 4$ and $b=28ak+(av-7)/4$. Choose k so that b is a prime and have $\left(\frac{-t}{b}\right)=\left(\frac{b}{t}\right)=\left(\frac{4b}{t}\right)=\left(\frac{-7}{t}\right)=\left(\frac{t}{7}\right)=1$ and an odd r exists such that $t+r^2\equiv 0 \pmod b$ and thus $t+r^2\equiv 0 \pmod{4b}$ and c exists.

2) If $a=7a'$ where $a'\equiv 1,2$ or $4 \pmod 7$ and $a'\equiv 3 \pmod 4$, take

$h=7a'x^2+4by^2+4cz^2+28ryz+28xz$ and seek integers b, c, r so that $H=28=7a'(16bc-14^2\cdot r^2)-4\cdot 14^2 b$, that is

$$28b=a't-1 \text{ where } t=4bc-49r^2.$$

Let $t=7\cdot 16k+v$ where v is chosen so that $a'v\equiv 29 \pmod{56}$. Thus $\left(\frac{t}{7}\right)=1$, $t\equiv 3 \pmod 4$ and $b=4a'k+(a'v-1)/28$. Choose k so that b is a prime >7 and have $\left(\frac{-t}{b}\right)=\left(\frac{b}{t}\right)=\left(\frac{28b}{t}\right)\left(\frac{7}{t}\right)=-\left(\frac{2}{t}\right)\left(\frac{t}{7}\right)=1$ and an odd $r'=7r$ exists so that $t+49r^2\equiv 0 \pmod{4b}$ as above.

3) $f=x^2+(2y-z)^2+7z^2$ and thus $f=49f$ and the proof is complete.

142. $f=(2,3,6,-2,0,-2)\neq 8n+5,\ 4^k(8n+1)$. $(H=28)$.

Apply method 1 as for form 108 for $7a\neq 8n+3, 4^k(8n+7)$ represented by $g=4x^2+y^2+z^2$ and find that 7a is represented by $g'=4x^2+(7Y+x)^2+(7Z+3x)^2$ and thus $g'/7$ represented exclusively all positive integers not of the form $8n+5$, $4^k(8n+1)$. The only properly reduced forms (representing odds) of Hessian 28 and minimum 2 are $h=(2,2,7)$ which does not represent 3, $h'=(2,3,5,-2,0,0)$ and $h''=(2,4,5,-4,-2,0)$ which represent 5 and f. Thus $g'/7$ is equivalent to f.

143. $f=(1,1,5,1,1,1)\neq 4^k(16n+2)$. $(H=28/8)$.

Form 108: $g=(2,2,3,2,2,2)\equiv 0 \pmod 2$ implies $z=2Z$ and thus $g'=(2,2,12,4,4,2)$ of Hessian 28 represents all evens $\neq 4^k(8n+1)$. The only reduced forms of Hessian 28 with all coefficients even are $(2,2,8,-2,-2,0)$ and $(2,4,4,0,0,-2)$ which represent 4 and f. Thus g' is equivalent to 2f.

144. $f=(1,5,8,-4,0,0)\neq 8n+3,\ 4^k(8n+7)$. $(H=36)$.

Apply method 1 as for form 111 for $9a\neq 8n+3, 4^k(8n+7)$

represented by $4x^2+y^2+z^2$ and find 9a is represented by $4x^2+(9Y+2x)^2+(9Z+x)^2$. Thus a is represented by $g=(x+2Y+Z)^2+5Y^2+8Z^2-4YZ$ which is equivalent to f.

145. $f=(3,4,4,-4,0,0) \neq 4n+1,\ 4n+2,\ 9^k(3n+2)$. (H=36).

$f=3x^2+(2y-z)^2+3z^2=(3,1,3) \equiv 0$ or $3 \pmod 4$.

146. $f=(1,1,6,0,0,-1) \neq 3n+2,\ 4^k(16n+14)$. (H=36/8).

$2f=g \equiv 0 \pmod 2$ where g is form 112.

147. $f=(1,2,3,-2,-1,0) \neq 4^k(16n+14)$. (H=36/8).

$2f=(2y-z)^2+2x^2+5z^2-2xz=g \equiv 0 \pmod 2$ where g is form 111.

148. $f=(2,2,2,1,2,2) \neq 9^k(3n+1)$. (H=36/8).

$2f=(2x+y+z)^2+3y^2+3z^2=(1,3,3) \equiv 0 \pmod 2$, for $X^2 \equiv y^2+z^2 \pmod 2$ implies $X \equiv y+z \pmod 2$.

149. $f=(1,2,3,0,-1,0) \neq 4^k(16n+10)$. (H=44/8).

$2f=g \equiv 0 \pmod 2$ where g is form 114.

150. $f=(1,6,9,-6,0,0) \neq 3n+2,\ 4^k(8n+3)$. (H=45).

$f=x^2+5y^2+(3z-y)^2=(1,5,1) \equiv 0$ or $1 \pmod 3$.

151. $f=(2,2,15,0,0,-2) \neq 9^k(3n+1),\ 25^k(25n \pm 5),\ 4^k(8n+3)$. (H=45).

$2f=(2x-y)^2+3y^2+30z^2=(1,3,30) \equiv 0 \pmod 2$.

152. $f=(1,8,8,-8,0,0) \neq 4n+2,\ 4n+3,\ 4^k(8n+5)$. (H=48).

$f=x^2+2(2y-z)^2+6z^2=(1,2,6) \equiv 0$, 1 or $4 \pmod 8$.

153. $f=(3,3,6,-2,-2,0) \neq 8n+1,\ 8n+2,\ 32n+4,\ 4^k(8n+5)$. (H=48).

$3f=(3x-z)^2+(3y-z)^2+16z^2=(1,1,16) \equiv 0 \pmod 3$.

154. $f=(3,3,7,-2,-2,-2) \neq 4n+1,\ 4n+2,\ 4^k(8n+5)$. (H=48).

$3f=(3x-y-z)^2+2(2y-z)^2+18z^2=X^2+2Y^2+18z^2 \equiv 9 \pmod{12}$, for $g=X^2+2Y^2+18z^2 \equiv 1 \pmod 4$ implies $Y+z \equiv 0 \pmod 2$, $g \equiv 0 \pmod 3$

implies $X \equiv \pm Y \pmod 3$ where one of the signs holds and $g = (1,2,2) \equiv 0 \pmod 3$.

$3f \neq 4n+3,\ 4n+2$ implies $f \neq 4n+1,\ 4n+2$.

156. $f = (2,5,6,0,0,-2) \neq 3n+1,\ 9^k(9n+3)$. (H=54).

$2f = (2x-y)^2 + 9y^2 + 12z^2 = (1,9,12) \equiv 0 \pmod 2$.

157. $f = (1,1,10,0,0,-1) \neq 9^k(9n+6),\ 25^k(25n\pm 5),\ 4^k(16n+2)$. (H=60/8).

$2f = g \equiv 0 \pmod 2$ where g is form 121.

158. $f = (1,3,3,1,1,1) \neq 4^k(16n+2)$. (H=60/8).

Applying method 2 to form 122 we see that all the evens represented by form 122 are represented by $g = (2,6,6,-2,-2,0)$. That is, g represents all evens not of the form $4^k(8n+1)$ and none others. The reduced forms of Hessian 60 with all coefficients even are $(2,2,16,-2,-2,0)$, $(2,4,8,0,-2,0)$, $(4,4,4,0,0,-2)$ and $(4,4,6,2,4,4,)$ which represent 4, form 2g' where g' is form 157 and thus does not represent 30, and f. Thus g, of Hessian 60 is equivalent to f.

159. $f\ (2,5,7,-2,-2,0) \neq 9n\pm 3,\ 4^k(8n+1)$. (H=63).

Apply method 1 as for form 108 for $7a \neq 9n\pm 3, 4^k(8n+7)$ represented by (9,1,1) to find 7a is represented by $9x^2 + (7Y+x)^2 + (7Z+2x)^2$ and thus <u>a</u> is represented by $2(x+Z)^2 + 7Y^2 + 5Z^2 + 2xY = g$ and replacing x by $-X-Z$, then interchanging Y and Z we find f is equivalent to g.

160. $f = (3,3,8,0,0,-2) \neq 4n+1,\ 4n+2,\ 4^k(8n+7)$. (H=64).

$3f = (3x-y)^2 + 8y^2 + 24z^2 = (1,8,24) \equiv 0 \pmod 3$.

161. $f=(1,9,9,-6,0,0)\neq 3n+2,\ 4n+3,\ 16n+6,\ 4^k(16n+14)$. $(H=72)$.

$$f=x^2+(3y-z)^2+8z^2=(1,1,8)\equiv 0 \text{ or } 1 \pmod 3.$$

162. $f=(1,1,12,0,0,-1)\neq 4n+2,\ 9^k(3n+2)$. $(H=72/8)$.

Reference to part 3) of the proof for form 136 shows that $g/3=3X^2+y^2+12Z^2-3Xy$ where g is form 136. Replace y by X+Y and find g/3 becomes f.

163. $f=(1,10,10,-10,0,0)\neq 9^k(9n+6),\ 25^k(5n\pm 2),\ 4^k(8n+5)$. $(H=75)$.

$$2f=2x^2+5(2x-y)^2+15y^2=(2,5,15)\equiv 0 \pmod 2.$$

164. $f=(1,8,12,-8,0,0)\neq 4n+2,\ 4n+3,\ 25^k(25n\pm 5)$. $(H=80)$.

$$f=x^2+2(2y-z)^2+10z^2=(1,2,10)\equiv 0 \text{ or } 1 \pmod 4.$$

166. $f=(2,3,20,0,0,-2)\neq 25^k(5n\pm 1),\ 4n+1$. $(H=100)$.

$$2f=(2x-y)^2+5y^2+40z^2=(1,5,40)\equiv 0 \pmod 2.$$

167. $f=(2,5,5,3,1,-1)\neq 169^k(13n+e)$ where $e=1,3,4,9,10$ or 12. $(H=338/8)$.

Reference to the proof for form 138 shows that $g/13=f$ where g is form 138.

V. Partial proofs for the six forms $f=ax^2+by^2+cz^2$ in tables I to IV not yet proved regular.

24. $f=(1,2,32)$ represents exclusively all positive integers not of the forms $8n+5$, $4^k(8n+7)$, $16n+10$, $16n+14$, or $32n+20$ <u>provided</u> it represents all $8n+3$, $8n+1$. (f represents all $8n+3$, $8n+1 < 1000$).

$f=2a$ implies $x=2X$ and $f/2=(2,1,16)$. f obviously represents no $8n+5$, $8n+7$.

36. $f=(1,8,32)$ represents exclusively all positive integers not of the forms $4n+3$, $8n+5$, $4n+2$, $32n+20$, or $4^k(8n+7)$ <u>provided</u> it represents all $8n+1$.

$f=2a$ implies $x=2X$ and $f/4=(1,2,8)$.

Note: complete results for this form would follow from complete results for form 24 since $f=x^2+2y^2+32z^2 \equiv 1 \pmod 8$.

38. $f=(1,8,64)$ represents exclusively all positive integers not of the form $4n+3$, $8n+5$, $4n+2$, $32n+20$, $32n+28$, $64n+40$ or $4^k(16n+14)$ <u>provided</u> it represents all $8n+1$. (f represents all $8n+1 < 1000$).

$f=2a$ implies $x=2X$ and $f/4=(1,2,16)$.

54. $f=(1,3,36)$ represents exclusively all positive integers not of the form $3n+2$, $4n+2$ or $9^k(9n+6)$ <u>provided</u> it represents all $24n+1$. (f represents all $24n+1 < 1000$).

$f=3a$ implies $x=3X$ and $f/3=(3,1,12)$ for which results are known.

$f=2a$ implies $a=2a'$ and $f/4=(1,3,9)$ using the corollary to lemma b.

f represents all $4n+3 \equiv 1 \pmod 3$ for $g = x^2+3y^2+9z^2 \equiv 3 \pmod 4$ implies $z=2Z$ and $f = g \equiv 3 \pmod 4$. It remains to prove

f represents all $a=24n+13$. We know $\underline{a}$ is represented by g.

1) If $g=a$ with z even, $\underline{a}$ is represented by f.

2) If $g=a$ with y and z odd, then x is odd and g represents $\underline{a}$ with $3(y^2+3z^2) \equiv 0 \pmod 4$ and thus, by the corollary to lemma b, g represents $\underline{a}$ with y and z even and thus $\underline{a}$ is represented by f.

3) If $g=a$ with $y=2Y$ and z odd, then $x=2X$ and g becomes $4X^2+12Y^2+9z^2 \equiv 5 \pmod 8$ which implies $X \not\equiv Y \pmod 2$ and thus $x+y=2y' \equiv 2 \pmod 4$ and $x-y=2x' \equiv 2 \pmod 4$ and $\underline{a}$ is represented by $(2x'-y')^2+3y'^2+9z^2$ where y' is odd and thus, from 2), f represents a.

64. $f=(1,12,36)$ represents exclusively all positive integers not of the form $3n+2$, $4n+2$, $4n+3$, or $9^k(9n+6)$ <u>provided</u> it represents all $24n+1$.

$f=2a$ implies $x=2X$ and $f/4=(1,3,9)$.

f represents all $a \equiv 5 \pmod 8$ not of the forms $3n+2$, $9^k(9n+6)$ since $g=x^2+3y^2+36z^2 \equiv 1 \pmod 4$ implies $y=2Y$ and thus $f = g \equiv 1 \pmod 4$. This also shows that complete results for form 64 will result from those for form 54.

67. $f=(1,48,144)$ represents exclusively all positive integers not of the form $3n+2$, $4n+2$, $4n+3$, $16n+8$, $16n+12$, $8n+5$, or

$9^k(9n+6)$ provided it represents all $24n+1$ and all $96n+4$.

$f=2a$ implies $x=2X$ and $f/4=(1,12,36)$ showing also that f represents all $96n+4$ would follow from the proven result that $(1,12,36)$ represents all $24n+1$.

Note: f represents all $24n+1$ if $(1,12,36)$ does by use of the corollary to lemma b.

PART C

SEMI-REGULAR FORMS

Since the number of semi-regular forms is so great and since, in many cases, proof may easily be derived from known results for regular forms, only a few proofs are given below to illustrate methods by which results may be obtained. It is to be noted that by the application of theorem 10 alone, proofs for an infinite number of semi-regular forms result. Also proofs for semi-regular forms without cross products often result from proofs for regular forms with cross products. Only the essentials of the proofs are given below - the details being analogous to previous methods described in detail.

1. $f=(1,1,7)$[1] $\equiv 0,1 \pmod 4) \neq 49^k(49n+7e)$ where $e=3$, 5, or 6.

   $g=(1,4,8,-4,0,0)=f\equiv 0$ or $1 \pmod 4$. g is form 141. (See method 2).

2. $f=(1,1,10)\equiv 0 \pmod 5) \neq 4^k(16n+6)$ is obtained by applying theorem 10 to (1,1,2) with $m=5$.

3. $f=(1,1,14)\equiv 0$ or $2 \pmod 8) \neq 49^k(49n+7e)$ where $e=3,5$, or 6.

   Apply method 2 to f to find $f/2=(1,1,7)$.

4. $f=(1,1,15)\equiv 0 \pmod 5) \neq 9^k(9n+3)$ is obtained by applying theorem 10 to (1,1,3) with $m=5$.

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1 Similar notation is used throughout this section to mean (for 1) f represents all positive integers $\equiv 0$ or $1 \pmod 4$ except $49^k(49n+7e)$ and none of the form $49^k(49n+7e)$.

5. $f=(1,1,18)\equiv 0 \pmod 2$ or $\equiv 0 \pmod 9 \neq 9n\pm 3,\ 4^k(16n+14)$.

Apply method 2 to f to find $f/2=(1,1,9)$.

$f/9=(1,1,2)$.

6. $f=(1,1,20)\equiv 0 \pmod 4$ or $\pmod 5 \neq 4^k(8n+3),\ 8n+7$.

$f/4=(1,1,5)$.

For multiples of 5 apply theorem 10 to $(1,1,4)$ with $m=5$.

7. $f=(1,1,25)\equiv 0 \pmod 5 \neq 4^k(8n+7)$ is obtained by applying theorem 10 to $(1,1,5)$ with $m=5$.

8. $f=(1,1,27)\equiv 0 \pmod 9 \neq 9^k(9n+6)$.

$f/9=(1,1,3)$.

9. $f=(1,1,30)\equiv 0 \pmod 5 \neq 9^k(9n+6)$ by application of theorem 10 to $(1,1,6)$ with $m=5$.

10. $f=(1,2,9)\equiv 0 \pmod 2$ or $\pmod 3 \neq 4^k(16n+14)$.

Setting $x=2Y-z$, $y=X$ in f we get $f/2=g$ where g is form 111.

$f=(1,2,1)\equiv 0 \pmod 3$ for $(1,2,1)=3a$ implies x or $z\equiv 0 \pmod 3$.

11. $f=(1,2,11)\equiv 0 \pmod 2$ or $\pmod{11} \neq 4^k(16n+10)$.

Applying method 2 to f by setting $y=X$, $x=2Y-z$ we get $f/2=g$ where g is form 114.

Applying theorem 10 to $(1,2,1)$ with $m=11$ we have the result for multiples of 11.

12. $f=(1,2,12)\equiv 0 \pmod 2$ or $\pmod 3 \neq 4^k(16n+10)$.

$f/2=(2,1,6)$.

Applying theorem 10 to $(1,2,4)$ with $m=3$ we have the result for multiples of 3.

13. $f=(1,2,15)\equiv 0 \pmod 3) \neq 25^k(25n\pm 5)$ by application of theorem 10 to (1,2,5) with $m=3$.

14. $f=(1,2,18)\equiv 0 \pmod 3)$ or $\pmod 4) \neq 4^k(8n+7)$.

For multiples of 3 apply theorem 10 to (1,2,6) with $m=3$.

$f/2=(2,1,9)$.

15. $f=(1,2,22)\equiv 0 \pmod 4)$ or $\pmod{11})$ or $\equiv 1 \pmod 8) \neq 4^k(8n+5)$.

$f/2=(2,1,11)$.

For multiples of 11 apply theorem 10 to (1,2,2) with $m=11$.

It remains to prove that f represents all $8n+1$. We know $g=x^2+2y^2+11z^2$ represents all $16n+2$. But $g\equiv 2 \pmod 8)$ implies $x=2X$, $z=2Z$ and $2X^2+y^2+22Z^2$ represents all $8n+1$.

16. $f=(1,3,5)\equiv 0 \pmod 3) \neq 25^k(25n\pm 10)$.

For $f=3a$ implies $3a=(3z-x)^2+3y^2+5x^2=3g$ where g is form 106, and conversely $g=a$ implies $f=3a$.

17. $f=(1,3,7)\equiv 0 \pmod 7) \neq 9^k(9n+6)$ by application of theorem 10 to (1,3,1) with $m=7$.

18. $f=(1,3,8)\equiv 0 \pmod 3)$ or $\pmod 4) \neq 4n+2,\ 4^k(16n+10)$.

$f=4a$ where $a\neq 4^k(16n+10)$ for $g=x^2+3y^2+2z^2$ represents all such 4a and $g=4a$ implies $z=2Z$.

$f=3a$ implies $3a=(3z-x)^2+3y^2+8x^2=3g$ where g is form 109 and conversely $g=a$ implies $f=3a$.

19. $f=(1,3,14)\equiv 0 \pmod 3) \neq 4^k(16n+6)$.

$f=3a$ implies $3a=(3x-z)^2+3y^2+14z^2=3g$ where g is form 119 and conversely $g=a$ implies $f=3a$.

20. $f=(1,3,16)\equiv 0 \pmod 2$ or $\equiv 1 \pmod 8) \neq 4n+2,\ 16n+8,\ 9^k(9n+6)$.

$f=2a$ implies $2a=(2x-y)^2+3y^2+16z^2=4g$ where g is form 136 and conversely $2g=a$ implies $f=2a$. (f is irregular as to $8n+5$, $8n+3$ for $f\neq 5, 11$).

$f=a\equiv 1 \pmod 8$ if $a\neq 9^k(9n+6)$ since $(1,48,16)$ represents such a.

21. $f=(1,3,20)\equiv 0 \pmod 3) \neq 4n+2,\ 25^k(25n\pm 10)$.

$f=3a$ implies $3a=(3x-z)^2+3y^2+20z^2=3g$ where g is form 128 and conversely $g=a$ implies $f=3a$.

22. $f=(1,4,5)\equiv 0 \pmod 4)$ or $\pmod 5) \neq 4^k(8n+3),\ 8n+7$, or $\equiv 1 \pmod 4$.

$f=a\equiv 0$ or $1 \pmod 4$. $g=x^2+y^2+5z^2=a$ implies that two of x, y, z are even and thus $f=a$.

$f=5a\neq 4^k(8n+3),\ 8n+7$ is obtained by applying theorem 10 to $(1,4,1)$ with $m=5$.

23. $f=(1,4,7)\equiv 0,1 \pmod 4) \neq 49^k(49n+7e)$ where $e=3,5$, or 6.

$f=(1,1,7)\equiv 0$ or $1 \pmod 4$.

24. $f=(1,5,6)\equiv 0 \pmod 2)$ or $\pmod 5) \neq 4^k(16n+2)$.

$f=2a$ implies $2a=(2x-y)^2+5y^2+6z^2=2g$ where g is form 122.

For multiples of 5 apply method 1 to prove that $f=5a$ implies $5a=(5x+2z)^2+5y^2+6z^2=5(y^2+2(x+z)^2+3x^2)=5g$ where g is equivalent to $(1,2,3)$ and conversely $g=a$ implies $f=5a$.

25. $f=(1,6,7)\equiv 0 \pmod 7) \neq 9^k(9n+3)$ is obtained by applying theorem 10 to $(1,6,1)$ with $m=7$.

26. $f=(1,6,8)\equiv 1 \pmod 2),\ \equiv 0 \pmod 3)$ or $\pmod 4) \neq 8n+3,\ 4^k(8n+5)$.

$f=a\equiv 7 \pmod 8$ for $g=(1,6,2)$ represents all such <u>a</u>

and $g=a$ implies $z=2Z$.

$f=a\equiv 1 \pmod 8$ since (1,24,8) does.

$f=3a$ implies $3a=(3x-z)^2+6y^2+8z^2=3g$ where g is form 124 and $g=a$ implies $f=3a$.

$f/4=(1,2,6)$. (f is irregular as to $4n+2$ as is evidenced by taking $k=2$).

27. $f=(1,6,42)\equiv 0 \pmod 7 \neq 9^k(3n+2)$, $8n+5$ by applying theorem 10 to (1,6,6) with $m=7$.

28. $f=(1,7,7)\equiv 0,3 \pmod 4 \neq 49^k(7n+e)$ where $e=3,5$ or 6 for $f=g/7$ where $g=(1,7,1)$.

29. $f=(1,7,12)\equiv 0 \pmod{28} \neq 9^k(9n+6)$.

$g=x^2+3y^2+7z^2$ represents all $28a\neq 9^k(9n+6)$ and $g=28a$ implies $x^2\equiv y^2+z^2 \pmod 4$. If $y=2Y$, f represents 28a. Otherwise $y\equiv x \pmod 2$ and by the corollary to lemma b, g represents 28a with x and y even, thus completing the proof.

30. $f=(1,7,24)\equiv 0 \pmod{28} \neq 9^k(9n+3)$.

$g=(1,6,7)$ represents all $28n\neq 9^k(9n'+3)$ and $g=28n$ implies $y=2Y$.

31. $f=(1,11,22)\equiv 0 \pmod{22} \neq 4^k(16n+14)$, for $f=11g$ where $g=(11,1,2)$.

32. $f=(2,2,5)\equiv 7 \pmod 8$ or $\equiv 0 \pmod 4$ or $\pmod 5 \neq 4^k(8n+3)$.

f represents all $8n+7$ for $g=x^2+y^2+5z^2\equiv 7 \pmod 8$ implies $x\equiv y \pmod 2$ and apply method 2 to prove $f=g\equiv 7 \pmod 8$.[1]

$f/2=(1,1,10)$ for which we know results.

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1 See J.G.A.Arndt, Göttingen Thesis, 1925, p.26.

$f=5a$ implies $5a=2(5x+2y)^2+2y^2+5z^2=5(z^2+2(y+2x)^2+2x^2)=5g$ where g is equivalent to (1,2,2), and conversely $g=a$ implies $f=5a$.

33. $f=(2,3,4)\equiv 0(\text{mod } 2)$ or $(\text{mod } 3)\neq 4^k(16n+10)$.

$f/2=(1,6,2)$.

For multiples of 3 apply theorem 10 to (2,4,1) with $m=3$ and $d=2$.

34. $f=(2,3,21)\equiv 0(\text{mod } 7)\neq 9^k(3n+1)$ for $2f=g$ where $g=(1,6,42)$.

35. $f=(2,3,24)\equiv 1(\text{mod } 2)$ or $\equiv 0(\text{mod } 4)\neq 3n+1,\ 8n+1,\ 4^k(8n+7)$.

f represents all $8n+5\not\equiv 1(\text{mod } 3)$ for $g=(2,3,6)$ represents all such and $g\equiv +5(\text{mod } 8)$ implies $z=2Z$.

f represents $a=24n+19$ since (8,3,24) does.

$f/4=(2,3,6)$.

36. $f=(2,11,22)\equiv 0(\text{mod } 44)$ or $\equiv 11(\text{mod } 88)\neq 4^k(8n+7)$, for $f=11g$ where $g=(22,1,2)$.

37. $f=(3,4,8)\equiv 0(\text{mod } 3)$ or $(\text{mod } 4)\neq 4n+1,\ 4n+2,\ 4^k(16n+10)$.

For multiples of 3 apply theorem 10 to (4,8,1) with $m=3$, $d=4$.

$f/4=(3,1,2)$.

38. $f=(3,8,21)\equiv 0(\text{mod } 28)\neq 9^k(3n+1)$.

$g=(2,3,21)$ represents all $28a\neq 9^k(3n+1)$ but $g=28a$ implies $x=2X$ and thus f represents all such $28a$.

PART D

TABLES

Table I.

Regular forms $ax^2+by^2+cz^2$ (i.e. (a,b,c)) where no two of a, b, c have a factor in common.

No.	Form	Represents exclusively all positive integers not of form -	Reference *
1	(1,1,1)	$4^k(8n+7)$**	1
2	(1,1,2)	$4^k(16n+14)$	2
3	(1,1,3)	$9^k(9n+6)$	1̸,3,2
4	(1,1,4)	$8n+3$, $4^k(8n+7)$	T
5	(1,1,5)	$4^k(8n+3)$	4,3̸,T
6	(1,1,6)	$9^k(9n+3)$	5,3̸,T
7	(1,1,8)	$4n+3$, $4^k(16n+14)$, $16n+6$	4
8	(1,1,9)	$9n\pm3$, $4^k(8n+7)$	4
9	(1,1,12)	$4n+3$, $9^k(9n+6)$	4
10	(1,1,16)	$8n+6$, $4n+3$, $32n+12$, $4^k(8n+7)$	T
11	(1,1,21)	$4^k(8n+3)$, $9^k(9n+6)$, $49^k(49n+7r)$ r=1,2 or 4	T
12	(1,1,24)	$4n+3$, $9^k(9n+3)$, $8n+6$	4
13	(1,2,3)	$4^k(16n+10)$	2,T
14	(1,2,5)	$25^k(25n\pm10)$	2,3̸
15	(1,3,4)	$4n+2$, $9^k(9n+6)$	5
16	(1,3,10)	$9^k(9n+6)$, $25^k(25n\pm5)$, $4^k(16n+2)$	T
17	(1,5,8)	$4n+3$, $8n+2$, $25^k(25n\pm10)$	T

*For references corresponding to the numbers given see bibliography after these tables.
1̸, 3̸ are used to denote partial proofs in references 1 and 3 respectively.
T: proved in this thesis - see preceeding pages.
**k integral and ≥ 0.

Table II.

Regular forms $ax^2+by^2+cz^2$ where two of a, b, c have a factor 2 in common but no two have a prime factor greater than 2 in common.

No.	Form	Represents exclusively all positive integers not of form-	Reference
18	(1,2,2)	$4^k(8n+7)$	2
19	(1,2,4)	$4^k(16n+14)$	2
20	(1,2,6)	$4^k(8n+5)$	T
21	(1,2,8)	$8n+5,\ 4^k(8n+7)$	5
22	(1,2,10)	$8n+7,\ 25^k(25n\pm5)$	T
23	(1,2,16)	$8n+5, 8n+7, 16n+10,\ 4^k(16n+14)$	T
24	(1,2,32)	*	
25	(1,4,4)	$4n+3, 4n+2, 4^k(8n+7)$	T
26	(1,4,6)	$16n+2,\ 9^k(9n+3)$	T
27	(1,4,8)	$4n+2, 4n+3,\ 4^k(16n+14)$	T
28	(1,4,12)	$4n+2, 4n+3,\ 9^k(9n+6)$	T
29	(1,4,16)	$4n+2, 4n+3, 32n+12,\ 4^k(8n+7)$	T
30	(1,4,24)	$4n+2, 4n+3,\ 9^k(9n+3)$	T
31	(1,4,36)	$4n+2, 4n+3, 9n\pm3,\ 4^k(8n+7)$	T
32	(1,6,16)	$8n\pm3, 16n\pm2, 64n+8\ 9^k(9n+3)$	T
33	(1,8,8)	$4n+2, 4n+3, 8n+5\ 4^k(8n+7)$	T
34	(1,8,16)	$4n+2, 4n+3, 4^k(16n+14), 8n+5$	T
35	(1,8,24)	$4n+2,\ 4n+3,\ 4^k(8n+5)$	T

*Only results completely proved are given in tables I to IV. See paragraph V in Part B for partial results.

(Table II continued)

No.	Form	Represents exclusively all positive integers not of form -	Reference
36	(1,8,32)	*	
37	(1,8,40)	$4n+2, 4n+3, 8n+5, 32n+28, 25^k(25n\pm 5)$	T
38	(1,8,64)	*	
39	(1,16,16)	$4n+2, 4n+3, 16n+12, 16n+8, 8n+5, 4^k(8n+7)$	T
40	(1,16,24)	$4n+2, 4n+3, 8n+5, 64n+8, 9^k(9n+3)$	T
41	(1,16,48)	$4n+2, 4n+3, 8n+5, 16n+8, 16n+12, 9^k(9n+6)$	T
42	(2,2,3)	$8n+1,\ 9^k(9n+6)$	5, ~~6~~
43	(2,3,8)	$8n\pm 1,\ 32n+4,\ 9^k(9n+6)$	T
44	(2,5,6)	$4^k(8n+1),\ 9^k(9n+3),\ 25^k(25n\pm 10)$	T
45	(3,4,4)	$4n+1,\ 4n+2,\ 9^k(9n+6)$	5
46	(3,8,8)	$4n+1, 4n+2, 8n+7, 32n+4,\ 9^k(9n+6)$	T
47	(5,8,24)	$4n+2, 4n+3, 4^k(8n+1), 9^k(9n+3), 25^k(25n\pm 10)$	T

*Only results completely proved are given in tables I to IV. See paragraph V in Part B for partial results.

Table III.

Regular forms $ax^2+by^2+cz^2$ where two of a, b, c have a factor 3 in common but no two have a prime factor greater than 3 in common.

No.	Form	Represents exclusively all positive integers not of form -	Reference
48	(1,3,3)	$9^k(3n+2)$	3,2
49	(1,3,6)	$3n+2$, $4^k(16n+14)$	T
50	(1,3,9)	$3n+2$, $9^k(9n+6)$	T
51	(1,3,12)	$4n+2$, $9^k(3n+2)$	5
52	(1,3,18)	$3n+2, 9n+6, 4^k(16n+10)$	T
53	(1,3,30)	$9^k(3n+2), 25^k(25n\pm 10), 4^k(16n+6)$	T
54	(1,3,36)	*	
55	(1,6,6)	$8n+3$, $9^k(3n+2)$	T
56	(1,6,9)	$3n+2$, $9^k(9n+3)$	T
57	(1,6,18)	$3n+2, 9n+3, 4^k(8n+5)$	T
58	(1,6,24)	$8n\pm 3$, $9^k(3n+2), 32n+12$	T
59	(1,9,9)	$3n+2$, $9n\pm 3$, $4^k(8n+7)$	T
60	(1,9,12)	$3n+2$, $4n+3$, $9^k(9n+6)$	T
61	(1,9,21)	$3n+2$, $9^k(9n+6)$, $4^k(8n+3)$, $49^k(49n+7r)$ where $r=1,2$ or 4	T
62	(1,9,24)	$3n+2, 8n+6, 4n+3, 9^k(9n+3)$	T
63	(1,12,12)	$4n+2, 4n+3, 9^k(3n+2)$	5
64	(1,12,36)	*	
65	(1,24,24)	$4n+3, 8n+5, 4n+2, 32n+12, 9^k(3n+2)$	T

*Only results completely proved are given in tables I to IV. See paragraph V in Part BI for partial results.

(Table III continued)

No.	Form	Represents exclusively all positive integers not of form -	Reference
66	(1,24,72)	$4n+2, 4n+3, 3n+2, 9n+3, 4^k(8n+5)$	T
67	(1,48,144)	*	
68	(2,3,3)	$9^k(3n+1)$	3
69	(2,3,6)	$3n+1,\ 4^k(8n+7)$	T
70	(2,3,9)	$3n+1, 9n+6,\ 4^k(16n+10)$	T
71	(2,3,12)	$16n+6,\ 9^k(3n+1)$	T
72	(2,3,18)	$3n+1, 8n+1,\ 9^k(9n+6)$	T
73	(2,3,48)	$16n\pm6, 9^k(3n+1),\ 8n\pm1,\ 64n+24$	T
74	(2,6,9)	$3n+1,\ 9n+3,\ 4^k(8n+5)$	T
75	(2,6,15)	$9^k(3n+1),\ 25^k(25n\pm5),\ 4^k(8n+3)$	T
76	(3,3,4)	$4n+1,\ 9^k(3n+2)$	3
77	(3,3,7)	$9^k(3n+2), 4^k(8n+1), 49^k(49n+7r)$ where $r=3,5$ or 6	T
78	(3,3,8)	$4n+1,\ 8n+2,\ 9^k(3n+1)$	T
79	(3,4,12)	$4n+1,\ 4n+2,\ 9^k(3n+2)$	5
80	(3,4,36)	$3n+2,\ 4n+1,\ 4n+2,\ 9^k(9n+6)$	T
81	(3,8,12)	$4n+1,\ 4n+2,\ 9^k(3n+1)$	T
82	(3,8,24)	$3n+1,\ 4n+1,\ 4n+2,\ 4^k(8n+7)$	T
83	(3,8,48)	$4n+1,\ 4n+2,\ 64n+24,\ 8n+7,\ 9^k(3n+1)$	T
84	(3,8,72)	$3n+1, 4n+1, 4n+2, 8n+7, 32n+4, 9^k(9n+6)$	T
85	(3,16,48)	$4n+1, 4n+2, 8n+7, 16n+4, 16n+8, 9^k(3n+2)$	T
86	(8,9,24)	$3n+1, 4n+2, 4n+3, 9n+3,\ 4^k(8n+5)$	T
87	(8,15,24)	$4n+1, 4n+2, 4^k(8n+3), 9^k(3n+1), 25^k(25n\pm5)$	T

*Only results completely proved are given in tables I to IV. See paragraph V in Part B for partial results.

Table IV.

Regular forms $ax^2+by^2+cz^2$ where two of a, b, c have a prime factor greater than 3 in common.

No.	Form	Represents exclusively all positive integers not of form -	Reference
88	(1,5,5)	$5n\pm 2,\ 4^k(8n+7)$	T
89	(1,5,10)	$25^k(5n\pm 2)$	T
90	(1,5,25)	$25n\pm 10,\ 4^k(8n+3),\ 5n\pm 2$	T
91	(1,5,40)	$4n+3,\ 8n+2,\ 25^k(5n\pm 2)$	T
92 [does not rep. 44 ?]	(1,5,200)	$5n\pm 2,\ 4n+3,\ 8n+2,\ 25^k(25n\pm 10)$	T [NOT REGULAR]
93	(1,10,30)	$9^k(9n+6),\ 25^k(5n\pm 2),\ 4^k(8n+5)$	T
94	(1,21,21)	$9^k(3n+2), 4^k(8n+7), 49^k(7n+r)$ where $r=3,5$ or 6	T
95	(1,40,120)	$4n+2, 4n+3, 4^k(8n+5), 9^k(9n+6), 25^k(5n\pm 2)$	T
96	(2,5,10)	$8n+3,\ 25^k(5n\pm 1)$	T
97	(2,5,15)	$9^k(9n+3), 25^k(5n\pm 1),\ 4^k(16n+10)$	T
98	(3,7,7)	$9^k(9n+6), 4^k(8n+5), 49^k(7n+r)$ where $r=1,2$ or 4	T
99	(3,7,63)	$3n+2, 9^k(9n+6), 4^k(8n+5), 49^k(7n+r)$ where $r=1,2$ or 4	T
100	(3,10,30)	$9^k(3n+2), 25^k(5n\pm 1),\ 4^k(8n+7)$	T
101	(3,40,120)	$4n+1, 4n+2, 4^k(8n+7), 9^k(3n+2), 25^k(5n\pm 1)$	T
102	(5,6,15)	$9^k(3n+1), 25^k(5n\pm 2),\ 4^k(16n+14)$	T
103	(5,8,40)	$4n+2, 4n+3, 8n+1,\ 32n+12,\ 25^k(5n\pm 1)$	T

Note that all forms $f=ax^2+by^2+cz^2$ not listed in tables I to IV are irregular when 1 is the greatest common divisor of a, b, and c.

Table V

Regular reduced positive forms $ax^2+by^2+cz^2+ryz+sxz+txy$ (i.e. (a,b,c,r,s,t)) of Hessian ≤20.

No.	Form		Represents exclusively all positive integers not of form -	Reference
104	(1,2,2,-2,0,0)	3.[1]	$4^k(8n+5)$	2,T
105	(1,1,1,1,1,1)	4/8.	$4^k(16n+14)$	T
106	(1,2,3,-2,0,0)	5.	$25^k(25n\pm5)$	2,T
107	(1,1,1,0,0,-1)	6/8.	$9^k(9n+6)$	5,T
108	(2,2,3,2,2,2)	7.	$4^k(8n+1)$	T
109	(1,3,3,-2,0,0)	8.	$4n+2$, $4^k(16n+14)$	T
110	(2,2,3,-2,-2,0)	8.	$4n+1, 16n+6$, $4^k(16n+14)$	T
111	(1,2,5,-2,0,0)	9.	$4^k(8n+7)$	T
112	(2,2,3,0,0,-2)	9.	$3n+1$, $4^k(8n+7)$	T
113	(1,1,2,1,1,1)	10/8.	$25^k(25n\pm5)$	5,T
114	(1,2,6,-2,0,0)	11.	$4^k(8n+5)$	T
115	(1,4,4,-4,0,0)	12.	$4n+2$, $4n+3$, $9^k(9n+6)$	T
116	(2,3,3,2,2,2)	12.	$8n+1$, $4^k(8n+5)$	T
117	(1,1,2,-1,-1,0)	12/8.	$9^k(9n+3)$	T
118	(1,1,2,0,0,-1)	12/8.	$4^k(16n+10)$	T
119	(1,3,5,-2,0,0)	14.	$4^k(16n+$[illegible]$)$	T
120	(1,1,2,0,-1,0)	14/8.	$49^k(49n+7e)$ where $e=3,5,$ or 6	5
121	(2,2,5,0,0,-2)	15.	$9^k(9n+3), 25^k(25n\pm10), 4^k(8n\pm1)$	T

[1] The number given after each form is the value of the Hessian.

(Table V continued)

	No.	Form		Represents exclusively all positive integers not of form -	Reference
alone	122	(2,3,3,0,0,-2)	15.[1]	$4^k(8n+1)$	T
alone	123	(1,4,5,-4,0,0)	16.	8n+2, 8n+3, 32n+12, $4^k(8n+7)$	T
alone	124	(2,3,3,-2,0,0)	16.	8n+1, $4^k(8n+7)$	T
alone	125	(3,3,3,-2,-2,-2)	16.	4n+1, 4n+2, $4^k(8n+7)$	T
alone	126	(1,1,3,1,1,1)	16/8.	4n+2, $4^k(64n+56)$	T
alone	127	(1,1,3,0,0,-1)	18/8.	$9^k(3n+2)$	5,T
alone	128	(1,3,7,-2,0,0)	20.	4n+2, $25^k(25n\pm5)$	T
alone	129	(2,3,4,0,0,-2)	20.	4n+1, $25^k(25n\pm5)$	T
alone	130	(3,3,3,2,2,2)	20.	4n+1, 4n+2, $25^k(25n\pm5)$	T
alone	131	(1,1,3,-1,-1,0)	20/8.	$4^k(16n+6)$	T
alone	132	(1,2,2,2,1,1)	20/8.	$25^k(25n\pm10)$	T

Note: The forms listed in this table are the only regular positive reduced ternary quadratic forms with cross products and Hessian ≤ 20.

[1] The number immediately after each form is the value of the Hessian.

Table VI

Certain regular reduced positive forms (a,b,c,r,s,t) of Hessian > 20.

	No.	Form		Represents exclusively all positive integers not of form -	Reference	
alone	133	(1,4,7,-4,0,0)	24.[1]	$4n+2,\ 9^k(9n+3)$	T	
alone	134	(2,2,7,-2,-2,0)	24.	$4n+1,\ 8n+6,\ 9^k(9n+3)$	T	
alone	135	(3,3,3,0,0,-2)	24.	$4n+1, 16n+2,\ 4^k(16n+10)$	T	
alone	136	(1,1,4,0,0,-1)	24/8.	$4n+2,\ 9^k(9n+6)$	T	
alone	137	(2,3,5,0,0,-2)	25.	$25^k(5n\pm 1)$	T	
alone	138	(1,2,2,-1,0,-1)	26/8.	$169^k(169n+13e)$ where $e=1,3,4,9,10$ or 12	T	
alone	139	(1,6,6,-6,0,0)	27.	$3n+2,\ 9n+3,\ 4^k(8n+5)$	T	
alone	140	(2,3,5,0,-2,0)	27.	$3n+1,\ 9^k(9n+6)$	T	
alone	141	(1,4,8,-4,0,0)	28.	$4n+2, 4n+3, 49^k(49n+7e)$ where $e=3,5$ or 6	T	
alone	142	(2,3,6,-2,0,-2)	28.	$8n+5,\ 4^k(8n+1)$	T	
alone	143	(1,1,5,1,1,1)	28/8.	$4^k(16n+2)$	T	
	144	(1,5,8,-4,0,0)	36.	$8n+3,\ 4^k(8n+7)$	T	Spurious
alone	145	(3,4,4,-4,0,0)	36.	$4n+1,\ 4n+2,\ 9^k(3n+2)$	T	
alone	146	(1,1,6,0,0,-1)	36/8.	$3n+2,\ 4^k(16n+14)$	T	
alone	147	(1,2,3,-2,-1,0)	36/8.	$4^k(16n+14)$	T	
alone	148	(2,2,2,1,2,2)	36/8.	$9^k(3n+1)$	T	
alone	149	(1,2,3,0,-1,0)	44/8.	$4^k(16n+10)$	T	
alone	150	(1,6,9,-6,0,0)	45.	$3n+2,\ 4^k(8n+3)$	T	

[1] The number given immediately after each form is the value of the Hessian.

(Table VI continued)

No.	Form		Represents exclusively all positive integers not of form -	Reference
151	(2,2,15,0,0,-2)	45.	$9^k(3n+1), 25^k(25n\pm5), 4^k(8n+3)$	T
152	(1,8,8,-8,0,0)	48.	$4n+2,\ 4n+3,\ 4^k(8n+5)$	T
153	(3,3,6,-2,-2,0)	48.	$8n+1, 8n+2, 32n+4, 4^k(8n+5)$	T
154	(3,3,7,-2,-2,-2)	48.	$4n+1, 4n+2, 4^k(8n+5)$	T
155	(2,2,2,-1,-1,-1)	50/8.	$25^k(5n\pm1)$	5
156	(2,5,6,0,0,-2)	54.	$3n+1,\ 9^k(9n+3)$	T
157	(1,1,10,0,0,-1)	60/8.	$9^k(9n+6), 25^k(25n\pm5), 4^k(16n+2)$	T
158	(1,3,3,1,1,1)	60/8.	$4^k(16n+2)$	T
159	(2,5,7,-2,-2,0)	63.	$9n\pm3,\ 4^k(8n+1)$	T
160	(3,3,8,0,0,-2)	64.	$4n+1, 4n+2, 4^k(8n+7)$	T
161	(1,9,9,-6,0,0)	72.	$3n+2, 4n+3, 16n+6, 4^k(16n+14)$	T
162	(1,1,12,0,0,-1)	72/8.	$4n+2,\ 9^k(3n+2)$	T
163	(1,10,10,-10,0,0)	75.	$9^k(9n+6), 25^k(5n\pm2), 4^k(8n+5)$	T
164	(1,8,12,-8,0,0)	80.	$4n+2, 4n+3,\ 25^k(25n\pm5)$	T
165	(1,2,7,0,0,-1)	98/8.	$49^k(7n+e)$ where $e=3,5,$ or 6	5
166	(2,3,20,0,0,-2)	100.	$4n+1,\ 25^k(5n\pm1)$	T
167	(2,5,5,3,1,-1)	338/8.	$169^k(13n+e)$ where $e=1,3,4,9,10$ or 12	T

Table VII.

Certain semi-regular forms (a, b, c).

No.	f	A[1]	B[1]	Reference
1.	(1,1,7)	4m, 4m+1	$49^k(49n+7e)$ where e=3,5 or 6	T
2.	(1,1,10)	2m	$4^k(16n+6)$	4
		5m	$4^k(16n+6)$	T
3.	(1,1,14)	8m, 8m+2	$49^k(49n+7e)$ where e=3, 5 or 6	T
4.	(1,1,15)	5m	$9^k(9n+3)$	T
5.	(1,1,18)	2m, 9m	$9n\pm3$, $4^k(16n+14)$	T
6.	(1,1,20)	4m, 5m	$4^k(8n+3)$, $8n+7$	T
7.	(1,1,25)	5m	$4^k(8n+7)$	T
8.	(1,1,27)	9m	$9^k(9n+6)$	T
9.	(1,1,30)	5m	$9^k(9n+6)$	T
10.	(1,2,9)	2m, 3m	$4^k(16n+14)$	T
11.	(1,2,11)	2m, 11m	$4^k(16n+10)$	T
12.	(1,2,12)	2m, 3m	$4^k(16n+10)$	T
13.	(1,2,15)	3m	$25^k(25n\pm5)$	T
14.	(1,2,18)	3m, 4m	$4^k(8n+7)$	T
15.	(1,2,22)	4m, 11m, 8m+1	$4^k(8n+5)$	T
16.	(1,3,5)	3m	$25^k(25n\pm10)$	T
17.	(1,3,7)	7m	$9^k(9n+6)$	T
18.	(1,3,8)	4m, 3m	$4n+2$, $4^k(16n+10)$	T
19.	(1,3,14)	3m	$4^k(16n+6)$	T
20.	(1,3,16)	2m, 8m+1	$4n+2, 16n+8$, $9^k(9n+6)$	T

[1] f represents all positive integers of the forms A except those of the forms B and f represents no integer of the forms B.

(Table VII continued)

No.	f	A[1]	B[1]	Reference
21.	(1,3,20)	3m	4n+2, $25^k(25n+10)$	T
22.	(1,4,5)	4m, 5m, 4m+1	8n+7, $4^k(8n+3)$	T
23.	(1,4,7)	4m, 4m+1	$49^k(49n+7e)$ where e=3, 5, or 6	T
24.	(1,5,6)	2m, 5m	$4^k(16n+2)$	T
25.	(1,6,7)	7m	$9^k(9n+3)$	T
26.	(1,6,8)	2m+1, 3m, 4m	8n+3, $4^k(8n+5)$	T
27.	(1,6,42)	7m	8n+5, $9^k(3n+2)$	T
28.	(1,7,7)	4m, 4m+3	$49^k(7n+e)$ where e=3,5 or 6	T
29.	(1,7,12)	28m	$9^k(9n+6)$	T
30.	(1,7,24)	28m	$9^k(9n+3)$	T
31.	(1,11,22)	22m	$4^k(16n+14)$	T
32.	(2,2,5)	8m+7	-	3
		4m, 5m	$4^k(8n+3)$	T
33.	(2,3,4)	2m, 3m	$4^k(16n+10)$	T
34.	(2,3,21)	7m	$9^k(3n+1)$	T
35.	(2,3,24)	2m+1, 4m	3n+1, 8n+1, $4^k(8n+7)$	T
36.	(2,11,22)	44m, 88m+11	$4^k(8n+7)$	T
37.	(3,4,8)	3m, 4m	4n+1, 4n+2, $4^k(16n+10)$	T
38.	(3,8,21)	28m	$9^k(3n+1)$	T

My addition (May 6 '93): (1, 1, 2t) t a sum of 2 sqs, reps. tm for m odd

Proof. (1,1,2) rep. m. Mult by t

Remark Also OK for ~~tm~~ even if not $4^k(16n+14)$.

[1] f represents all positive integers of the forms A except those of the forms B and f represents no integer of the form B.

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