I will adopt the language of (quadratic) lattices and spaces. If L is a $\mathbb{Z}$-lattice, then $L^{(n)}$ denotes the $\mathbb{Z}$-lattice whose underlying set is L but with the quadratic form scaled by the factor n. We say that a $\mathbb{Z}$-lattice M represents another $\mathbb{Z}$-lattice L if M has a sublattice M' which is isometric to L, or equivalently, there exists an isometry sending L into M. Our basic assumption is:

L and M are $\mathbb{Z}$-lattices such that $d(M) = nd(L)$, where n is a positive integer, and M represents $L^{(n)}$.

(This is the same as the hypothesis in your Conjecture stated in terms of polynomials and "homotheties".)

Notice that by the assumption we may assume that $L^{(n)}$ and M are $\mathbb{Z}$-lattices on the same space, say V. There are couple of immediate consequences. First, if M' is a sublattice of M which is isometric to $L^{(n)}$, then $[M : M'] = n$. Second, we have the "dual" statement which says that L also represents $M^{(n)}$. As you already indicated in your notes, the fact that n is an integer is crucial here.

Now, let K be a $\mathbb{Z}$-lattice in $\text{gen}(M)$. We claim that K represents $H^{(n)}$ for some $H \in \text{gen}(L)$. Let τ be an isometry of V which sends $L^{(n)}$ into M, and let $M' = \tau(L^{(n)})$. Since $K \in \text{gen}(M)$, for each prime p there exists an isometry σ_p of V_p which sends M_p to K_p. For almost all p (in fact, for those p that do not divide n), $M'_p = M_p$, and so there is a sublattice K' of K with $K'_p = \sigma_p(M'_p)$ for all p. Moreover, $L_p^{(n)}$ is isometric to K'_p at each p; $\sigma_p \tau$ is an isometry between them. Therefore, $K' \in \text{gen}(L^{(n)})$ and so K' is isometric to $H^{(n)}$ for some $H \in \text{gen}(L)$.

Now we can "reverse" the steps to show that for every $H \in \text{gen}(L)$ there will be an $K \in \text{gen}(M)$ which represents $H^{(n)}$. Since $H^{(n)}$ is in $\text{gen}(L^{(n)})$, there will be isometry σ_p of V_p such that $\sigma_p(H_p^{(n)}) = L_p^{(n)}$ at each prime p. Then $\tau\sigma_p(H_p^{(n)}) = M'_p$ for all p. Now, for $p \nmid n$ we have $M_p = M'_p$ and so $(\tau\sigma_p)^{-1}(M_p) = H_p^{(n)}$. Therefore there will be a $\mathbb{Z}$-lattice K on V such that $K_p = (\tau\sigma_p)^{-1}(M_p)$. Obviously, $K \in \text{gen}(M)$ and K represents $H^{(n)}$, in fact, K contains $H^{(n)}$!