February 4, 2004

In the manuscript "Discriminant bounds for spinor regular ternary quadratic lattices" (hereafter referred to as [CE]), bounds are obtained for the prime-power factors of positive integers that could possibly occur as discriminants of spinor regular ternary quadratic lattices. The purpose of these notes is to state and prove several results that, together with the results of the on-going computer search for spinor regular ternary forms, can be used to eliminate some potential discriminants (integers for which the prime-power bounds are met) from further consideration.

In order to conveniently switch back and forth between the languages of forms and lattices, let us first set some notations. In keeping with the convention adopted in the listing of spinor regular ternary forms produced by Jagy (hereafter referred to as [J]), the discriminant of a ternary quadratic form f will be $\frac{1}{2}$ times the determinant of the matrix of second partial derivatives of the form. This discriminant will be denoted by df. According to the correspondence described on page 2 of [CE], there is an even quadratic lattice L_f associated to such a form f. The lattice L_f is "normalized", in the terminology of [CE], precisely when the form f is primitive. The discriminant dL_f used in [CE] equals the determinant of the matrix of second partial derivatives of f; thus, $dL_f = 2df$. A positive integer d will be said to be a "regular discriminant" ("spinor regular discriminant", respectively) if there is a regular (spinor regular, respectively) normalized ternary lattice L for which $dL = d$ (or, equivalently, there is a regular (spinor regular, respectively) primitive ternary form f for which $2df = d$). The sets of all regular and spinor regular discriminants will be denoted by $\mathcal{R}$ and $\mathcal{S}$, respectively.

The best general upper bound obtained in [CE] for the power of 2 appearing in an integer in $\mathcal{S}$ is 28. Our first result will allow us to eliminate certain integers of the form $2^t d_0$ as possible elements of $\mathcal{S}$ based on the results of the calculations already completed in [J].

RESULT 1: Let d_0 be an odd positive integer. If there exists an integer $k \geq 4$ for which none of $2^k d_0$, $2^{k+1} d_0$, $2^{k+2} d_0$ and $2^{k+3} d_0$ lies in $\mathcal{S}$, then no integer of the form $2^t d_0$ with $t \geq k$ lies in $\mathcal{S}$.

The proof of this result rests on several results that can be extracted from [CE]. These are summarized as follows:

(i) Suppose that L is spinor regular and $\text{ord}_2 dL \geq 8$, then $\lambda_2(L)$ is spinor regular. (This follows from Prop. 3.2 and Prop. 5.5(a) of [CE].)

(ii) If $8|dL$ and L_2 is not split by $\mathbb{H}$ or $\mathbb{A}$, then $d(\lambda_2(L)) = \frac{1}{2}dL$ or $d(\lambda_2(L)) = \frac{1}{16}dL$. (This result can be isolated from the proof of Lemma 2.5 in [CE].)

Proof of Result 1: If the result is not true, then there exists a smallest integer t_0 exceeding k for which $2^{t_0}d_0$ lies in $\mathcal{S}$. By the assumption that none of $2^k d_0$, $2^{k+1}d_0$, $2^{k+2}d_0$ and $2^{k+3}d_0$ lies in $\mathcal{S}$, it must be that $t_0 \geq k+4 \geq 8$. Then $\lambda_2(L)$ is spinor regular, by (i). But then $d(\lambda_2(L)) \in \mathcal{S}$ and $d(\lambda_2(L)) = 2^s d_0$ where s equals either $t_0 - 1$ or $t_0 - 4$, by (ii). In either case, $k \leq s < t_0$. This contradicts the minimality of t_0, and the proof is complete.

Our second result limits the combinations of distinct odd prime divisors that can occur in a spinor regular discriminant.

RESULT 2: If p and q are distinct odd primes such that some element of $\mathcal{S}$ is divisible by pq, then there exists an element of $\mathcal{R}$ divisible by pq.

Proof of Result 2: Recall that in [CE], a lattice L is said to "behave well at r", for a prime r, if either L_r is split by $\mathbb{H}$ or $2r^2$ does not divide dL. Let L be a spinor regular ternary lattice with $pq|dL$. Applying the argument in the proof of Lemma 4.1 of [CE] to all prime divisors of dL different from p and q, we may assume without loss of generality that L behaves well at all primes $r \neq p, q$. In particular, $\theta(O^+(L_r)) \supseteq \mathfrak{u}_r \dot{\mathbb{Q}}_r^2$ holds for all $r \neq p, q$, by Lemma 3.4 of [CE].

Now consider the splitting of L_p, which has the form

$$L_p \cong \langle a, p^\beta b, p^\gamma c \rangle, \text{ where } 0 \leq \beta \leq \gamma \text{ are integers and } a, b, c \in \mathfrak{u}_p.$$

Suppose first that $\beta = 1$. Then it follows from repeated application of Lemma 2.7 of [CE] that

$$\lambda_p^{(\gamma-1)}(L_p) \cong \langle a', pb', pc \rangle, \text{ where } a', b' \in \mathfrak{u}_p.$$

Each of $\lambda_p(L), \lambda_p^2(L), \ldots, \lambda_p^{(\gamma-1)}(L)$ is spinor regular by Prop. 3.2, and $\theta(O^+(\lambda_p^{(\gamma-1)}(L_p))) \supseteq u_p\dot{\mathbb{Q}}_p^2$ holds since the p-modular component of $\lambda_p^{(\gamma-1)}$ has rank two. Now consider the case $\beta \geq 2$. If β is even and $\beta \neq \gamma$, then

$$\lambda_p^{\frac{\beta}{2}}(L_p) \cong \langle a, b, p^{\gamma-\beta}c\rangle,$$

$\lambda_p^{\frac{\beta}{2}}(L)$ is spinor regular, and $\theta(O^+(\lambda_p^{\frac{\beta}{2}}(L_p))) \supseteq u_p\dot{\mathbb{Q}}_p^2$ holds since the unimodular component of $\lambda_p^{\frac{\beta}{2}}(L_p)$ has rank two. If β is even and $\beta = \gamma$, then

$$\lambda_p^{\frac{\beta-2}{2}}(L_p) \cong \langle a, p^2b, p^2c\rangle.$$

So $\lambda_p^{\frac{\beta-2}{2}}(L)$ is spinor regular, and $\theta(O^+(\lambda_p^{\frac{\beta-2}{2}}(L_p))) \supseteq u_p\dot{\mathbb{Q}}_p^2$ holds since the p^2-modular component of $\lambda_p^{\frac{\beta-2}{2}}(L_p)$ has rank two. Finally, if β is odd, then

$$\lambda_p^{\frac{\beta-1}{2}}(L_p) \cong \langle a, pb, p^{\gamma-\frac{\beta-1}{2}}c\rangle.$$

Applying the argument above for the case $\beta = 1$ to the lattice $\lambda_p^{\frac{\beta-1}{2}}(L)$ then leads to a spinor regular lattice L' for which $\theta(O^+(L'_p)) \supseteq u_p\dot{\mathbb{Q}}_p^2$. In all cases, we ultimately obtain a spinor regular lattice L' for which $p|dL'$ and $\theta(O^+(L'_p)) \supseteq u_p\dot{\mathbb{Q}}_p^2$. Moreover, $\theta(O^+(L'_r)) \supseteq u_r\dot{\mathbb{Q}}_r^2$ holds for all $r \neq p, q$ since applying the transformation λ_p only results in scaling the form on L_r and thus leaves the spinor norm group $\theta(O^+(L_r))$ unchanged.

Repeating the steps in the preceding paragraph for the prime q then leads to a lattice K for which $pq|dK$, K is spinor regular, and $\theta(O^+(K_t)) \supseteq u_t\dot{\mathbb{Q}}_t^2$ holds for all primes t. The latter containments imply that the genus and spinor genus of K coincide; thus, K is regular. Hence, dK is an element of $\mathcal{R}$ divisible by pq. This completes the proof of Result 2.