Integral Positive Ternary Quadratic Forms

William C. Jagy

January 16, 2014

1 Introduction

I do not expect to publish this. I've just been adding material since about 1996, whenever the mood struck. The date above is the most recent day I edited the file, placed there by the typesetting program.

News in 2008: This and all the related unpublished notes by Kap are now on

`http://zakuski.math.utsa.edu/~kap/`

including [39] under the filename Kap_Classification_1996.pdf, which explains why you should believe that there are no other regular quadratic forms than our list of 891 proved and 22 candidates. Kap's classification rules are an easier and more tractable version of the rules in Chan and Earnest [8]. I'm keeping, and updating, this document on the same site under the filename Jagy_Encyclopedia.pdf. Meanwhile, Byeong-Kweon Oh [44] of Seoul National University, has proved 8 of the 22 regular. From the "odd" forms in Table 1 of [31] Oh has proved one of the discriminant 1125 forms (1125: 2 7 22 -6 1 1) and discriminant 4500 regular. From the "even" forms in Table 2 of [31] he has proved discriminants 1008, 2112, 2880, 6336, 8000, 14400 regular. So at this point there are still 13 forms in Table 1 without proof but just one form in Table 2 needing proof.

The article [31] by Jagy, Kaplansky, and Schiemann identifies all possible regular positive ternary quadratic forms with integer coefficients. The word ternary just means three variables, the word positive means that the associated symmetric matrix is positive definite. So, writing the form as $g(x, y, z)$, the value of g is an integer when (x, y, z) are integers, and is a positive integer ('natural number') unless x, y, z are all zero. There are at most 913 regular ones (up to equivalence). During preparation of that article, many of the new forms we found were proved regular, but the proofs are not included in [31]. We never mentioned the word 'equivalence' in that article, instead giving a representative of each class according to the reduction scheme in Schiemann's dissertation. Alexander did publish his reduction criteria [48]. Some of the significance of the work is discussed by Andrew G. Earnest in [19].

I will prove several forms regular in this note. Meanwhile, 22 of the forms in our list of 913 were not proved regular; we call those 'candidates'.

It seems to have become standard [4] to describe a ternary form by its six coefficients in a specific order: given variables named (x, y, z), the 6-tuple $a\ b\ c\ d\ e\ f$ refers to $T(x, y, z) = ax^2 + by^2 + cz^2 + dyz + ezx + fxy$. Notice the funny yz, zx, xy sort of 'cyclic' order. I suspect that this order was chosen because it respects permutations of the variables x, y, z, but it takes some getting used to. Anyway, if all of d, e, f are even numbers we will call the form 'even'. John Horton Conway has suggested calling our 'even' forms "integer-matrix," or perhaps "matrix-integral"-see page 3 of [10]. This is reasonable, as then the usual symmetric matrix for $a\ b\ c\ d\ e\ f$, that is

$$\begin{pmatrix} a & f/2 & e/2 \\ f/2 & b & d/2 \\ e/2 & d/2 & c \end{pmatrix},$$

is made up of integers. If at least one of d, e, f is odd we will call the form 'odd'. I suspect Conway would say these 'odd' forms were "integer-valued but not integer-matrix." The 'discriminant' of a form is the determinant of the associated symmetric matrix for even forms, but 4 times that for odd forms. Meanwhile, we say that the form 'represents' an integer n if there are integers x, y, z such that $n = ax^2 + by^2 + cz^2 + dyz + ezx + fxy$. The word 'regular' indicates that all natural numbers not ruled out by congruence considerations are represented. As we put it in [31], $g(x, y, z)$ is regular if the solvability of $g \equiv n \bmod k$ for every k implies the solvability of $g = n$. Here n and k denote positive integers ('natural numbers'). If there is a natural number n such that $g \equiv n \bmod k$ is solvable for every k but $g = n$ is not solvable in integers, we call n 'sporadic' with regard to the form g.

There is a milder condition on positive ternaries called "spinor regularity." A reference showing how to find all of them is [8]. I found the spinor regular ternaries with discriminant up to a large finite bound, see `http://www.research.att.com/~njas/lattices/Jagy.txt` .

Notice that a form does represent each of its 'diagonal' coefficients (a, b, c); just set the appropriate one of the variables (x, y, z) to 1 and the others to 0. Furthermore, when the form given is a 'reduced' representative for its equivalence class, using *any* of the methods for reduction that have been proposed, the number a is the smallest natural number represented by the form. In the article [31], each form is preceded by its discriminant, so that the form $x^2 + 2y^2 + 3z^2$ appears as 6: 1 2 3 0 0 0.

Articles with bits and pieces about quadratic forms in three or more variables are: [4]. [17]. Recent survey [16]. [27] [28]. [30]. [31]. [33] [34]. [32]. [36]. [37] [38] . There are also the books [35], [47], [15], and recent [10].

All our efforts depend heavily on the works of George Leo Watson; see, for example, [59], [58]. In particular, see part 7 in Watson's unpublished dissertation [55] if you can find a copy.

First I describe the techniques used. Then I present the proofs, first some 'odd' forms, then a larger number of 'even' forms.

2 Inequalities

Alexander Schiemann [48] defines a positive ternary quadratic form

$$T(x, y, z) \quad = \quad ax^2 + by^2 + cz^2 + dyz + ezx + fxy$$

as **reduced** when:

$$0 < a \leq b \leq c,$$

$$-b < d \leq b,$$

$$0 \leq e \leq a,$$

$$0 \leq f \leq a,$$

$$a + b \geq -d + e + f,$$

$$\text{if } \; e = 0 \; \text{ or } \; f = 0 \; \text{ then } \; d \geq 0,$$

$$\text{if } \; a = b \; \text{ then } \; |d| \leq e,$$

$$\text{if } \; b = c \; \text{ then } \; e \leq f,$$

$$\text{if } \; f = a \; \text{ then } \; e \leq 2d,$$

$$\text{if } \ e = a \ \text{ then } \ f \leq 2d,$$

$$\text{if } \ d = b \ \text{ then } \ f \leq 2e,$$

$$\text{if } \ a + b + d - e - f = 0 \ \text{ then } \ 2a - 2e - f \leq 0.$$

The last condition is funny looking, but I checked the software he sent me and that is exactly what he intended. He proves in his dissertation that there is one and only one reduced form in each equivalence class.

If we define Δ as the absolute value of the Brandt-Intrau discriminant, so that

$$\Delta = 4abc + def - ad^2 - be^2 - cf^2,$$

it follows from the inequalities above that

$$2abc \leq \Delta \leq 4abc.$$

I wish people would tell me these things. I found the statement and most of a proof in Watson's book [56], in the section on Minkowski reduction, pages 27-29.

In [31] and in most of this note, the "discriminant" is given as Δ when at least one of d, e, f is odd, but $\frac{\Delta}{4}$ when d, e, f are all even. I hope you can adjust.

3 More inequalities

Hanke once asked me how I got nice bounds on the variables in programming a computer search on the ellipsoid $T(x, y, z) \leq M$ for some large positive M, where $T(x, y, z) = ax^2 + by^2 + cz^2 + dyz + ezx + fxy$ is a positive ternary. Well,

$$T(x, y, z) = (x \ y \ z) \cdot \begin{pmatrix} a & f/2 & e/2 \\ f/2 & b & d/2 \\ e/2 & d/2 & c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

It is simple enough to confirm that the gradient of T, written as a column vector, is

$$\nabla T(x, y, z) = \begin{pmatrix} 2ax + fy + ez \\ fx + 2by + dz \\ ex + dy + 2cz \end{pmatrix} = 2 \begin{pmatrix} a & f/2 & e/2 \\ f/2 & b & d/2 \\ e/2 & d/2 & c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

We are going to use the method of Lagrange multipliers. It follows from the compactness of the ellipsoid $T \le M$ (the Gram matrix has positive eigenvalues) that any of the variables x, y, z achieves its maximum. It follows from the strict convexity of the ellipsoid that these maxima are achieved at boundary points where $T = M$. Finally it follows from the smoothness of the boundary that Lagrange multipliers will locate all such points.

Give a name F to the matrix, so

$$F = \begin{pmatrix} a & f/2 & e/2 \\ f/2 & b & d/2 \\ e/2 & d/2 & c \end{pmatrix}.$$

We need the other gradients,

$$\nabla x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = e_1, \nabla y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e_2, \nabla z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = e_3.$$

So, given

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

we are solving the system

$$\begin{aligned} 2FX &= \lambda e_i \\ X'FX &= M \end{aligned}$$

$X' = (x\ y\ z)$ being the transpose of X.

The matrix F has an inverse that we will cleverly name F^{-1}. So we find

$$X = \left(\frac{\lambda}{2}\right) F^{-1} e_i.$$

The fraction doesn't help or hurt, so we will name $t = \left(\frac{\lambda}{2}\right)$ and get

$$X = tF^{-1}e_i.$$

Notice that F and so F^{-1} are symmetric. Next we use $X'FX = M$, or $te_i'F^{-1}FF^{-1}e_it = M$, whence $te_i'F^{-1}e_it = M$. Now $e_i'F^{-1}e_i$ is the i, i entry of F^{-1}, which we write as F_{ii}^{-1}. So we find

$$t^2 F_{ii}^{-1} = M.$$

or

$$t = \sqrt{\frac{M}{F_{ii}^{-1}}}.$$

Recalling $X = tF^{-1}e_i$ gives us

$$X = \begin{pmatrix} tF_{1i}^{-1} \\ tF_{2i}^{-1} \\ tF_{3i}^{-1} \end{pmatrix},$$

So, maximizing $x_1 = x, x_2 = y, x_3 = z$ leads us to the value

$$x_i = tF_{ii}^{-1} = \sqrt{\frac{M}{F_{ii}^{-1}}} F_{ii}^{-1},$$

or

$$x_i = \sqrt{MF_{ii}^{-1}}$$

In conclusion,

$$|x| \leq \sqrt{MF_{11}^{-1}}, \qquad |y| \leq \sqrt{MF_{22}^{-1}}, \qquad |z| \leq \sqrt{MF_{33}^{-1}}.$$

If supreme efficiency is needed, one then fixes, say, a value of z, and notes that the ellipsoid section described is an ellipse. The Lagrange multiplier method can be repeated to find, say, the maximum and minimum of y, which are no longer of the same absolute value. Finally, with values of y, z chosen, bounds on x come from the quadratic formula.

I worked up an example to illustrate the possible need. What follows is an ellipsoid of revolution of a cigar shape, long in the direction of the vector (1,1,1) and narrow in any orthogonal direction. As a result, the volume of the cube given by the bounds $|x| \leq \sqrt{MF_{11}^{-1}}, |y| \leq \sqrt{MF_{22}^{-1}}, |z| \leq \sqrt{MF_{33}^{-1}}$ is quite large compared with the volume of the ellipsoid. The volume of the ellipsoid is very close to the number of integer triples to be checked that satisfy $T(x, y, z) \leq M$. Think about it.

Given a large integer $W > 0$, let

$$\begin{aligned} T(x, y, z) &= (x + y + z)^2 + 3W(x - y)^2 + W(x + y - 2z)^2 \\ &= (4W + 1)(x^2 + y^2 + z^2) - (4W - 2)(yz + zx + xy). \end{aligned}$$

In the ellipsoid $T \leq 9W^2$, we find the integer point (W, W, W), at a distance of $\sqrt{W^2 + W^2 + W^2} = W\sqrt{3}$ from the origin. However, in the plane $x+y+z =$

0, we get a circular section of the ellipsoid, and letting t now be the distance of a point from the origin, taking $x = t/\sqrt{2}, y = -t/\sqrt{2}, z = 0$ tells us that $\sqrt{x^2 + y^2 + z^2} \leq \sqrt{\frac{3W}{2}}$. Anyway much smaller than $W\sqrt{3}$.

As to the comparison of volumes, the cube given by the raw x, y, z bounds has volume at least $8W^3$, being at least $2W$ on a side. Using the discriminant recipe $\Delta = 4abc + def - ad^2 - be^2 - cf^2$ gives $\Delta = 432W^2$. The volume of the ellipsoid $T \leq M$ should be

$$\frac{8\pi M^{3/2}}{3\sqrt{\Delta}}.$$

With $T \leq 9W^2$, we have $M = 3W^2$, so the ellipsoid has volume $2\pi\sqrt{3}W^2$. Finally the volume of the cube divided by the volume of the cigar is

$$\frac{4W}{\pi\sqrt{3}} = \left(\frac{4}{\pi\sqrt{3}}\right) W,$$

a bit larger than $\frac{11W}{15}$.

4 Tables

The first thing to look at is a copy of table 5, pages 111-113 in Dickson's book [15]. This shows the 102 'diagonal' regular forms $(ax^2 + by^2 + cz^2)$, each together with the numbers not representable by the listed form. I know of no other list that puts the forms and congruence obstructions side by side.

It is also impossible to make sense of any of this without seeing some tables of ternary forms in some reduction scheme. One needs to see when forms are alone in their genera, like $x^2 + y^2 + z^2$, but just as importantly genera with two or more equivalence classes. Jones and Pall [36] give a list of complete genera for all the 102 regular diagonal forms.

Tables for *binary* forms (of small discriminant) are readily available in undergraduate number theory books, given in Gauss reduced form. Rose's book [47] includes indefinite binary forms. The book of Buell [5] gives considerable detail on both definite and indefinite forms, including reduction, composition, the Gauss method of cycles for equivalence classes of indefinite forms, and so on.

For positive ternary quadratic forms, the quickest method is probably to check Neil Sloane's web site,

```
www.research.att.com/~njas
```

and

```
http://www.research.att.com/~njas/lattices/Jagy.txt
```

Then click on Catalogue of Lattices. Then Brandt-Intrau Ternary Forms. Then Ternary Quadratic Forms. Then either Odd TQF or Even TQF . Caution: the website had the words 'Odd' and 'Even' backwards last time I checked. Furthermore: the number before the colon preceding each form is NOT the discriminant, it is just some cumulative count of the number of forms in the table. Finally, the discriminants for even forms are inflated by a factor of 4 (most authors do that, actually). So the discriminant of $x^2+y^2+z^2$ is given as 4. The tables were computed by Alexander Schiemann.

5 Recognizing Irregular Forms

Once a table of forms is available, split up into genera, recognizing the failure of a form to be regular is pretty easy: most irregular (positive) ternary forms miss some very small number that they should get. We call such numbers 'sporadics' to save space; it appears helpful to use the word 'sporadic' as both noun and adjective, depending upon context.

For example: even discriminant 7. There is a genus of two classes, with reduced representatives $x^2 + y^2 + 7z^2$ and $x^2 + 2y^2 + 2yz + 4z^2$. The genus represents all natural numbers other than 7^{2k+1}(nonresidues mod 7). That is, all such numbers are represented by at least one of the two forms given, usually by both. However, there is no solution in integers to $x^2+y^2+7z^2 = 3$, so the first form is irregular, and the number 3 is 'sporadic' for the first form. There is no solution in integers to $x^2+2y^2+2yz+4z^2 = 7$, so the second form is irregular, and the number 7 is sporadic for the second form. Further, it was not necessary to know precisely what numbers are eligible for the genus! The first form misses 3, but the second form gets it ($x = 1, y = 1, z = 0$), so the first form fails to be regular. The second form misses 7, which the first form gets ($x = 0, y = 0, z = 1$), so the second form fails to be regular. That's enough. Details on this are in [37].

In a genus of three or more classes, one may compare the (reduced) forms in pairs, or collect together all numbers represented by at least one of the forms up to some convenient bound, then check whether each of the forms gets everything in that list.

A word on bounds: these are positive forms, so a finite set of triples (x, y, z) needs to be checked to find out what numbers a given form represents up to some bound M. Indeed, as *real numbers* in R^3 the set $\{\text{form} \leq M\}$ is an ellipsoid, and the number of lattice points in the ellipsoid is well approximated by its volume. The volume is just $CM^{3/2}/D^{1/2}$, where the constant C can be inferred from the volume of a sphere, and D is the discriminant. So I evolved the idea of checking a constant number of lattice points by making M proportional to $D^{1/3}$.

Here is a situation that gives some idea of conservatively large bounds: the even discriminant 8000 has many genera. One of those is a genus of two classes, given by A: 3 27 107 -26 2 2 , and B: 12 27 28 -12 8 4 . Now in reduced forms, the first coefficient is the smallest natural number represented, meaning that we are told that B misses the number 3 and is irregular. Not obvious: the form B (and so the genus) gets the number 803, but form A misses the number 803, so A is also irregular. That 803 is the largest among forms I checked; we could call 803 the largest 'first sporadic' I ever found. It is also the largest in the sense of $D^{1/3}$ that I mentioned earlier. As a result, I came to the experimental result that any form that represents all eligible numbers up to $41D^{1/3}$ was very likely regular. Also noteworthy in this regard is 29: 1 2 4 1 1 0, which has 87 as its 'first sporadic'. Now, there are some 'candidate' forms in [31]. The candidates each hit their eligible numbers up to 2 million. If one of the candidates is actually irregular, that throws my experimental result out the window.

6 Representing Prescribed Numbers

Irving Kaplansky wrote an article on forms that represent all odd natural numbers [38]. This question is a finite search, for an elementary bound on determinants of matrices shows that any form representing the numbers $\{1, 3, 5, 7\}$ has a discriminant no larger than 77. That bound is achieved, by $x^2 + xy + 3y^2 + 7z^2$, but this form misses 13 and 17 and probably other odd numbers. For even forms, the discriminant is no larger than 15, achieved by $x^2 + 3y^2 + 5z^2$, but that one misses 11 and 15.

If a ternary form hits the numbers $\{1, 2, 3, 5\}$, its discriminant is bounded by 40 (or 10 for even forms in my convention). Examples are $x^2 + 2y^2 + 5z^2$, which is regular, and $x^2 + 2y^2 + yz + 5z^2$, which is not.

The record for representing consecutive small numbers is $x^2 + 2y^2 + yz +$

$4z^2$, of discriminant 31. It is not regular. It represents all the numbers from 1 to 30, then misses the number 31. Indeed, it appears that it only misses numbers divisible by 31 (I can't be certain about that). The form is called 'Little Methuselah' in John Horton Conway's book [10].

Another form with prime discriminant that appears to represent all natural numbers prime to its discriminant is 29: 1 2 4 1 1 0, or $2y^2 + yz + 4z^2 + zx + x^2$. This form is given incorrectly in William Duke's survey [16]. Anyway, if one adds a particular binary to it, giving $2y^2+yz+4z^2+zx+x^2+29u^2+29uv+29v^2$, the form in five variables represents all natural numbers except 290. Conway has referred to this form as 'Methuselah' in talks. Conway's conjecture is that this quinary form gives the world record, that any positive quadratic form that represents the numbers from 1 to 290 represents all natural numbers (see [16]). The Fifteen Theorem [10] is for even forms only, and states that a positive even form that represents the numbers from 1 to 15 does represent all natural numbers. Any such positive 'universal' form has at least four variables. An acceptable proof of the Fifteen Theorem was eventually given by Manjul Bhargava [3]. Conway himself did not publish his work with Schneeberger on the problem, instead writing a short article [11] to serve as a preface for Bhargava's paper.

A positive ternary form must miss an infinite set of natural numbers, containing at least one arithmetic progression. In contrast, an indefinite ternary form such as $xy + z^2$ may be universal (including negative integers) yet be irreducible. An indefinite binary that represents a nonzero proportion of integers is reducible (such as $x^2 - y^2$ or xy.) A positive binary fails to represent almost all natural numbers. Indeed, in the last chapter of volume II of his book 'Topics in Number Theory', William Judson LeVeque [42] shows that the count of natural numbers less than n that are expressible as $x^2 + y^2$ is approximately $\frac{Cn}{\sqrt{\log n}}$, where the constant $C = 0.7642...$ is given by a certain infinite product. I like to think of the expression $\frac{Cn}{\sqrt{\log n}}$ as being the geometric mean between n and the number of primes less than n, at least up to a constant multiple.

7 Diagonal Forms

A diagonal form has zero coefficients for the yz, zx, xy terms. The terminology is reasonable, as the associated symmetric matrix is then diagonal.

The discriminant of a diagonal form is just the product of the three coefficients (of x^2, y^2, z^2). It turns out that 102 diagonal forms are regular, a result in Jones's dissertation [32]. Proofs that all 102 specified are regular were published in [36].

The beginning of wisdom is the theorem that all positive integers n that are not of the form $4^k(8t+7)$ are expressible as the sum of three squares; so we say that any such n is represented by the quadratic form $x^2+y^2+z^2$. This result is due to Legendre(1798). Gauss proved it independently at roughly the same time, but his proof first appeared in his book Disquisitiones Arithmeticae (1801), section 291. It is proved in section 9.2 of Harvey Rose's book [47]. Rose gives the simplest-looking proof, which uses Dirichlet's famous result(1837) on primes in arithmetic progressions. It is also proved in Dickson's book [15]. Indeed, Dickson proves several related results, and gives a list of all 102 regular diagonal forms, together with the numbers each fails to represent. Finally, Burton Wadsworth Jones [35] gives a proof that illustrates the use of genus theory, but apparently refers eventually to Dirichlet as well. Conway gives a quick proof [10].

Once one form is proved regular, others follow: A:$x^2+2y^2+6z^2$ represents all positive integers except $4^k(8m+5)$. B:$x^2+y^2+5z^2$ represents all positive integers except $4^k(8m+3)$. C:$x^2+y^2+2z^2$ represents all positive integers except $4^k(16m+14)$. D:$x^2+2y^2+3z^2$ represents all positive integers except $4^k(16m+10)$. E:$x^2+y^2+4z^2$ represents all positive integers except $4^k(8m+7)$ and $8m+3$. You see, most forms have more than one congruence obstruction attached. F:$x^2+y^2+9z^2$ represents all positive integers except $4^k(8m+7)$ and $9m \pm 3$.

If we leave diagonal forms for a moment: G:$x^2+y^2+3z^2+yz+zx$ represents all positive integers except $4^k(16m+6)$. The form has discriminant 10. Note that form G appears in our shorthand as 10 : 1 1 3 1 1 0. Next, H:$2x^2+2y^2+3z^2+2yz+2zx+2xy$ represents all positive integers except $4^k(8m+1)$. The form has discriminant 7, of all things. To provide a form that misses only those numbers of the form $4^k(16m+2)$, we may use either the even form I: 1 3 5 2 0 0 or the odd form J: 1 1 5 1 1 1. Both I and J are considered discriminant 14 in our notation.

For each form A-J, there is a short proof of regularity, using explicit formulas for expressing the form (multiplied by its discriminant) as the sum of three squares.

8 Ramanujan and $x^2 + y^2 + 10z^2$

Apparently Dickson became interested in positive ternary forms because of a 1916 article of Ramanujan [46]. Ramanujan was investigating diagonal forms in four variables; in that article he commented that $x^2 + y^2 + 10z^2$ represents all even numbers not of the form $4^k(16m+6)$, but the odd numbers (3,7,21,31,...) not expressible as $x^2 + y^2 + 10z^2$ appeared to follow no definite pattern. Later computations confirmed that the form misses odd numbers as large as 679 and 2719, but appears to miss no odd numbers larger than 2719. Jumping to the present day, Ono and Soundararajan [45] have shown that all odd numbers missed (these are 'sporadic') are squarefree, and that an appropriate Generalized Riemann Hypothesis implies that 2719 is the largest sporadic.

If a positive integer n is even but not of the form $4^k(16m + 6)$, then $n/2$ is not of the form $4^k(8m + 3)$, so that we may write $\frac{n}{2} = r^2 + s^2 + 5t^2$. It follows that $n = (r - s)^2 + (r + s)^2 + 10t^2$.

If a number n is divisible by 5 and not of the form $4^k(16m+6)$, then $n/5$ is not of the form $4^k(16m + 14)$. We may write $\frac{n}{5} = r^2 + s^2 + 2t^2$. It follows that $n = (2r - s)^2 + (r + 2s)^2 + 10t^2$. This one was pointed out by Ono and company.

Now there is just one other equivalence class in the same genus as 1 1 10 0 0 0, containing the form 2 2 3 2 0 0, which refers to $2x^2+2y^2+2yz+3z^2$. The results of Jones on 'regularity of a genus' show that every eligible number n is represented either by 1 1 10 0 0 0 or by 2 2 3 2 0 0. Usually both.

Just to make trouble: passing from even forms back to a Regular odd form. You see, the form G from an earlier section, $x^2 + y^2 + 3z^2 + yz + zx$ represents all positive integers except $4^k(16m + 6)$. Fix the letters x, y, z for this paragraph. If z is even, write $x = r-t, y = s-t, z = 2t$, the quantity t is an integer, and the polynomial G becomes $r^2+s^2+10t^2$. If the sum $(x+y)$ is even, switch to $x = r+s, y = -r+s, z = t$, with result $2r^2+2s^2+2st+3t^2$. If z and $(x+y)$ are both odd, use $x = r+s+t, y = r-s, z = -t$; again, r, s, t will be integers, the result is again $2r^2 + 2s^2 + 2st + 3t^2$. We have just explicitly constructed the Jones result, that every eligible number is represented by either 1 1 10 0 0 0 or 2 2 3 2 0 0.

How did I decide on the quantities z and $x+y$ in the preceding paragraph? Well, we know that $u^2 \equiv u \bmod 2$, so that $x^2 + y^2 \equiv x + y \bmod 2$. Thus the parity of z and $x+y$ clearly influences the parity of

$$x^2+y^2+3z^2+yz+zx = 3z^2+z(x+y)+(x^2+y^2);$$

the form cannot be even unless both z and $x+y$ are even. Note that x^2+y^2 might be 2 mod 4, in which case the result need not be divisible by 4 even when z and $x+y$ are both even.

9 Hsia , $x^2+y^2+10z^2$ and me

John S. Hsia, in a letter to Irving Kaplansky(dated June 1993), proved that $x^2+y^2+10z^2$ represents all eligible numbers of the form $3m+2$. This result is not mentioned in [45]. As all odd numbers are eligible, being not of the form $4^k(16m+6)$, it follows that $x^2+y^2+10z^2$ represents the entire arithmetic progression $6m+5$. Here's my proof of this fact.

We begin by assuming that a natural number $n \equiv 2 \bmod 3$ is represented by the genus, in particular by the genus mate $2x^2+2y^2+2yz+3z^2$. That is, we assume $n = 2x^2+2y^2+2yz+3z^2$, and show how to express n as $u^2+v^2+10w^2$.

First, check all 27 triples (x,y,z) with values mod3. With $n \equiv 2 \bmod 3$, one of two things happens: (A) $x \equiv \pm z \bmod 3$, or (B) $y \equiv z \equiv 0 \bmod 3$, but x is not divisible by 3.

For case (A), two formulas suffice:

$$9n = (2x-3y+z)^2+(2x+3y+4z)^2+10(x-z)^2,$$

$$9n = (2x-3y-4z)^2+(2x+3y-z)^2+10(x+z)^2.$$

Note that either one of the formulas suffices to prove that $x^2+y^2+10z^2$ represents all eligible multiples of 9; this result is a tiny part of the Ono square-free result. Now we're in the case where $x \equiv \pm z \bmod 3$. So, in one of the two formulas, all three of the linear combinations given describe numbers divisible by 3. Pretend it's the first one: we get

$$n = \left(\frac{2x-3y+z}{3}\right)^2+\left(\frac{2x+3y+4z}{3}\right)^2+10\left(\frac{x-z}{3}\right)^2.$$

It would be similar for the second formula.

In case (B), the above formulas don't immediately help. Here $y \equiv z \equiv 0 \bmod 3$, but x is not divisible by 3. Recall $n = 2x^2+2y^2+2yz+3z^2$. If

y, z are actually 0, then $n = 2x^2 = x^2 + x^2 + 10(0)^2$. If at least one of y, z is nonzero, we can apply the following result:

Lemma Given $m = 2y^2 + 2yz + 3z^2$, with m nonzero, and with m divisible by 3, we can construct $m = 2s^2 + 2st + 3t^2$, with s, t prime to 3. **Proof:** Induction on the power of 9 dividing m, combined with specific instances of Gaussian composition for binary forms.

(i) If $m \equiv \pm 3 \mod 9$, then $\frac{m}{3}$ satisfies the conditions to be represented as $\frac{m}{3} = a^2 + 5b^2$. Now one of a, b will be divisible by 3 and the other will not. We have explicitly $m = 2s^2 + 2st + 3t^2$, with $s = -a + 2b$ and $t = a + b$.

(ii) If $m \equiv \pm 9 \mod 27$, then $\frac{m}{9}$ satisfies the conditions to be represented as $\frac{m}{9} = 2c^2 + 2cd + 3d^2$. Both c and $c + d$ will be prime to 3. We have explicitly $m = 2s^2 + 2st + 3t^2$, with $s = c + 4d$ and $t = 2c - d$.

(iii) If $m \equiv 0 \mod 27$, (the induction step) then we will have $\frac{m}{9} = 2e^2 + 2ef + 3f^2$. Both e and f will be prime to 3. We have explicitly $m = 2s^2 + 2st + 3t^2$, with $s = e - 3f$ and $t = 2e + 3f$.

This establishes the Lemma, which is very much in the spirit of the Jones dissertation [32]. So we now have $n = 2x^2 + 2s^2 + 2st + 3t^2$, with the important point being $x \equiv \pm t \mod 3$. That means we are back in case (A), and can use one of the two formulas given. Done.

10 Homotheties: the equation $P'AP = B$

Here the notation P' refers to the transpose of P. The matrices A and B are symmetric positive definite and have integer or half-integer entries, so that in any case $2A$ and $2B$ have integer entries. The matrix P is required to have integer entries but is probably not symmetric. Furthermore P may have determinant other than ± 1, so that we allow A and B to have different determinants. There will simply be no solution if $\det B / \det A$ is not an integer square.

The rows of P find their way into substitution formulas relating one quadratic form to another. For example, if A is the matrix of the form 26:1 3 9 2 0 0 with B being the matrix of 26: 2 2 7 0 2 0, which is in the same genus. These forms are not equivalent, so there will be no solutions without some multiplier. That is, we will show a solution to $P'AP = 9B$. We will then show the same information in formulas with substitutions.

$$\begin{pmatrix} 1 & -2 & 1 \\ 2 & -1 & -1 \\ 7 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 9 \end{pmatrix} \begin{pmatrix} 1 & 2 & 7 \\ -2 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} = 9 \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 7 \end{pmatrix}.$$

Let

$$g(X, Y, Z) = X^2 + 3Y^2 + 2YZ + 9Z^2.$$

Let

$$h(x, y, z) = 2x^2 + 2y^2 + 7z^2 + 2zx.$$

Then

$$g\left(x + 2y + 7z, -2x - y + z, x - y + z\right) = 9 \; h(x, y, z),$$

which means that

$$g\left(\frac{x + 2y + 7z}{3}, \frac{-2x - y + z}{3}, \frac{x - y + z}{3}\right) = h(x, y, z).$$

So, if A represents the form f and B represents the form g, we are describing a linear change of variable. If we use capital V to refer to the column vector with entries x, y, z, then a solution P corresponds to the formula $f(PV) = g(V)$. Notice that when the determinant of P is equal to 1, so that P is called 'unimodular', then the inverse of P is also unimodular. In this case the forms are called 'equivalent', as the change of variables is invertible.

Often the word 'homothety' is used for the mapping $A \longmapsto P'AP$ when the determinant of P is not restricted to 1. Many examples of this occur in the present article. With H, M matrices associated with forms in the same genus, I get some cheap proofs of regularity from multiple solutions of the equation $P'HP = 4M$. In considering $x^2 + y^2 + 10z^2$, I used two solutions of $P'HP = 9M$. To show that $x^2+y^2+10z^2$ represents eligible even numbers, we solved $P'HP = 2R$, where the matrix R is associated with the known regular form $x^2 + y^2 + 5z^2$. For multiples of 5, we solved $P'HP = 5R$, where this time the matrix R is associated with the known regular form $x^2 + y^2 + 2z^2$. In general, a solution of $P'HP = qR$, with q prime and R representing a regular form, shows that the form represented by the matrix H represents eligible multiples of the prime q. A solution of $P'AP = q^2B$, with q prime and B representing another form in the same genus, shows that the form represented by the matrix A dominates that of B on multiples of q^2, but much more can sometimes be proven from such expressions. It's worth emphasizing the importance of finding *several* solutions of $P'AP = C$. See especially the section on 26: 1 3 9 2 0 0.

11 Genera

Quadratic forms of a fixed number of variables, fixed as to the collective parity of mixed coefficients, and fixed discriminant split into a finite set of equivalence classes. Here equivalence is by matrices of integers with determinant $+1$.

For binary forms, we get Gaussian composition and a group structure, and the basic question is which primes are represented by which forms of the discriminant. Deciding on whether a composite number is represented by a certain form is effectively a question of composition. The set of elements in this 'class group' that are squares in the group is called the 'principal genus'. The identity in the group is called the 'principal form'. The class group is commutative, so the principal genus is a normal subgroup. Each genus is a left coset of the principal genus.

For ternary positive forms, there is no group structure. In two papers in the same issue of the AMS Transactions [33] and [34], Burton W. Jones showed that any number satisfying the relevant congruence conditions for representation by the forms of a genus is, in fact, represented by at least one form in the genus. A form that represents all numbers 'eligible for its genus' is called 'regular' after Leonard Eugene Dickson [14]. It follows that any form in a genus containing only one equivalence class of forms is necessarily regular.

I should say that the equivalence relation of being in the same genus is slightly stronger than rational equivalence, it is rational equivalence 'without essential denominator'. If two forms are given by symmetric matrices A and B, and the forms agree as to 'odd'/'even' and have the same discriminant, then the forms lie in the same genus if there are an integer matrix P of determinant k such that $P'AP = k^2 B$ and $\mathrm{GCD}(k\ ,\ 2\ \mathrm{disc}) = 1$, as well as the reverse: $Q'BQ = j^2 A$ with $\mathrm{GCD}(\ j\ ,\ 2\ \mathrm{disc}\ ) = 1$. If we divide the matrix P by k , we see a rational equivalence between A and B that is 'without essential denominator' in the phrase of Siegel. This definition of genus is discussed in detail in Jones's book [35]. With three variables, the question of whether j, k are positive is irrelevant. But for binary forms, a change of variable with determinant -1 gives us the important 'opposite' form, which is not necessarily equivalent to the original but is in the same genus.

It turns out that regular positive ternary quadratic forms are all contained in genera with four or fewer equivalence classes. Indeed, only twice do we find four classes: the diagonal form 1 48 144 0 0 0 of discriminant 6912 is in a genus of four classes. Also, the odd form 1 3 37 3 1 0 of discriminant 432 is in a genus of four classes. The latter was proved regular by Earnest et al [1]. All other regular forms are in genera with three or fewer classes. I. Kaplansky has found a unified method for proofs of regularity that avoids the use of spinor genera or of Tits buildings, and handles some thirty-six forms. These include all the forms proved regular in [50].

Returning to the 432 form, the formula

$$4(x^2 + 3y^2 + 37z^2 + 3yz + zx) = (2x + z)^2 + 3(2y + z)^2 + 36(2z)^2$$

allows us to deduce its regularity from that of the diagonal form (1,3,36), along with (1,3,9) and (1,3,12) for special cases.

12 Jones-Pall, Theorem 5

In the massive article [36], great use is made of their Theorem 5, which is (partly) proved elsewhere. The theorem refers to $x^2 + y^2 + cz^2$ for $c = 1, 2, 3$.

For $c = 1$: If n is *not* a square , $n \equiv 1 \bmod 8$, then the number of solutions to $n = x^2 + y^2 + z^2$ with one of x, y, z being divisible by 4 is equal to the number of solutions with one of x, y, z being congruent to $2 \bmod 4$. Notice that two out of three of x, y, z are even and they are congruent $\bmod 4$. For example, taking $n = 41$, each of the triples (0,4,5) and (3,4,4) contributes 24 solutions to the $0 \bmod 4$ pile, taking into account permutations and sign changes. The collection of $2 \bmod 4$ solutions gets 48 total from (1,2,6), giving the required equality.

For $c = 2$: If n is *not* a square , $n \equiv 1 \bmod 8$, then the number of solutions to $n = x^2 + y^2 + 2z^2$ with one of x, y being divisible by 8 is equal to the number of solutions with one of x, y being congruent to $4 \bmod 8$. Notice that we ignore z. For example, taking $n = 73$, each of the triples (0,1,6) and (3,8,0) contributes 8 solutions to the $0 \bmod 8$ pile, and (1,8,2) contributes another 16, making 32 total. The collection of $4 \bmod 8$ solutions gets 16 each from (4,5,4) and (4,7,2).

For $c = 3$, we require $1 \bmod 24$: If n is *not* a square , $n \equiv 1 \bmod 24$, then the number of solutions to $n = x^2 + y^2 + 3z^2$ with one of x, y being divisible by 6 is equal to the number of solutions with one of x, y being congruent to $3 \bmod 6$. Notice that we ignore z. For example, taking $n = 193$, each of the triples (7,12,0) and (0,1,8) contributes 8 solutions to the $0 \bmod 6$ pile, while (1,12,4) and (6,7,6) each contribute another 16, making 48 total. The collection of 3 mod 6 solutions gets 16 each from (9,10,2),(8,9,4), and (2,9,6).

13 Jones-Pall, Theorem 4 : Squares

On the same page 177 of [36], Theorem 4 discusses rewriting squares as $x^2+y^2+cz^2$ for $c = 1, 2, 3$. We will give the results for primes only. Nontrivial expressions $p^2 = x^2 + y^2 + cz^2$ can be shown to exist using various special cases of the identity $(r^2 + As^2 + Bt^2 + ABu^2)^2 = (r^2 - As^2 - Bt^2 + ABu^2)^2 + A(2rs - 2Btu)^2 + B(2rt + 2Asu)^2$.

For $c = 1$ and an odd prime p, we are writing $p^2 = x^2 + y^2 + z^2$ with at least two of $\{x, y, z\}$ nonzero. Permuting labels so that x is odd, Theorem 4 says that if $p \equiv 1 \bmod 4$, then $y \equiv z \equiv 0 \bmod 4$. However if $p \equiv 3 \bmod 4$, then $y \equiv z \equiv 2 \bmod 4$.

For $c = 2$ and an odd prime p, we are writing $p^2 = x^2 + y^2 + 2z^2$ with at least two of $\{x, y, z\}$ nonzero. Permuting labels so that x is odd, note $y \equiv 0 \bmod 4$ and z is even. Theorem 4 says that if $p \equiv 1, 3 \bmod 8$, then $y \equiv 0 \bmod 8$. However if $p \equiv 5, 7 \bmod 8$, then $y \equiv 4 \bmod 8$.

For $c = 3$ and an odd prime $p \neq 3$, we are writing $p^2 = x^2 + y^2 + 3z^2$ with at least two of $\{x, y, z\}$ nonzero. Note that z must be even. Permuting labels so that x is odd, Theorem 4 says that if $p \equiv 1 \bmod 3$, then $y \equiv 0 \bmod 6$. However if $p \equiv 2 \bmod 3$, then $y \equiv 3 \bmod 6$.

14 Rewriting Squares 2

Occasionally it is necessary to show that there is a way to write some n^2 in a nontrivial way as $x^2 + y^2 + cz^2$, especially in the cases $c = 1, 2, 3$. If $n = 1$, there will be no alternative expression from the obvious. If, however, $n \geq 2$, we consider an appropriate prime p that divides n. Next we show that there is an expression $p^2 = x^2 + y^2 + cz^2$ such that at least two out of three of the numbers x, y, z are nonzero. Finally, the appropriate expression for n^2 results from multiplying all three of x, y, z by the integer n/p.

From my little Acta Arithmetica paper, suppose

$$n = r^2 + As^2 + Bt^2 + ABu^2.$$

Then

$$n^2 = (r^2 - As^2 - Bt^2 + ABu^2)^2 + A(2rs - 2Btu)^2 + B(2rt + 2Asu)^2.$$

The first example is $x^2 + y^2 + 2z^2$. We will show that for all $n \geq 2$, we can write $n^2 = x^2 + y^2 + 2z^2$ with at least two of the numbers x, y, z nonzero. As mentioned, it suffices to consider the question for all primes p. For example,

$$2^2 = 1^2 + 1^2 + 2 \cdot 1^2,$$

$$3^2 = 0^2 + 1^2 + 2 \cdot 2^2,$$
$$5^2 = 3^2 + 4^2 + 2 \cdot 0^2,$$
$$7^2 = 1^2 + 4^2 + 2 \cdot 4^2.$$

Generally, if $p \equiv 1 \bmod 4$, then $p = x^2 + y^2$ with $xy \neq 0$ and $x^2 \neq y^2$. Then $p^2 = (x^2 - y^2)^2 + (2xy)^2$.

If $p \equiv 3 \bmod 8$, then $p = x^2 + 2y^2$ with $xy \neq 0$. Then $p^2 = (x^2 - 2y^2)^2 + 2(2xy)^2$.

If $p \equiv 7 \bmod 8$, then $p = x^2 + y^2 + 2z^2$ with $xyz \neq 0$. Then $p^2 = (x^2 - y^2 - 2z^2)^2 + (2xy)^2 + 2(2zx)^2$. Note that the first term $(x^2 - y^2 - 2z^2)$ must be odd, therefore nonzero. ◯ ◯ ◯◯

The second example is $x^2 + y^2 + z^2$. We will show that we can write $n^2 = x^2 + y^2 + z^2$ with at least two of the numbers x, y, z nonzero, *unless* n is a power of 2. Note that the only expression for 4 as the sum of three squares is the trivial

$$2^2 = 0^2 + 0^2 + 2^2.$$

For odd primes: if $p \equiv 1 \bmod 4$, then $p = x^2 + y^2$ with $xy \neq 0$ and $x^2 \neq y^2$. Then $p^2 = (x^2 - y^2)^2 + (2xy)^2$.

If $p \equiv 3 \bmod 8$, then $p = x^2 + 2y^2$ with $xy \neq 0$. Then

$$p^2 = (x^2 - 2y^2)^2 + 2(2xy)^2 = (x^2 - 2y^2)^2 + (2xy)^2 + (2xy)^2.$$

If $p \equiv 7 \bmod 8$, then $p = w^2 + x^2 + y^2 + z^2$ with $wxyz \neq 0$. Then

$$p^2 = (w^2 - x^2 - y^2 + z^2)^2 + (2wx - 2yz)^2 + (2wy + 2zx)^2.$$

Note that the first term must be odd, therefore nonzero. ◯ ◯ ◯◯

The last example is $x^2 + y^2 + 3z^2$. We will show that we can write $n^2 = x^2 + y^2 + 3z^2$ with at least two of the numbers x, y, z nonzero, *unless* n is a power of 3. Note that the only expression for 9 is the trivial

$$3^2 = 0^2 + 3^2 + 3 \cdot 0^2.$$

On the other hand,

$$2^2 = 1^2 + 0^2 + 3 \cdot 1^2.$$

For odd primes: if $p \equiv 1 \bmod 4$, then $p = x^2 + y^2$ with $xy \neq 0$ and $x^2 \neq y^2$. Then $p^2 = (x^2 - y^2)^2 + (2xy)^2$.

If $p \equiv 1 \bmod 3$ then $p = x^2 + 3y^2$ with $xy \neq 0$. Then

$$p^2 = (x^2 - 3y^2)^2 + 3(2xy)^2.$$

If $p \equiv 11 \bmod 12$, then $p = +x^2 + y^2 + 3z^2$ with $xyz \neq 0$. Then

$$p^2 = (x^2 - y^2 - 3z^2)^2 + (2xy)^2 + 3(2zx)^2.$$

$\bigcirc\bigcirc\bigcirc\bigcirc$

15 The Jones lemmas

The following is Theorem 9 (page 51) in the unpublished Ph.D. dissertation of Burton Jones, U. of Chicago 1928 [32].

Lemma(Jones). If$f = x^2 + ky^2$ represents an odd prime p, where k is a positive integer prime to p, then every mp represented by f (m a positive integer) is represented by f with x and y prime to p .

The lemma fails when $m = 0$.

Next, from page 52 of Jones' dissertation: if n is positive, n and p are represented by $x^2 + ky^2$, and $n = mp^a$ with m prime to p, then m is also represented by $x^2 + ky^2$. For this separate argument it is not necessary to require $k \neq p$, and we permit $p = 2$. Indeed, the argument can be modified to apply to forms of the type $x^2 + xy + ky^2$, so the fundamental restriction in this paragraph is simply that the binary form represent 1. For example, if a nonzero number n is represented by $x^2 + xy + 3y^2$ and n is divisible by 11, then $n/11$ is also represented, and so on if n was divisible by $11^a, a \geq 2$.

Corollary(Jones). Finally, as a corollary, if n and p are both represented by $x^2 + ky^2$, p is an odd prime and k is prime to p, and n is nonzero and divisible by p, then we have a representation $n = x^2 + ky^2$ with x, y prime to p.

Recently, these results have been used in this form [38] : if n is nonzero, divisible by 5 and is the sum of two squares, then there is an expression $n = x^2 + y^2$ with x and y prime to 5. Another example [37] is: if n has an expression as $x^2 + 2y^2$, n is nonzero and n is divisible by 3, then we can take $n = x^2 + 2y^2$ with x, y both prime to 3. In an earlier section of this note, I showed: Given $m = 2y^2 + 2yz + 3z^2$, with m nonzero, and with m divisible by 3, we can construct $m = 2s^2 + 2st + 3t^2$, with s, t prime to 3.

16 Difficulties with forms not proved regular

The main process in the work of [31] was a means of taking all forms known regular and producing new ones with higher discriminants. Prof. Kaplansky eventually put some of the process into recognizable theorems:

Theorem For odd forms only: if the discriminant D is divisible by 4, and the form is regular, there is a regular form of discriminant $\frac{D}{4}$.

This can be placed in very concrete terms: The original form is equivalent to one with coefficients a b 4c 2d 2e f, where f is odd. The form with discriminant $D/4$ is equivalent to a b c d e f.

Theorem For odd forms only: if the discriminant D is divisible by p^k, for $k \geq 2$ and p an odd prime, and the form is regular, there is a regular form of discriminant $\frac{D}{p}$ or $\frac{D}{p^2}$ or $\frac{D}{p^4}$.

Eventually it turned out that it was not necessary to keep the $\frac{D}{p^2}$ bit, but perhaps $\frac{D}{p^4}$ must remain: there is a form, odd of discriminant $2592 = 32 \cdot 81$, that is probably regular. Its coefficients are 5 9 17 6 5 3. There are two regular odd forms of discriminant 648, which is $\frac{2592}{4}$; I forget which is relevant. But there is no odd regular form with discriminant 864 or 288 or 96, those being the quotients of 2592 by 3, 9, and 27 respectively. There is, however, a regular odd form of discriminant 32, and there's the p^4 quotient.

We used a variety of ways for constructing even regular forms out of odd ones, and several ways for using even forms to get new even regular ones with higher discriminants. The regular odd forms with squarefree discriminant were found by Watson [55], and the results above show how to descend from any odd regular form to one with squarefree discriminant.

In the present note, I have been lazy about even numbers and numbers that have common divisors with the discriminant. This is a *metatheorem*:

for a form that does turn out to be regular, even numbers and multiples of primes dividing the discriminant should cause no difficulties.

Here's an example: even 5184, the form $3x^2+16y^2+16yz+112z^2$. Suppose we examine multiples of three: set $3x^2+16y^2+16yz+112z^2 = 3A$. Change variables by $y \to y+z$, giving $3x^2+16y^2+48yz+144z^2 = 3A$. It follows that y is divisible by 3, so we set $y \to 3t$, leading to $3x^2+144t^2+144tz+144z^2 = 3A$. Simply divide through by 3, resulting in $x^2+48t^2+48tz+48z^2 = A$. This new form, 1 48 48 48 0 0, discriminant 1728, is regular, so it does represent A if we know that A is eligible according to congruences. Here's the magic part: the number A is in fact eligible to be represented by the new form, which follows from diagonalizing both forms over the 3-adic numbers. Indeed, the process of reducing questions to questions about forms with lower discriminant (that divide the original) is at the center of the effort in [31].

The bad news is that half of the metatheorem does not apply for several of the 'candidate forms'; we get no help if the form of lower discriminant appealed to is not known to be regular. I have a list of co-dependencies: the odd candidate form of discriminant 240 is regular if and only if the odd 720 form is; each attends to multiples of 3 for the other. Similar for odd 8232 and 24696. Next are even 2112 and 6336. Finally even 2880 and 14400, where the relevant prime is 5.

Among the odd forms there are also some one-way implications. If the 2160 candidate is regular, so is the 720 (and therefore the 240). The 1620 implies the 405. The 4500 implies one of the 1125 candidates, I forget which.

Not really news is the fact that irregular forms sometimes miss eligible numbers that are even or have a prime factor in common with the discriminant. The form $x^2 + 4y^2 + 9z^2$ is close to being regular, but misses the number 2. The form 31: 1 2 4 1 0 0 misses the number 31; it should only miss quadratic nonresidues times 31 to an odd power. 29: 1 2 4 1 1 0 misses 87, which is $3 \cdot 29$; this form should only miss residues times 29 to an odd power, but 3 is not a quadratic residue of 29.

17 Audience Request: 2 2 5 2 0 0 is nearly regular

Recall the funny order used for the six coefficients: we consider

$$g(x, y, z) = 2x^2 + 2y^2 + 5z^2 + 2yz$$

The other class in this genus is diagonal 1 1 18 0 0 0. The eligible numbers for the genus are those not $9m \pm 3, 4^k(16u+14)$. We will show that $2x^2 + 2y^2 + 5z^2 + 2yz$ misses the number 1 but represents all other eligible numbers.

Our basic tool is the identity

$$2x^2 + 2y^2 + 5z^2 + 2yz = (y+2z)^2 + (y-z)^2 + 2x^2.$$

For any eligible number $n > 1$ we will show how to write $n = r^2 + s^2 + 2t^2$ such that $r \equiv s \bmod 3$. This will allow us to solve for integer values of x, y, z in $n = (y+2z)^2 + (y-z)^2 + 2x^2$.

CASE I. n is divisible by 9 , $n \neq 4^k(16u+14)$. It happens that $9 \cdot 14 \equiv 14 \bmod 16$. As a result $\frac{n}{9} \neq 4^j(16v+14)$. Without knowing anything else about the power of 3 that divides n, we still get to write

$$\frac{n}{9} = a^2 + b^2 + 2c^2.$$

Therefore

$$n = (3a)^2 + (3b)^2 + 2(3c)^2.$$

Insofar as $3a \equiv 3b \equiv 0 \bmod 3$, we know we can solve the system $y + 2z = 3a, y - z = 3b, x = 3c$ over the integers. Indeed we get $x = 3c, y = a + 2b, z = a - b$. So

$$n = (y+2z)^2 + (y-z)^2 + 2x^2$$

as required.

From now on we consider n not divisible by 3, as the genus does not represent any numbers $9m \pm 3$. We begin with some expression

$$n = r^2 + s^2 + 2t^2.$$

CASE II. If r, s are both nonzero modulo 3 , a choice of $\pm s$ forces $r \equiv s \bmod 3$ and we are done , as we can solve for integer values of x, y, z. As $r^2 + s^2$ is two modulo three, in this subcase $n \equiv 2, 1 \bmod 3$ depending on the value of t modulo 3.

The same happens if r, s are both divisible by 3, as then $r \equiv s \equiv 0 \bmod 3$ as required for integer x, y, z. In this subcase, $n \neq 0 \bmod 3$ means that $t \neq 0 \bmod 3$ and in fact $n \equiv 2 \bmod 3$.

Remaining Cases: With $n = r^2 + s^2 + 2t^2$, if one of r, s is divisible by 3 and the other is not, we permute the letters so that $r \equiv \pm 1 \bmod 3$ but $s \equiv 0 \bmod 3$.

CASE III. $n \equiv 1 \bmod 3$, $r \equiv \pm 1 \bmod 3$ and $s \equiv 0 \bmod 3$. **Furthermore** $s^2 + 2t^2 > 0$. Note that we have here $t \equiv 0 \bmod 3$ as well because $n \equiv 1 \bmod 3$. We quote Lemma 3 from Kap's "First Nontrivial Genus" [37]: **Lemma** Suppose that w is a nonzero integer divisible by 3 and expressible as $s^2 + 2t^2$. Then w can be so written with s and t both prime to 3.**Proof** Induction on the power of 3 that divides w, using the crucial fact that if $v \equiv 0 \bmod 3$ and there is some expression $v = s_0^2 + 2t_0^2$, then there is an expression $\frac{v}{3} = s_1^2 + 2t_1^2$. Since the original s, t are both divisible by 3 and at least one is nonzero, we can revise them, giving

$$n = r^2 + S^2 + 2T^2,$$

with all three prime to 3. Finally choosing $\pm S$ allows us to force $r \equiv S \bmod 3$ and we can solve for x, y, z as required.

CASE IV. $n \equiv 1 \bmod 3$, $r \equiv \pm 1 \bmod 3$ and $s \equiv 0 \bmod 3$. **However** $s^2 + 2t^2 = 0$, so that both s, t are 0 and n is a square not divisible by 3. **Furthermore** $n > 1$. The point in this case is to rewrite n in a nontrivial way. It suffices to consider primes! Any square is a prime squared or the product of a squared prime times some other square. Without loss of generality, we take $n = p^2$ with $p \neq 3$ a prime number.

CASE IVa. If $p = 2$, $4 = 1^2 + 1^2 + 2 \cdot 1^2$ while $1 \equiv 1 \bmod 3$.

CASE IVb. If $p \equiv 1 \bmod 4$, we know there is an expression

$$p = X^2 + Y^2.$$

Here we have $XY \neq 0$ and $X^2 - Y^2 \neq 0$. We get

$$p^2 = (X^2 - Y^2)^2 + (2XY)^2.$$

Since $p^2 \equiv 1 \bmod 3$,one of the pair $2XY, X^2 - Y^2$ is divisible by 3 and the other is not. Thus we are back in case III, with r chosen to be the expression not divisible by 3, s being the other, and t being 0.

CASE IVc. If $p \neq 3$ but $p \equiv 3 \bmod 8$, we know there is an expression

$$p = X^2 + 2Y^2$$

with $X, Y \neq 0$. Next

$$p^2 = (X^2 - 2Y^2)^2 + 2(2XY)^2.$$

Again , as p^2 is one modulo 3, we must have $XY \equiv 0 \bmod 3$ while $X^2 - 2Y^2 \neq 0 \bmod 3$. We are in case III, with $r = X^2 - 2Y^2, s = 0, t = 2XY$.

CASE IVd. If $p \equiv 7 \bmod 8$, we know there is an expression

$$p = X^2 + Y^2 + 2Z^2$$

with $X, Y, Z \neq 0$. Think about it. Next

$$p^2 = (X^2 - Y^2 - 2Z^2)^2 + (2XY)^2 + 2(2XZ)^2.$$

The first of the terms squared is odd and thus nonzero, while the second and third terms are nonzero because $X, Y, Z \neq 0$. Since p^2 is one modulo 3, either all three terms are nonzero modulo 3 or we are back in case III.

18 Example: diagonal 1,4,9 is nearly regular

To illustrate, we show that the form $x^2 + 4y^2 + 9z^2$ is nearly regular. Congruence considerations show that it must fail to represent all integers of the forms $9n \pm 3, 8n+3, 4^k(8n+7)$. The only other number it misses is 2, so I call the form 'nearly regular'; we say that the only sporadic number is 2. This proof is essentially the same as my treatment of an even form of discriminant 18 in the article [30].

It is easy to show that the form gets all eligible numbers that are divisible by 4 or 9, as well as eligible numbers that are congruent to 1 mod 4.

The difficult case is $n \equiv 2 \bmod 4$, with $n \neq 2$ and n prime to 3. Here, $n/2$ is odd, prime to 3, and larger than 1. We will show that we can find

$$\frac{n}{2} = r^2 + s^2 + 2t^2$$

with

$$r \equiv s \bmod 3.$$

There are two types of trouble, if we do not immediately have the mod3 congruence. First, we could have $r \equiv 0, s \equiv \pm 1 \bmod 3$. But in this case n itself would be divisible by 3 and therefore by 9. Secondly, we could have $r \equiv 0, s \equiv \pm 1, t \equiv 0 \bmod 3$. If at least one of $r, t \neq 0$ then $r^2 + 2t^2$ is a nonzero number divisible by 3. We may use the Jones lemmas to find revised values r', t' that are prime to 3, finishing this case. If, however, r, t are actually both 0, then $n/2$ is a perfect square larger than 1, and we can rewrite it in a nontrivial way, with at least two of r, s, t not equal to 0, so we are in a previous case.

We finally have

$$\frac{n}{2} = r^2 + s^2 + 2t^2$$

with $n/2$ odd and prime to 3, and

$$r \equiv s \bmod 3.$$

Then

$$n = (r+s)^2 + 4t^2 + 9((r-s)/3)^2.$$

19 M. Bhargava and 26: 1 3 9 2 0 0 on arithmetic progressions

In another section I discuss Hsia's result that $x^2 + y^2 + 10z^2$ gets all eligible $3m+2$, including the entire arithmetic progression $6m+5$. One could say that $x^2 + y^2 + 10z^2$ is "regular on the arithmetic progression $3m+2$."

In September 1999 Manjul Bhargava brought to my attention the form 26: 1 3 9 2 0 0. We will call it g, and display it as

$$g(X,Y,Z) = X^2 + 3Y^2 + 2YZ + 9Z^2.$$

The form g misses $4^k(16m+6)$ generically. It also has numerous sporadics, the first few (those up to 5000) being 2, 5, 8, 20, 32, 62, 80, 122, 128, 248, 320, 488, 512, 992, 1280, 1952, 2048, 3968. Notice all the sporadics are congruent to 2 mod 3. In addition, it is possible that g represents all odd numbers except 5. I can't prove quite so much, but I will show that g does represent all eligible numbers congruent to $0, 1 \bmod 3$, including *all* the odd numbers $6m+1$ and $6m+3$.

The form g is in a genus of three classes, with reduced representatives

$$\left\{ \begin{array}{cccccc} 1 & 3 & 9 & 2 & 0 & 0 \\ 1 & 1 & 26 & 0 & 0 & 0 \\ 2 & 2 & 7 & 0 & 2 & 0 \end{array} \right\}.$$

We have Jones' result, any target $n \neq 4^k(16m+6)$ is represented by at least one form in the genus. So we have two cases to consider. First we show that if n is congruent to zero or one mod 3 and is represented by 1 1 26 0 0 0, it is also represented by g. Second, we must show that if n is congruent to zero or one mod 3 and is represented by 2 2 7 0 2 0, it is also represented by g.

I was quite surprised when the proof appeared. I should point out that I used a computer program that gives me all homotheties between two given forms, a task which is difficult and annoying by hand. This is a finite search, as all our forms are positive, the associated matrices are positive definite, and so on. In the matrix equation $P'AP = C$, the columns of P are vectors whose values by the quadratic form A are the diagonal elements of C. This gives easy bounds on the elements of P, either through considering eigenvalues of A or using Lagrange multipliers. If v stands for the column vector with entries (x, y, z) then the gradient of the function $v'Av$ is just $2Av$, or would be $(A + A')v$ if A itself were not symmetric.

In the present proof, I display a number of homotheties of the type $P'AP = 9B$, where A is the matrix for 1 3 9 2 0 0 and B is the matrix of one of the other two forms in the genus. I write these below as formulas with substitution, possibly aiding clarity.

CASE I. Take target number $n \neq 4^k(16m + 6)$ and $n \equiv 0, 1 \bmod 3$. Let $n = x^2 + y^2 + 26z^2$. Note that there are restrictions mod 3 on the triple (x, y, z). Perhaps (x, y, z) are all 0 mod 3, or all are nonzero mod 3, or one of (x, y) is 0 mod 3 and the other is not. Accordingly, it is guaranteed that $x \equiv \pm z \bmod 3$ or that $y \equiv \pm z \bmod 3$. We have:

$$g(X, Y, Z) = X^2 + 3Y^2 + 2YZ + 9Z^2,$$

$$n \equiv 0, 1 \bmod 3,$$
$$n \neq 4^k(16m + 6),$$
$$n = x^2 + y^2 + 26z^2,$$
$$n = g\left(y, 3z, \frac{x - z}{3}\right),$$
$$n = g\left(y, -3z, \frac{x + z}{3}\right),$$
$$n = g\left(x, 3z, \frac{y - z}{3}\right),$$
$$n = g\left(x, -3z, \frac{y + z}{3}\right).$$

Since we know that $x \equiv \pm z \bmod 3$ or that $y \equiv \pm z \bmod 3$, at least one of the above rational representations for n by the form g consists entirely of integers, thus showing that g also represents n.

CASE II. Take target number $n \neq 4^k(16m+6)$ and $n \equiv 0, 1 \bmod 3$. Let $n = 2x^2 + 2y^2 + 7z^2 + 2zx$. There are only six triples (x, y, z) mod 3 for which $2x^2+2y^2+7z^2+2zx \equiv 2 \bmod 3$, those being 010,020,100,101,200,202. Those six are the triples we are avoiding. All the other 21 triples, for which the value of $2x^2 + 2y^2 + 7z^2 + 2zx \equiv 0, 1 \bmod 3$, have the property that at least one of the four linear expressions $\{x+y, x-y, x+y+z, x-y+z\}$ is divisible by 3. We have:

$$g(X, Y, Z) = X^2 + 3Y^2 + 2YZ + 9Z^2,$$

$$n \equiv 0, 1 \bmod 3,$$

$$n \neq 4^k(16m+6),$$

$$n = 2x^2 + 2y^2 + 7z^2 + 2zx,$$

$$n = g\left(\frac{x-2y-6z}{3}, \frac{-2x+y-3z}{3}, \frac{x+y}{3}\right),$$

$$n = g\left(\frac{x+2y-6z}{3}, \frac{-2x-y-3z}{3}, \frac{x-y}{3}\right),$$

$$n = g\left(\frac{x-2y+7z}{3}, \frac{-2x+y+z}{3}, \frac{x+y+z}{3}\right),$$

$$n = g\left(\frac{x+2y+7z}{3}, \frac{-2x-y+z}{3}, \frac{x-y+z}{3}\right).$$

We know that at least one of the four linear expressions $\{x+y, x-y, x+y+z, x-y+z\}$ is congruent to 0 mod 3. It follows that at least one of the above rational representations for n by the form g consists entirely of integers, thus showing that g also represents n.

That's it. The form $g(X, Y, Z) = X^2 + 3Y^2 + 2YZ + 9Z^2$, also known as 26: 1 3 9 2 0 0, behaves as though regular on the arithmetic progressions $3m$ and $3m+1$. In particular, it represents the progressions $6m+3$ and $6m+1$ in their entirety. ◯ ◯ ◯◯

It's worth a word on the meaning of this result for the original intended problem, that of proving that certain even (integer-matrix) forms in four or more variables represent all odd numbers. This effort eventually became Bhargava's 33 Theorem, see page 26 of [11]. At least this is the way the problem was being discussed at the time. The first case is $1 \oplus 3\ 5\ 8\ 4\ 2\ 0$ of discriminant 103, that is

$$h_1(W, X, Y, Z) = W^2 + 3X^2 + 5Y^2 + 8Z^2 + 4YZ + 2ZX.$$

We have the homothety

$$h_1(x, -3w + y, 17w - z, 9w + z) = 2678w^2 + x^2 + 3y^2 + 2yz + 9z^2.$$

This formula shows that h_1 dominates the latter form, that is h_1 represents every number represented by $2678w^2 + x^2 + 3y^2 + 2yz + 9z^2$ and probably a few others. As $2678 \equiv 2 \bmod 6$, we may take $w = 0, 1$ and conclude that h_1 represents all odd numbers larger than 2678. It is easy to check that h_1 represents all odd numbers up to 5000. For that matter, 26:1 3 9 2 0 0 represents all odd numbers except 5 up to 5000. Anyway, this proves that h_1 represents all odd numbers. Bhargava commented that this shows that h_1 actually represents all numbers other than 2. I'll think about it.

The second case is $1 \oplus 3\ 5\ 9\ 2\ 2\ 0$ of discriminant 127, that is

$$h_2(W, X, Y, Z) = W^2 + 3X^2 + 5Y^2 + 9Z^2 + 2YZ + 2ZX.$$

We have the homothety

$$h_2(x, w + y, 26w, -3w + z) = 3302w^2 + x^2 + 3y^2 + 2yz + 9z^2.$$

This formula shows that h_2 dominates the latter form, that is h_2 represents every number represented by $3302w^2 + x^2 + 3y^2 + 2yz + 9z^2$. Again, $3302 \equiv 2 \bmod 6$, we may take $w = 0, 1$ and conclude that h_2 represents all odd numbers larger than 3302. Done.$\bigcirc$

Just as the 290 conjecture for representing all natural numbers using odd forms is not finished, so the problem of representing all odd numbers with odd forms in not finished. However there is a big obstacle in the latter problem, as there are three remaining odd ternary forms that appear to represent all odd natural numbers but for which no proof is available. See Kaplansky [38]. The three 'candidates', each preceded by discriminant, are:

38: 1 2 5 0 1 0, or $x^2 + 2y^2 + 5z^2 + zx$; this form seems to miss only $14 \cdot 4^k$ compared with its genus. The other form in the genus is 38: 1 1 13 1 1 1, which represents exactly the same numbers as 1 3 13 2 0 0.

62: 1 3 6 2 0 1, or $x^2 + 3y^2 + 6z^2 + 2yz + xy$; this form seems to miss only $26 \cdot 4^k$ compared with its genus. The other form in the genus is 62: 1 1 21 1 1 1 which represents exactly the same numbers as 1 3 21 2 0 0.

74: 1 3 7 1 1 1, or $x^2 + 3y^2 + 7z^2 + yz + zx + xy$. this form seems to miss only $2 \cdot 4^k$ and $50 \cdot 4^k$ compared with its genus. The other forms in the genus are 74: 1 1 19 1 1 0, not sure about this one, and 74: 1 1 25 1 1 1 which represents exactly the same numbers as 1 3 25 2 0 0.

20 A reason to like odd numbers

What follows is a method for showing quickly that a positive quaternary form represents all natural numbers up to some large finite bound. The work was prompted by contact with Manjul Bhargava and Jonathan Hanke.

Given a positive ternary $T(x, y, z) = ax^2 + by^2 + cz^2 + dyz + ezx + fxy$, suppose that T represents very many odd numbers, because it represents all of $1, 3, 5, 7 \bmod 8$, and all values $\bmod p$ and $\bmod p^2$ for any odd prime p that divides the discriminant of T. For that extra feeling of security, suppose that the spinor genus of T and the genus of T coincide.

What we are hoping for is a form that represents all sufficiently large odd numbers. I sent e-mail to Rainer Schulze-Pillot asking if I had correctly described sufficient conditions for this desirable property. He replied the next day, 17 November 2004, in the affirmative. The reference is [51], with a later survey article [52].

In short, suppose that there are odd numbers $m < m + M$ such that $T(x, y, z)$ represents all the odd numbers from $m+2$ to $m+M$ inclusive, and where M is much larger than m. I guess this makes M even. Think of m as "the largest known odd miss."

Now consider the quaternary form

$$Q(x, y, z, w) = T(x, y, z) + Rw^2,$$

where we insist that R also be odd! We want M much larger than both R and m. Indeed we require

$$\frac{M^2}{16R} > m + M,$$

so that $M > 16R$. Sorry to be annoying, but we also require

$$4M > 9R + 2m + \frac{m^2}{R}.$$

In the Theorem that follows, forget about genera or spinor genera. All that matters is that a particular positive ternary represents a very long set of consecutive odd numbers.

Theorem: Given a positive ternary $T(x, y, z) = ax^2 + by^2 + cz^2 + dyz + ezx + fxy$, and positive integers m, M with m odd and M even. Suppose that $T(x, y, z)$ represents all the odd numbers from $m+2$ to $m+M$ inclusive. Suppose further that there is an odd number $R > 0$ such that $M^2 > 16R(m+$

M) and $4MR > 9R^2 + 2Rm + m^2$. Define $Q(x,y,z,w) = T(x,y,z) + Rw^2$. **Then** $Q(x,y,z,w)$ represents all natural numbers from $m + R + 2$ up to $\lfloor \frac{M^2}{16R} \rfloor$.

Proof: we will of course solve $n = T(x,y,z) + Rw^2$ by solving $n - Rw^2 = T(x,y,z)$ first.

Case 1: If $m + R + 2 \leq n \leq m + M$, then either n itself or $n - R$ is odd (we insisted that R be odd), and both values lie in the interval from $m + 2$ to $m + M$. So either n or $n - R$ is represented by $T(x,y,z)$, which is to say that n is represented by $Q(x,y,z,w)$ where w is either 0 or 1.

Notation for cases 2 and 3: For $n > m + M$, but $n \leq M^2/(16R)$, so that $\sqrt{n} \leq M/(4\sqrt{R})$. Pick W so that $RW^2 \leq n$ but $R(W+1)^2 > n$. We will need

$$W \leq \sqrt{\frac{n}{R}} \leq \frac{M}{4R},$$

and

$$RW \leq \frac{M}{4}.$$

Case 2: If $n > m + M$ and $n - RW^2 \geq m + 1$, it is still true that $n - R(W+1)^2 < 0$, or

$$n - RW^2 < 2RW + R.$$

That is $n - RW^2 < \frac{M}{2} + R < \frac{9M}{16} < m + M$. This is small enough, but what if $n - RW^2$ is an even number? Then we need to switch to the (larger) odd number $n - R(W-1)^2 = n - RW^2 + 2RW - R < 4RW < M < m + M$.

Case 3: If $n > m + M$ and $n - RW^2 \leq m$, we consider $W - 1$ and $W - 2$. That is, either $n - R(W-1)^2$ or $n - R(W-2)^2$ is an odd number. It follows from combining $R(W+1)^2 > m + M$ and $4M > 9R + 2m + \frac{m^2}{R}$ that $2WR - R > m$, then adding $n - RW^2 \geq 0$ gives $n - R(W-1)^2 > m$, so that $n - R(W-2)^2 > m$ as well. How big could they be?

$$n - R(W-2)^2 = n - RW^2 + 4RW - 4R < m + M - 4R < m + M.$$

○○○○○

Example: take

$$T(x,y,z) = 9x^2 + 19y^2 + 35z^2 + 18yz + 8zx + 8xy,$$

with Gram matrix

$$\begin{pmatrix} 9 & 4 & 4 \\ 4 & 19 & 9 \\ 4 & 9 & 35 \end{pmatrix}$$

with $m = 124499 \approx 1.24 \cdot 10^5$, and $R = 5$. With these figures, our result above can be applied when $M > 775062310 \approx 7.75 \cdot 10^8$. A long computer calculation (19 November - 22 November , 2004) confirmed matters for $M = 940042422 \approx 9.40 \cdot 10^8$. That is, it confirmed that $T(x, y, z)$ represents all odd numbers from $m + 2 = 124501$ up to $m + M = 940166921$.

It follows that the quaternary $Q(x, y, z, w) = T(x, y, z) + 5w^2$ represents all natural numbers from $m+R+2$ up to $\lfloor \frac{M^2}{16R} \rfloor$, or 124506 up to $\lfloor \frac{M^2}{80} \rfloor$. That is,

$$Q(x, y, z, w) = 9x^2 + 19y^2 + 35z^2 + 18yz + 8zx + 8xy + 5w^2$$

represents the numbers from 124506 to $11045996939495326 \approx 1.104 \cdot 10^{16}$.

The payoff is a subset relation: $Q(x, y, z, w)$, with Gram matrix

$$\begin{pmatrix} 9 & 4 & 4 & 0 \\ 4 & 19 & 9 & 0 \\ 4 & 9 & 35 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix},$$

represents a subset of the integers represented by

$$U(x, y, z, w) = x^2 + 2y^2 + 4z^2 + 31w^2 + 3zw - wy + yz,$$

with Gram matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1/2 & -1/2 \\ 0 & 1/2 & 4 & 3/2 \\ 0 & -1/2 & 3/2 & 31 \end{pmatrix}.$$

This is because of the homothety

$$Q(r, s, t, u) = U(-r + 3s + t + u, 2r + 2s + t, -s + u, -t),$$

which is the same as saying that the matrix product

$$\begin{pmatrix} -1 & 2 & 0 & 0 \\ 3 & 2 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1/2 & -1/2 \\ 0 & 1/2 & 4 & 3/2 \\ 0 & -1/2 & 3/2 & 31 \end{pmatrix} \begin{pmatrix} -1 & 3 & 1 & 1 \\ 2 & 2 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

is equal to

$$\begin{pmatrix} 9 & 4 & 4 & 0 \\ 4 & 19 & 9 & 0 \\ 4 & 9 & 35 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix},$$

which is true.

So, the numbers from 124506 to 11045996939495326 $\approx 1.1046 \cdot 10^{16}$ are represented by $U(x, y, z, w)$. It is expected that analytic methods will finish a proof that $U(x, y, z, w)$ represents all the natural numbers.

21 When odd numbers are not good enough

After the success with

$$U(x, y, z, w) = x^2 + 2y^2 + 4z^2 + 31w^2 + 3zw - wy + yz,$$

I asked Hanke for some more examples to try. He ignored me, so I started on $V(x, y, z, w) = x^2 + 3y^2 + 5z^2 + 7w^2$. I was surprised at being unable to find any ternary quadratic form "represented by V," that in turn represented almost all odd numbers. A synonym for "ternary quadratic form represented by V" is "ternary sublattice."

There was good reason for this, and all the relevant material is in a wonderful book by Cassels [6]. By certain more or less global relations (page 76, Lemma 1.1), any positive ternary is anisotropic at at least one, and in fact an odd number of, finite primes, as "positive" means anisotropic at infinity. In particular, on page 59 we find Lemmas 2.5 and 2.6, and all necessary material on $c_p(f)$ on pages 55-58. Put together with the fact that 105 is a square in the 2-adic numbers, and any ternary sublattice of V is isotropic at 2 and anisotropic at some odd finite prime.

22 Quaternaries representing binaries

In early 2006 I heard a talk by Jordan Ellenberg on positive quadratic forms of dimension n representing forms of dimension m. Under fairly mild extra assumptions, when $n \geq m+7$, Q is positive of dimension n and Q' is positive of dimension m, and if Q represents Q' locally and Q' has a sufficiently large minimum value, then Q represents Q' over the integers. The work, with

Ashkay Venkatesh, has now appeared [23]. They cite Schulze-Pillot [52] as a valuable survey. Both mention the earlier result [29], in full strength (you still need sufficiently large minimum for Q'), for $n \geq 2m+3$, and Schulze-Pillot mentions a result [9] requiring extra hypotheses but including the best possible rank comparison, $n \geq m+3$.

Schulze-Pillot [53] himself has recently provided his own improvements that include $n \geq m+3$, with a nice description in matrix form (Theorem 11).

Ellenberg said that things could possibly be improved further to $n \geq m+3$, but under no circumstances would $n = m+2$ work. This fits, we know that positive ternaries ($n = 3$) have generic obstructions in representing integers ($m = 1$); for this topic, one should think of any integer a as either a one-by-one matrix or the one-variable form $f(t) = at^2$, with 'minimum' a.

I got to thinking about $n = 4, m = 2$, positive quaternaries representing binaries. I found a nice reference by Earnest [18]. He calls a positive quaternary "2-regular" if it represents all binaries it locally represents. He shows that, just as with regular ternaries, there are a finite number of 2-regular quaternaries (up to equivalence, or isometry of integer lattices as he calls it).

For positive quaternaries representing binaries, I took quaternaries from the book of Gordon L. Nipp [43]. The first few examples I tried missed entire discriminants of binaries. But it turned out to be a simple lemma based on the fact that I was checking diagonal quaternaries with square determinant and some additional restrictions, allowing a relationship with quaternion multiplication. The lemma described below should also be useful generalized to the p-adic diagonalization of a quaternary.

See Cassels [6, pages 171-178] for a better formalism that reveals generalizations; we take an aggressively low-budget approach. In short, take positive integers A, B, C. Consider the positive quaternary form

$$Q(z_0, z_1, z_2, z_3) = z_0^2 + BCz_1^2 + CAz_2^2 + ABz_3^2$$

For a quaternion

$$z = z_0 + z_1 i\sqrt{BC} + z_2 j\sqrt{CA} + z_3 k\sqrt{AB},$$

define

$$z' = z_0 - z_1 i\sqrt{BC} - z_2 j\sqrt{CA} - z_3 k\sqrt{AB},$$

so that

$$zz' = z'z = Q(z_0, z_1, z_2, z_3),$$

the usual squared norm. We will also need the "real part"

$$\Re z = z_0.$$

The matrix multiplication of our quaternary form representing some binary is:

$$\begin{pmatrix} p & q & r & s \\ t & u & v & w \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & BC & 0 & 0 \\ 0 & 0 & CA & 0 \\ 0 & 0 & 0 & AB \end{pmatrix} \begin{pmatrix} p & t \\ q & u \\ r & v \\ s & w \end{pmatrix} = \quad M.$$

We introduce more quaternions,

$$x = p + qi\sqrt{BC} + rj\sqrt{CA} + sk\sqrt{AB},$$

and

$$y = t + ui\sqrt{BC} + vj\sqrt{CA} + wk\sqrt{AB}.$$

We calculate

$$xx' = p^2 + BCq^2 + CAr^2 + ABs^2,$$
$$yy' = t^2 + BCu^2 + CAv^2 + ABw^2,$$

and a real part

$$\Re(xy') = pt + BCqu + CArv + ABsw.$$

Earlier we denoted by M the Gram matrix of a represented binary. With the new notation we find

$$M = \begin{pmatrix} xx' & \Re(xy') \\ \Re(xy') & yy' \end{pmatrix}$$

The determinant of M is

$$\det M = xx'yy' - \Re^2(xy').$$

Now, real numbers commute with quaternions, so

$$(xy')(xy')' = xy'yx' = x(y'y)x' = xx'(y'y) = xx'yy'.$$

If we now use our variable z in a substitution, taking

$$z = xy',$$

we get $xx'yy' = zz'$, so that

$$\det M = zz' - \Re^2 z.$$

Finally, we have a lemma: we find that the determinant of M is restricted to the values of a certain diagonal positive ternary:

$$\det M = zz' - \Re^2 z = Q(z_0, z_1, z_2, z_3) - \Re^2 z = Q(z_0, z_1, z_2, z_3) - z_0^2,$$

or

$$\det M = BCz_1^2 + CAz_2^2 + ABz_3^2.$$

23 Kaplansky's proofs: one of his methods

We discuss some two-parameter families of forms. In a 1997 email to Alexander Schiemann, Kap conjectured that these, together with a few pairs of regular forms, give all possible examples of pairs of positive ternaries that represent the same numbers (ignoring multiplicities). Schiemann had already proven that the theta series of a positive ternary (which includes multiplicities) categorizes the form up to equivalence. A 1963 article by Timofeev [54] displays a subset of Kap's forms.

EDIT, December 2013: I decided to give Kap's conjecture a little computer trial and was surprised to find counterexamples plentiful. I did, eventually, find two examples of pairs of irregular forms of the same discriminant (and genus) representing the same numbers, at least up to 10^6. So I am reproducing those genera whole. Proof would be difficult or impossible. I also found, with varying discriminants, two quadruples of forms, and a few pairs, where the forms in each grouping miss the same numbers. Checked up to 10^6.

```
( 4n + 2) , 4^k * (16 n + 6 )
=====Discriminant  232  ==Genus Size==   3
```

```
   Discriminant   232
  Spinor genus misses     no exceptions
       232:    1     4         15      2    1    0 misses   3  7  11  31  43
       232:    3     3          7      1    2    1 misses   1  single squareclass
       232:    3     5          5      3    1    3 misses   1  single squareclass
--------------------------size 3
Last two: also 4^k * {1}


(9 n +- 3), (81 n +- 27), (4 n + 2), 4^k * (16 n + 14)
=====Discriminant  648  ==Genus Size==   6
   Discriminant   648
  Spinor genus misses     no exceptions
       648:    1     4         41      2    1    0 misses   7  11  19  23  31
       648:    1     5         35      4    1    1 misses   8  13  29  31
       648:    1    11         17      9    1    1 misses   5  7  8  65  179
       648:    4     5          9      3    0    2 misses   1  8
       648:    5     5          8      0    4    3 misses   1  40
       648:    5     7          7      6    1    5 misses   1  40
--------------------------size 6
Last two: also 4^k * {1,40}



             4^k * (16 n + 2) ; 4^k * {1}
        78 :     3     3          3      1    1    3
       142 :     3     3          5      2    3    1
       158 :     3     3          5     -1    2    1
       190 :     3     5          5      5    2    3

             4^k * (8 n + 1) ; 4^k * {2}
       156 :     3     3          5      2    2    0
       284 :     3     5          6      4    2    2
       316 :     3     5          6      0    2    2
       380 :     3     5          7      2    0    2

             9^k * (9 n + 6) ; 9^k * {2,3}
        75 :     1     4          5      1    1    0
```

```
111 :      1     4          7      1    0    0

      9^k * (9 n + 3) ; 9^k * {1,6}
177 :      2     4          7      4    2    1
213 :      2     4          7      0    1    1

      9^k * (3 n + 2) ; 9^k * {1,6}
225 :      3     4          7      4    3    3
333 :      3     4          7      1    0    0

      9^k * (3 n + 2) ; 9 n + 3; {1}
324 :      4     4          6      0    3    2
567 :      4     6          7      3    2    3
```

As long as I am putting in verbatim computer output, after a month or so I have what I suspect to be a complete list of the irregular and non-Kaplansky pairs of forms that represent the same numbers. No proofs. Let me see what I need to do to get this narrow enough that Latex will put it in the document without wrapping lines:

```
111 :   1   4     7   1  0  0       75 :  1   4     5   1  1  0
142 :   3   3     5   2  3  1       78 :  3   3     3   1  1  3
158 :   3   3     5  -1  2  1       78 :  3   3     3   1  1  3
158 :   3   3     5  -1  2  1      142 :  3   3     5   2  3  1
190 :   3   5     5   5  2  3       78 :  3   3     3   1  1  3
190 :   3   5     5   5  2  3      142 :  3   3     5   2  3  1
190 :   3   5     5   5  2  3      158 :  3   3     5  -1  2  1
213 :   2   4     7   0  1  1      177 :  2   4     7   4  2  1
216 :   2   4     8   4  1  1       54 :  2   2     4   1  2  0
232 :   3   5     5   3  1  3      232 :  3   3     7   1  2  1
284 :   3   5     6   4  2  2      156 :  3   3     5   2  2  0
316 :   3   5     6   0  2  2      156 :  3   3     5   2  2  0
316 :   3   5     6   0  2  2      284 :  3   5     6   4  2  2
333 :   3   4     7   1  0  0      225 :  3   4     7   4  3  3
380 :   3   5     7   2  0  2      156 :  3   3     5   2  2  0
380 :   3   5     7   2  0  2      284 :  3   5     6   4  2  2
```

380 :	3	5	7	2	0	2	316 :	3	5	6	0	2	2
567 :	4	6	7	3	2	3	324 :	4	4	6	0	3	2
639 :	5	5	8	-1	2	4	531 :	5	5	6	0	3	2
648 :	2	6	14	3	1	0	162 :	2	2	14	1	2	2
648 :	5	7	7	6	1	5	648 :	5	5	8	0	4	3
999 :	5	8	8	-5	1	4	675 :	5	5	8	-1	4	2
1944 :	2	6	41	3	1	0	486 :	2	2	41	1	2	2
2592 :	4	7	25	-4	2	2	648 :	4	7	7	5	2	2

These are pairs of positive quadratic forms that represent the same numbers, and violate a Kaplansky conjecture.

Delta : A B C R S T   means

$f(x,y,z) = A x^2 + B y^2 + C z^2 + R y z + S z x + T x y,$

and $\text{Delta} = 4ABC + RST - A R^2 - B S^2 - C T^2.$

The two pair within a genus each are

232 :	3	5	5	3	1	3	232 :	3	3	7	1	2	1
648 :	5	7	7	6	1	5	648 :	5	5	8	0	4	3

The most productive discriminant ratio is 4,
which includes Kap's two infinite families, also

24 :	1	2	4	2	1	1	6 :	1	1	2	1	1	0
72 :	2	2	5	1	1	1	18 :	2	2	2	1	2	2
216 :	2	5	6	3	0	1	54 :	2	2	5	1	2	2
648 :	2	6	14	3	1	0	162 :	2	2	14	1	2	2
1944 :	2	6	41	3	1	0	486 :	2	2	41	1	2	2

or

48N-24:	2	6	N	3	1	0	12N-6:	2	2	N	1	2	2

where $N = (1+ 3^k)/2$, and the pairs for N = 1,2,5 are regular and have been Schiemann reduced.

```
Reminder: Kap's two infinite familes, are

4D : A    3A    C   0   0  0    D :   A   A   C    0    0    A

4D:  A  2A-R  2A+R  0  2R  0    D :   A   A   A    R    R    R
```

It is well known that the positive binary forms $f(x,y) = x^2 + xy + y^2$ and $g(x,y) = x^2 + 3y^2$ represent the same numbers. Assume $0 < s < t$. So, the **"quasi-diagonal"** form

$$\{s, s, t, 0, 0, s\}$$

represents exactly the same numbers as

$$\{s, 3s, t, 0, 0, 0\}.$$

Also

$$\{s, t, t, t, 0, 0\}$$

represents the same numbers as

$$\{s, t, 3t, 0, 0, 0\}.$$

Next we get four related families of such pairs. Irving Kaplansky announced one of these in a letter to John Hsia and Dennis Estes dated 18 May, 1994. In later documents he displayed more variants, especially [41].

Consider the form with Brandt-Intrau coefficient sextuple $\{t, t, t, s, s, s\}$. This form has $\Delta = 4t^3 - 3ts^2 + s^3 = (2t - s)^2(t + s)$. Furthermore the binary section $B(x,y) = tx^2 + sxy + ty^2$ is positive if $2t > |s|$. Therefore $\{t, t, t, s, s, s\}$ is positive definite if $2t > s > 0$ or if $t > -s \geq 0$. It is reduced in the sense of Schiemann only if $t > s \geq 0$.

So, define

$$g(x,y,z) = tx^2 + ty^2 + tz^2 + syz + szx + sxy.$$

Next define

$$h(x,y,z) = tx^2 + (2t - s)y^2 + (2t + s)z^2 + 2szx$$

with Brandt-Intrau sextuple $\{t, 2t - s, 2t + s, 0, 2s, 0\}$.

We prove that g and h represent exactly the same numbers. First, g dominates h, because

$$g(X, Y + Z, -Y + Z) = h(X, Y, Z).$$

But h also dominates g. Given some target number N, suppose

$$g(X, Y, Z) = N.$$

We get three formulas that show $4N$ represented by g.

$$h(2X, Y - Z, Y + Z) = 4N.$$

$$h(2Y, X - Z, X + Z) = 4N.$$

$$h(2Z, -X + Y, X + Y) = 4N.$$

As we have three integers to consider, X, Y, Z, it follows by the pigeonhole principle that two of them share the same parity. That is, at least one of $Y + Z, Z + X, X + Y$ is even. Then the relevant one of the three formulas above has even numbers as arguments, and those can be divided by two to show an integer representation for N by g. We have shown:

Theorem. $\{t, t, t, s, s, s\}$ and $\{t, 2t - s, 2t + s, 0, 2s, 0\}$ represent exactly the same numbers.

Notice that $\{t, 2t - s, 2t + s, 0, 2s, 0, \}$ or $h(x, y, z) = tx^2 + (2t - s)y^2 + (2t + s)z^2 + 2szx$ decomposes into a binary plus a unary, to be specific

$$h(x, y, z) = \Big(tx^2 + 2sxz + (2t + s)z^2\Big) + (2t - s)y^2.$$

So it seems reasonable to refer to $\{t, t, t, s, s, s\}$ as **"quasi-decomposable."** I have no idea whether it matters, but the mixed coefficient $2s$ in $2sxz$ is even.

There are other fairly common ways for $\{t, t, t, s, s, s\}$ to be disguised, as it is not necessarily Schiemann reduced if t is too small.

Theorem. $\{a, a, b, a, a, a\}$ is equivalent to $\{b, b, b, 2b - a, 2b - a, 2b - a\}$. Note that $\Delta = a^2(3b - a)$ and the form is positive when $3b > a > 0$.

Proof. Let

$$g(x, y, z) = ax^2 + ay^2 + bz^2 + ayz + azx + axy.$$

Let

$$h(x, y, z) = bx^2 + by^2 + bz^2 + (2b - a)yz + (2b - a)zx + (2b - a)xy.$$

Then

$$g(X, Y, -X - Y - Z) = h(X, Y, Z)$$

is given by an invertible map, with matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & -1 \end{pmatrix}.$$

Theorem. $\{t, t, t, -s, s, s\}$ is equivalent to $\{t, t, t, -s, -s, -s\}$. Note that $\Delta = (2t + s)^2(t - s)$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Corollary.$\{t, t, t, -s, s, s\}$ represents the same numbers as $\{t, 2t + s, 2t - s, 0, -2s, 0\}$

Finally, given $\{t, t, t, -s, s, s\}$ with $t > s > \frac{2t}{3}$, Schiemann reduces this to $\{3(t - s), t, t, -s, 2(t - s), 2(t - s)\}$. So we need

Theorem. $\{3(t - s), t, t, -s, 2(t - s), 2(t - s)\}$ and $\{t, t, t, -s, -s, -s\}$. are equivalent. Note that $\Delta = (2t + s)^2(t - s)$.

Proof. Let

$$g(x, y, z) = tx^2 + ty^2 + tz^2 - syz + szx + sxy.$$

Let

$$h(x, y, z) = 3(t - s)x^2 + ty^2 + tz^2 - syz + 2(t - s)zx + 2(t - s)xy.$$

Then

$$g(X + Y, -X - Z, -X) = h(X, Y, Z)$$

gives an invertible map, matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}.$$

To summarize, the forms $\{t, t, t, s, s, s\}$, $\{a, a, b, a, a, a\}$, $\{t, t, t, -s, s, s\}$, $\{3(t - s), t, t, -s, 2(t - s), 2(t - s)\}$, are **"quasi-decomposable,"** each represents exactly the same numbers as an appropriate form; this form can be written as a binary plus a unary, where the mixed coefficient of the binary is even.

24 Kaplansky's proofs: a few specifics

On October 4, 1999, Kap wrote down short elementary proofs for three regular forms where the original proofs used spinor genus methods. These are:
$\Delta = 108$, coefficients $\{1, 3, 10, 3, 1, 0\}$, original proof in [28];
$\Delta = 432$, coefficients $\{1, 3, 37, 3, 1, 0\}$, original proof in [1];
$\Delta = 972$, coefficients $\{1, 7, 36, 0, 0, 1\}$, original proof in [28]. In this section I give an elementary proof for $\Delta = 289$, coefficients $\{3, 5, 6, 1, 2, 3\}$, original proof in [50]. My own original proofs for some 27 forms follow this section. These proofs, together with those indicated in [40] and [41], give elementary proofs for all regular forms found (and proved) using spinor genus methods, at least to date, those being in [28, 50, 1]. There are still 22 forms, seeming to be regular, for which no proof is known.

Kaplansky made a list of all known proofs of regularity in [40]. Some of the proofs are contained in three appendices. There are 119 forms that appear to be regular that are also contained in genera with more than one equivalence class (so that a proof of regularity for the candidate form is required). Of course 22 forms have no proofs, and several forms have well-known proofs (diagonal or quasi-diagonal). On lists of the relevant multi-class genera, Kap [40] lists 75 forms and the location of the original proof. We give details of two proofs using the technique of the previous section, but not indicated in [41].

Our first proof is that of item number 27 in [40]. Kap's original proof is in [40, Appendix II]. We have discriminant $\Delta = 121$, candidate form $\{1, 3, 11, 0, 0, 1\}$, with genus mate $\{3, 4, 4, -3, 2, 2\}$. But $\{3, 4, 4, -3, 2, 2\}$ is quasi-decomposable: it is equivalent to $\{4, 4, 4, -3, -3, -3\}$, and in turn this represents the same numbers as $\{4, 11, 5, 0, -6, 0\}$. Finally, $\{1, 3, 11, 0, 0, 1\}$ dominates $\{4, 11, 5, 0, -6, 0\}$; given

$$f(x, y, z) = x^2 + 3y^2 + 11z^2 + xy,$$

$$f(2X - 2Z, Z, Y) = 4X^2 + 11Y^2 + 5Z^2 - 6ZX.$$

Our second proof is that of item number 38 in [40]. Schulze-Pillot's original proof is in [50]. We have discriminant $\Delta = 289$, candidate form $\{3, 5, 6, 1, 2, 3\}$, with genus mate $\{3, 6, 6, -5, 2, 2\}$. But $\{3, 6, 6, -5, 2, 2\}$ is equivalent to $\{6, 6, 6, -5, -5, -5\}$, which represents exactly the same numbers as $\{6, 17, 7, 0, -10, 0\}$. Given

$$f(x, y, z) = 3x^2 + 5y^2 + 6z^2 + yz + 2zx + 3xy,$$

$$f(Y+Z,-2Y,X-Z) = 6X^2+17Y^2+7Z^2-10ZX.$$

Sometimes the genus mate is quasi-decomposable but the candidate for regularity does not have a homothety to the revised form. This happens, for instance , for item number 14, $\Delta = 50$, coefficients $\{1,2,7,2,1,0\}$, mate $\{1,1,17,1,1,1\}$. Kap's proof is in [40, Appendix I]. The same problem for item number 45, $\Delta = 484$, sextuple $\{1,3,44,0,0,1\}$, mate $\{5,5,5,-1,1,1\}$. Kap's proof is in [40, Appendix II].

Another five proofs by this technique, no more difficult, are indicated in Appendix III and then, with a just a little more detail but different ID numbers, in [41].

These are: item 34, $\Delta = 216$, $\{3,5,5,2,3,3\}$, with mate $\{3,3,8,0,0,3\}$; of course $\{3,3,8,0,0,3\}$ represents the same numbers as $\{3,8,9,0,0,0\}$. But with

$$f(x,y,z) = 3x^2+5y^2+5z^2+2yz+3zx+3xy,$$

$$f(X-Z,Y+Z,-Y+Z) = 3X^2+8Y^2+9Z^2.$$

Next, item 40, $\Delta = 392$, $\{3,3,12,-2,2,1\}$, with mate $\{5,5,5,3,3,3\}$; of course $\{5,5,5,3,3,3\}$ represents the same numbers as $\{5,7,13,0,6,0\}$. But with

$$f(x,y,z) = 3x^2+3y^2+12z^2-2yz+2zx+xy,$$

$$f(X+Y+Z,-X+Y-Z,-Z) = 5X^2+7Y^2+13Z^2+6ZX.$$

Next, item 41, $\Delta = 400$, $\{3,3,12,2,2,1\}$, with genus mate $\{5,5,7,5,5,5\}$; of course $\{5,5,7,5,5,5\}$ is equivalent to $\{7,7,7,9,9,9\}$ and represents the same numbers as $\{7,5,23,0,18,0\}$. But with

$$f(x,y,z) = 3x^2+3y^2+12z^2+2yz+2zx+xy,$$

$$f(X+Y+Z,X-Y+Z,Z) = 7X^2+5Y^2+23Z^2+18ZX.$$

Next, item 43, $\Delta = 432$, $\{3,5,9,3,0,3\}$, with genus mate $\{3,3,17,3,3,3\}$; of course $\{3,3,17,3,3,3\}$ is equivalent to $\{17,17,17,31,31,31\}$ and represents the same numbers as $\{17,3,65,0,62,0\}$. But with

$$f(x,y,z) = 3x^2+5y^2+9z^2+3yz+3xy,$$

$$f(X+Y+2Z,-2X-4Z,Z) = 17X^2+3Y^2+65Z^2+62ZX.$$

Finally, item 46, $\Delta = 600$, $\{5,7,7,6,5,5\}$, with mate $\{5,5,8,0,0,5\}$; of course $\{5,5,8,0,0,5\}$ represents the same numbers as $\{5,8,15,0,0,0\}$. But with

$$f(x,y,z) = 5x^2 + 7y^2 + 7z^2 + 6yz + 5zx + 5xy,$$
$$f(X-Z, Y+Z, -Y+Z) = 5X^2 + 8Y^2 + 15Z^2.$$

25 My original proofs: Odd 44 $= 4 \cdot 11, \Delta = 44$

Several of my own proofs (in this and the following sections) can be improved by the methods of [41] (as described in the previous two sections), or in the appendices of [40]. What can you do?

This is item number 11 in [40]. Recall that I eventually began using a uniform value for the discriminant: given

$$T(x,y,z) = ax^2 + by^2 + cz^2 + dyz + ezx + fxy,$$

define

$$\Delta = 4abc + def - ad^2 - be^2 - cf^2.$$

We prove here that h (below) is regular.

$$h(x,y,z) = 4x^2 + y^2 + yz + 3z^2$$

is based in an evident way on

$$r(x,y,z) = x^2 + y^2 + yz + 3z^2.$$

That is

$$r(2x,y,z) = h(x,y,z).$$

The form r was proved regular by Jones and Pall [36].

Now h does not represent any numbers congruent to 2 mod 4. So among even numbers, suppose n is eligible and divisible by 4. First find $\frac{n}{4} = r(x,y,z)$. Then we have $n = r(2x,2y,2z) = h(x,2y,2z)$, so that n really is represented by h.

So we need to show that any eligible odd number can be represented by r with the value of x even. First, note that if $n = r(x,y,z)$ and $n \equiv 3 \bmod 4$, we already have x even. We continue with $n \equiv 1 \bmod 4$. We're assuming x odd; it follows that y,z are even.

Case I: If both y, z are divisible by 4, we get a new even value for x by applying a rational automorph of r,

$$r\left(\quad \frac{2y+z}{2}, \quad \frac{2x-z}{2}, \quad z \right) = r(x,y,z).$$

The new value for x is $x' = (2y+z)/2$, and this will be even.

Case II: If x odd, $y \equiv z \equiv 2 \bmod 4$. After choosing $\pm x$ so that $2x+y+6z$ is divisible by 8, we get a new even value for x in the identity:

$$r\left(\quad \frac{2x+y+6z}{4}, \quad -y, \quad \frac{-2x+y+2z}{4} \right) = r(x,y,z).$$

Case III: If x odd, one of y, z is 2 mod 4, the other 0 mod 4. After choosing $\pm x$ so that $2x+y-5z$ is divisible by 8, we get a new even value for x in the identity:

$$r\left(\quad \frac{2x+y-5z}{4}, \quad y+z, \quad \frac{2x-y+z}{4} \right) = r(x,y,z).$$

This completes the proof. ◯

26 Odd 189 =27 $\cdot 7, \Delta = 189$

This is item number 32 in [40].

$$h = 2x^2 + 3y^2 + 8z^2 + zx.$$

The only other form in the genus of h is

$$m = 3x^2 + 3y^2 + 8z^2 + 3yz + 3zx + 3xy.$$

We get three similar expressions,

$$4m(x,y,z) = h(y-3z, 2x+y+z, -y-z),$$

$$4m(x,y,z) = h(x-3z, x+2y+z, -x-z),$$

$$4m(x,y,z) = h(x+y+4z, x-y, -x-y).$$

This gives a proof!!! Let n be represented by the mate m, so that $n = 3x^2 + 3y^2 + 8z^2 + 3yz + 3zx + 3xy$. There must be an agreement mod2 in one of the pairs $(x,y), (x,z)$, or (y,z). Therefore one of the three formulas above has all even entries in the right hand side, we can divide the entries by 2, thereby dividing the value $4n$ by 4. The result is a representation of n itself by the candidate form h.◯

27 Odd 243 $= 3^5, \Delta = 243$

This is item number 37 in [40].

The form considered is

$$h(x, y, z) = 2x^2 + 3y^2 + 11z^2 + 3yz + zx.$$

The form h must miss numbers of the forms $3m+1, 27m+9, 9^k(9m+6)$. As to multiples of 3, we have the homothety

$$h(y+3z, x+y, -y) = 3(x^2 + 4y^2 + 6z^2 + 3yz + xy).$$

The latter form is regular of discriminant 81, listed as 81: 1 4 6 3 0 1.

We proceed to show how to represent a number $n \equiv 2 \bmod 3$. We use the identity

$$4\, h = (2x - 4z)^2 + (2x + 5z)^2 + 3\,(2y + z)^2.$$

We will be able to represent n if we can arrange

$$4n = u^2 + v^2 + 3\, w^2,$$

$$u \equiv 0 \bmod 2, v \equiv w \bmod 2, u \equiv v \bmod 9.$$

Begin with

$$n = a^2 + b^2 + 3\, c^2,$$

choosing $\pm b$ so that $a \equiv b \bmod 3$. Then choose values for u, v, w based on values for $a, b, 3c$ mod 9; see the following table for $a \equiv 1 \bmod 9$.

a	b	3 c	u	v	w
1	1	0,3,6	2 a	2 b	2 c
1	4	0	2 b	-a + 3 c	-a - c
1	4	3	2 a	-b - 3c	-b+c
1	4	6	2 a	-b+3c	-b-c
1	7	0	2a	-b+3c	-b-c
1	7	3	2b	-a-3c	-a +c
1	7	6	2b	-a+3c	-a-c

If, instead, $n \equiv 3 \bmod 9$, note that $a, b \equiv 0 \bmod 3$, but c cannot be divisible by 3, so that $3c \equiv 3, 6 \bmod 9$. That allows us to modify the table and catch the cases where one of a, b is divisible by 9 but the other is not.$\bigcirc$

28 Odd 648 = 8 · 81 , $\Delta = 648$

This is item number 47 in [40]. A form h with coefficients (1,7,25,5,1,1), that is

$$h(x,y,z) = x^2 + 7y^2 + 25z^2 + 5yz + zx + xy.$$

The form is eligible to represent all positive integers other than $3n+2, 9n \pm 3, 4n+2, 4^k(16n+14)$. On even numbers, we need worry only about eligible multiples of 4, attended to by the homothety

$$h(2x+y, y+2z, -y) = 4(x^2 + 7y^2 + 7z^2 + 5yz + zx + xy).$$

The latter form is 162: 1 7 7 5 1 1. As to multiples of three, we need only consider multiples of 9, and

$$h(3x+y, 3z, y) = 9(x^2 + 3y^2 + 7z^2 + 2yz + zx + xy).$$

The latter form is 72: 1 3 7 2 1 1 . We will show that h represents all $6n+1$. It will be necessary to use the fact that the diagonal form $x^2+4y^2+9z^2$ is nearly regular. $x^2+4y^2+9z^2$ is not eligible to represent $8n+3, 9n \pm 3, 4^k(8n+7)$. Among eligible numbers, $x^2+4y^2+9z^2$ misses only the number 2.

We get a formula

$$2h = (x-y-4z)^2 + (x+2y+5z)^2 + (3y-3z)^2.$$

If a, b, c are the three parenthesized quantities, then

$$12x = 7a + 5b - c, 12y = -a + b + 3c, 12z = -a + b - c.$$

Therefore h represents all n for which we can arrange

$$2n = a^2 + b^2 + c^2$$

with $c \equiv 0 \bmod 3$,$a \equiv b \bmod 3$, and $a - b + c \equiv 0 \bmod 4$.

Notice that h represents the number 1. Now let $n \geq 7$ and $n \equiv 1 \bmod 6$. Then $2n \equiv 2 \bmod 4$, further $2n \equiv 2 \bmod 3$. As mentioned earlier, we can write

$$2n = r^2 + 4s^2 + 9t^2.$$

Both r and s will of necessity be prime to 3.

From this expression, let $a = r$, and choose $b = \pm 2s$ so that we have $a \equiv b \bmod 3$. Now $a - b$ is odd and t is odd, largely because $2n \equiv 2 \bmod 4$. Therefore, we may choose $c = \pm 3t$ to fulfill $a - b + c \equiv 0 \bmod 4$. ◯

29 Odd $1080 = 8 \cdot 27 \cdot 5, \Delta = 1080$

This is item number 51 in [40].

$$h = 3x^2 + 9y^2 + 11z^2 + 3yz + 3zx.$$

Candidate should miss $4m+2, 4^k(8m+2), 3m+1, 9^k(9m+6), 25^k(25m \pm 5)$. The only other form in the genus of h is

$$m = 3x^2 + 3y^2 + 41z^2 + 3yz + 3zx + 3xy.$$

We get three similar expressions,

$$4m(x, y, z) = h(x - y + 2z, -x - y, -4z),$$

$$4m(x, y, z) = h(x + 2y - z, -x - z, 4z),$$

$$4m(x, y, z) = h(2x + y - z, -y - z, 4z).$$

This gives a proof!!! Let n be represented by the mate m, so that $n = 3x^2 + 3y^2 + 41z^2 + 3yz + 3zx + 3xy$. There must be an agreement mod2 in one of the pairs $(x, y), (x, z)$, or (y, z). Therefore one of the three formulas above has all even entries in the right hand side, we can divide the entries by 2, thereby dividing the value $4n$ by 4. The result is a representation of n itself by the candidate form. $\bigcirc$

30 Odd 1323 $=27 \cdot 49, \Delta = 1323$

This is item number 52 in [40].

$$h = 2x^2 + 8y^2 + 21z^2 + xy.$$

The only other form in the genus of h is

$$m = 8x^2 + 8y^2 + 8z^2 - 5yz + 5zx + 5xy.$$

We get three similar expressions,

$$4m(x, y, z) = h(x - y + 4z, x - y, -x - y),$$

$$4m(x, y, z) = h(x + 4y - z, x - z, x + z),$$

$$4m(x,y,z) = h(4x+y+z, y+z, y-z).$$

This gives a proof. Let n be represented by the mate m, so that $n = 8x^2+8y^2+8z^2-5yz+5zx+5xy$. There must be an agreement mod2 in one of the pairs $(x,y),(x,z)$, or (y,z). Therefore one of the three formulas above has all even entries in the right hand side, we can divide the entries by 2, thereby dividing the value $4n$ by 4. The result is a representation of n itself by the candidate form h.◯

31 Odd 1800 = 8 ·9 · 25 , $\Delta = 1800$

This is item number 53 in [40].

$$h = 5x^2+11y^2+11z^2+7yz+5zx+5xy.$$

The only other form in the genus of h is

$$m = 5x^2+5y^2+24z^2+5xy.$$

We get three similar expressions,

$$4m(x,y,z) = h(x+2y-2z, x+2z, -x+2z),$$

$$4m(x,y,z) = h(2x+y-2z, y+2z, -y+2z),$$

$$4m(x,y,z) = h(x-y-2z, -x-y+2z, x+y+2z).$$

This gives a proof. Let n be represented by the mate m, so that $n = 5x^2+5y^2+24z^2+5xy$. At least one of the numbers $x, y, x+y$ is even. Therefore one of the three formulas above has all even entries in the right hand side, we can divide the entries by 2, thereby dividing the value $4n$ by 4. The result is a representation of n itself by the candidate form h.◯

32 Odd 5400 = 8 ·27 · 25 , $\Delta = 5400$

This is item number 54 in [40].

$$h = 7x^2+7y^2+28z^2-2yz+2zx+xy.$$

The only other form in the genus of h is

$$m = 13x^2+13y^2+13z^2+11yz+11zx+11xy.$$

We get three similar expressions,

$$4m(x,y,z) = h(-2y-2z, 2x+2z, x+y),$$

$$4m(x,y,z) = h(-2y-2z, 2x+2y, x+z),$$

$$4m(x,y,z) = h(-2x-2y, 2x+2z, y+z).$$

This gives a proof. Let n be represented by the mate m, so that $n = 13x^2 + 13y^2 + 13z^2 + 11yz + 11zx + 11xy$. At least one of the numbers $x+y, x+z, y+z$ is even. Therefore one of the three formulas above has all even entries in the right hand side, we can divide the entries by 2, thereby dividing the value $4n$ by 4. The result is a representation of n itself by the candidate form h.◯

33 Even 64 = $8^2 = 2^6$, $\Delta = 256$

This is item number 56 in [40]. Recall that I eventually began using a uniform value for the discriminant: given

$$T(x,y,z) = ax^2 + by^2 + cz^2 + dyz + ezx + fxy,$$

define

$$\Delta = 4abc + def - ad^2 - be^2 - cf^2.$$

$$h = x^2 + 5y^2 + 13z^2 + 2yz.$$

The only other form in the genus of h is

$$m = 4x^2 + 5y^2 + 5z^2 + 4yz + 4xy.$$

Even numbers are taken care of by the homothety

$$h(2x+y+z, y, z) = 2(2x^2 + 3y^2 + 7z^2 + 2yz + 2zx + 2xy).$$

The latter form is 32: 2 3 7 2 2 2.

We ignore the mate, and concentrate on

$$h = x^2 + (y-3z)^2 + (2y+2z)^2.$$

For eligible numbers $n = a^2 + b^2 + c^2$, we need only arrange that

$$c - 2b \equiv 0 \bmod 8.$$

This will follow if b is odd and $c \equiv 2 \bmod 4$. Then, $c - 2b \equiv 0 \bmod 4$, and if it should happen that $c - 2b \equiv 4 \bmod 8$, merely negate b, as $4b \equiv 4 \bmod 8$ and $c - (-2b) = c + 2b = c - 2b + 4b \equiv 0 \bmod 8$. If $n \equiv 6 \bmod 8$, we must have (up to permutation) that a, b are odd but $c \equiv 2 \bmod 4$. Similarly, if $n \equiv 5 \bmod 8$, we must have $a \equiv 0 \bmod 4$, with b odd and $c \equiv 2 \bmod 4$. Third, if n is not a square but $n \equiv 1 \bmod 8$, Jones and Pall showed that we may force $a, c \equiv 2 \bmod 4$ with b odd. Finally, if n is an odd square, we may take b, c to be 0. ◯

34 Even $108 = 4 \cdot 27$, $\Delta = 432$

This is item number 57 in [40]. An even form with coefficients (1,4,28,4,0,0), that is

$$h = x^2 + 4y^2 + 4yz + 28z^2.$$

The form h can not represent any of the numbers $4n+2, 4n+3, 9n+3, 9^k(9n+6)$. Even numbers (therefore multiples of 4) are handled by

$$h(4y, x+z, -z) = 4(x^2 + 4y^2 + 7z^2 + zx),$$

a homothety to 108: 1 4 7 0 1 0. Multiples of 3(therefore of 9) are handled by

$$h(3x, 3y+z, z) = 9(x^2 + 4y^2 + 4z^2 + 4yz),$$

a homothety to 12: 1 4 4 4 0 0. We find a useful formula,

$$h = x^2 + (2y+z)^2 + 3(3z)^2.$$

So we need to be able to write

$$n = a^2 + b^2 + 3c^2$$

with $b \equiv c \bmod 2$ and $c \equiv 0 \bmod 3$.

CASE A. If a, b, c are all odd, we already have the mod2 agreement. Here $n \equiv 5 \bmod 8$. At least one of a, b is prime to 3; relabel so that one is called b. If c is also prime to 3, choose $\pm c$ so that $b \equiv c \bmod 3$. Then switch to

$$n = a^2 + \left(\frac{b+3c}{2}\right)^2 + 3\left(\frac{b-c}{2}\right)^2.$$

Since $b + 3c \equiv b - c \bmod 4$, the fractions displayed agree modulo 2.

CASE B. If a is odd but the others even, but we continue to consider $n \equiv 5 \bmod 8$ in this paragraph, it follows that one of b, c is 2 mod 4 and the other is 0 mod 4. Thus when we switch to

$$n = a^2 + \left(\frac{b+3c}{2}\right)^2 + 3\left(\frac{b-c}{2}\right)^2,$$

everything is now odd. We return to case A to finish, so we are done with $n \equiv 5 \bmod 8$.

CASE C. If $n \equiv 1 \bmod 8$, but $n \equiv 2 \bmod 3$, both a, b must be prime to 3. Furthermore c and one of the others (call it b) are even. If it should happen that c is prime to 3, choose $\pm b$ so that $b \equiv c \bmod 3$, then switch to

$$n = a^2 + \left(\frac{b+3c}{2}\right)^2 + 3\left(\frac{b-c}{2}\right)^2.$$

CASE D. If $n \equiv 1 \bmod 8$, but $n \equiv 1 \bmod 3$, we have the very popular $n \equiv 1 \bmod 24$. If n is a square, it is obviously represented by h. If n is not a square, the Jones-Pall Theorem 5 says that we can write

$$n = a^2 + b^2 + 3c^2$$

with $a \equiv 3 \bmod 6$. Then b, c are even and b is prime to 3. If c is prime to 3, choose $\pm b$ so that $b \equiv c \bmod 3$, then switch to

$$n = a^2 + \left(\frac{b+3c}{2}\right)^2 + 3\left(\frac{b-c}{2}\right)^2.$$

○

35 Even 256 $= 16^2 = 2^8$, $\Delta = 1024$

This is item number 58 in [40].

$$h(x, y, z) = 3x^2 + 3y^2 + 32z^2 + 2xy.$$

The only other form in the genus of h is

$$m = 4x^2 + 8y^2 + 11z^2 + 8yz + 4zx.$$

The form h does not represent anything 2 mod 4, so even targets are handled by

$$h(x+y,-x+y,2z) = 4(x^2+2y^2+32z^2).$$

The latter is one of the famous diagonal regular forms 64: 1 2 32 0 0 0, proved regular in [36].

We ignore the mate. We simply show how to represent all $n \equiv 3 \bmod 8$ by h, which follows readily from

$$h = (x+y+4z)^2 + (x+y-4z)^2 + (x-y)^2.$$

If $n \equiv 3 \bmod 8$, we name $n = a^2+b^2+c^2$. From the expressions $z = (a-b)/8$, with $y = (a+b-2c)/4$ and $x = (a+b+2c)/4$, we know that we need merely arrange

$$a \equiv b \bmod 8, \quad a+b \pm 2c \equiv 0 \bmod 4.$$

We know that a, b, c are odd. To arrange $a \equiv b \bmod 8$, merely permute and change signs: for example, if the three values are $1, 3, 5 \bmod 8$, ignore the 1 but negate the 3, as $-3 \equiv 5 \bmod 8$.

Next, with the values chosen for a, b, we have $a+b \equiv 2c \equiv 2 \bmod 4$, so that $a+b \pm 2c \equiv 0 \bmod 4$ is immediate. ◯

36 Even 324 $= 18^2 = 4 \cdot 81$, $\Delta = 1296$

This is item number 59 in [40]. An even form with coefficients (3,4,28,4,0,0), that is

$$h(x,y,z) = 3x^2+4y^2+4yz+28z^2.$$

or

$$h = (y-4z)^2 + 3(y+2z)^2 + 3x^2. \tag{1}$$

Equation (1) shows that h can not represent any numbers of the form $9^k(3n+2)$. Furthermore, h does not represent any numbers congruent to 1 mod 4, 2 mod 4, or 6 mod 9.

For even numbers, multiples of 4 suffice, and we have

$$h(2y,x,z) = 4(x^2+3y^2+7z^2+zx),$$

being a homothety to 81: 1 3 7 0 1 0. For multiples of 3, we use

$$h(x,3y+z,z) = 3(x^2+12y^2+12z^2+12yz),$$

to the quasi-diagonal 108: 1 12 12 12 0 0.

We will concentrate on $n \equiv 7 \bmod 12$. We need to arrange

$$n = a^2 + 3b^2 + 3c^2$$

with

$$a \equiv b \bmod 6.$$

Then we will have

$$x = c, \;\; y = \frac{a+2b}{3}, \;\text{ and }\; z = \frac{-a+b}{6}.$$

Let

$$n = r^2 + 3s^2 + 3t^2.$$

As $n \equiv 3 \bmod 4$, possibly all three are odd. Otherwise, r and one of the others are even. Either way, we can require that $r \equiv s \bmod 2$. Furthermore, r is prime to 3, since $n \equiv 1 \bmod 3$. If s is also prime to 3, we choose $\pm r$ so that $r \equiv s \bmod 6$. If s is divisible by 3, switch to

$$n = \left(\frac{r-3s}{2}\right)^2 + 3\left(\frac{r+s}{2}\right)^2 + 3t^2.$$

This gives the required congruence modulo 6. ◯

37 Even 400 = $20^2 = 16 \cdot 25$, $\Delta = 1600$

This is item number 60 in [40]. An even form with coefficients (3,3,51,-2,2,2), that is

$$h = 3x^2 + 3y^2 + 51z^2 - 2yz + 2zx + 2xy.$$

The form h is not eligible to represent any numbers $4n+1, 4n+2, 4^k(8n+7), 25n \pm 5, 25n \pm 10$. The last two may be summarized by saying that h cannot represent any number that is divisible by 5 but not by 25.

We have a formula

$$h = (x+y+5z)^2 + (x+y-5z)^2 + (x-y+z)^2.$$

It follows that h represents all numbers n for which we can write

$$n = a^2 + b^2 + c^2$$

with $a \equiv b \equiv c \bmod 2$, and $a \equiv b \bmod 5$. For targets that are even (so divisible by 4), simply arrange the mod 5 conditions (see below) in representing $\frac{n}{4}$ as the sum of three squares, then double everybody to represent n itself. For multiples of 5 (therefore 25), use

$$h(5x + 2z, 5y - 2z, z) = 25(3x^2 + 3y^2 + 3z^2 - 2yz + 2zx + 2xy),$$

a map to 16: 3 3 3 -2 2 2.

We concentrate on a number $n \equiv 3 \bmod 8$. There is an expression

$$n = r^2 + s^2 + t^2,$$

for which we know that all three of r, s, t must be odd. Thus the mod2 equivalence is automatic. Considering r, s, t modulo 5, we can use $\pm$ signs to arrange two of them to agree mod5 unless one of them is 0, another is ± 1, and the last is ± 2. However, in that case n itself is divisible by 5 (and thus by 25). ◯

38 Even 448= $64 \cdot 7$, $\Delta = 1792$

This is item number 61 in [40].

$$h(x, y, z) = 5x^2 + 8y^2 + 12z^2 + 4zx.$$

Candidate must miss $4n + 2, 4n + 3, 4^k(8n + 1)$. The only other form in the genus of h is

$$m(x, y, z) = 5x^2 + 5y^2 + 20z^2 + 4yz + 4zx + 2xy.$$

We get three similar expressions,

$$h(2y, x - 2z, x + 2z) = 4m(x, y, z),$$
$$h(2x, y - 2z, y + 2z) = 4m(x, y, z),$$
$$h(4z, x - y, x + y) = 4m(x, y, z).$$

This gives a proof!!! We need merely show that every number represented by m is also represented by the candidate form h. Let n be represented by the mate m, so that $n = 5x^2 + 5y^2 + 20z^2 + 4yz + 4zx + 2xy$. The three formulas above show us three ways to represent $4n$ by h. Now, at least one of the three numbers $x, y, x - y$ is even. Therefore one of the three formulas above has all even entries in the left hand side. In that formula, we can divide the entries by 2, thereby dividing the value $4n$ by 4. The result is a representation of n itself by the candidate form. ◯

39 Even 1024 = 2^{10} , $\Delta = 4096$

This is item number 62 in [40].

$$h = 3x^2 + 2xy + 11y^2 + 32z^2.$$

The only other form in the genus of h is

$$m = 11x^2 + 11y^2 + 12z^2 - 4yz + 4zx + 10xy.$$

We get three similar expressions,

$$4m(x, y, z) = h(x + 4z, -x - 2y, -x),$$

$$4m(x, y, z) = h(y - 4z, -2x - y, -y),$$

$$4m(x, y, z) = h(x - y + 4z, -x + y, -x - y).$$

This gives a proof: let n be represented by the mate m, so that $n = 11x^2 + 11y^2 + 12z^2 - 4yz + 4zx + 10xy$. At least one of the three numbers $x, y, x + y$, must be even. ○

40 Even 1280 = $2^8 \cdot 5$, $\Delta = 5120$

This is item number 63 in [40].

$$h = 7x^2 + 4xy + 12y^2 + 16z^2.$$

The only other form in the genus of h is

$$m = 7x^2 + 7y^2 + 28z^2 - 4yz + 4zx + 2xy.$$

We get three similar expressions,

$$4m(x, y, z) = h(2y, x - 2z, -x - 2z),$$

$$4m(x, y, z) = h(2x, y + 2z, -y + 2z),$$

$$4m(x, y, z) = h(4z, x - y, -x - y).$$

This gives a proof: let n be represented by the mate m, so that $n = 7x^2 + 7y^2 + 28z^2 - 4yz + 4zx + 2xy$. At least one of the three numbers $x, y, x + y$, must be even. Therefore one of the three formulas above has all even entries in the right hand side, we can divide the entries by 2, thereby dividing the value $4n$ by 4. The result is a representation of n itself by the candidate form. ○

41 Even 1296 $= 36^2 = 16 \cdot 81$, $\Delta = 5184$

This is item number 64 in [40].

$$h = 5x^2 + 8y^2 + 36z^2 + 4xy.$$

The only other form in the genus of h is

$$m = 8x^2 + 9y^2 + 20z^2 + 8zx.$$

We ignore the mate. We simply show how to represent all $n \equiv 5 \bmod 12$ by h, which follows readily from

$$h = (x - 2y)^2 + (2x + 2y)^2 + (6z)^2.$$

If $n \equiv 5 \bmod 12$, we name $n = a^2 + b^2 + c^2$. To find integer values for x, y, z, we need to arrange that b is even, c is divisible by 6, and $a + b \equiv 0 \bmod 3$. That this is possible follows directly from the regularity of the diagonal form $x^2 + 4y^2 + 36z^2$, proved by Jones and Pall. That is, we may take c divisible by 6 and b even. The last detail comes from the realization that a, b are both prime to 3, so we choose $\pm a$ to arrange $a + b \equiv 0 \bmod 3$. ◯

42 Even 1728 $= 12^3 = 64 \cdot 27$, $\Delta = 6912$

This is item number 66 in [40].

$$h = x^2 + 16y^2 + 112z^2 + 16yz.$$

This form represents all eligible multiples of three (actually 9) by a reduction to the quasidiagonal (1,16,16,16,0,0). Even numbers work out by a related technique. From the expression

$$h = x^2 + (4y + 2z)^2 + 3(6z)^2,$$

we see that we need merely arrange

$$n = a^2 + b^2 + 3c^2, \quad c \equiv 0 \bmod 6, \quad b \equiv c \bmod 4.$$

The easier case is $n \equiv 17 \bmod 24$. Let a be odd. Note that both a, b are prime to 3, and that b, c are even and equivalent mod4. If c is not divisible by 3, choose $\pm b$ so that $b \equiv c \bmod 3$, then create new values $b' = (b+3c)/2$ and $c' = (b-c)/2$.

The harder case is $n \equiv 1 \bmod 24$. Choose a to be divisible by 3. Jones and Pall showed that we may choose whether a is equivalent to 0 or 3 mod6, that is odd or even. We elect $a \equiv 3 \bmod 6$. Then b is prime to 3 but even, and it follows that c is also even with $b \equiv c \bmod 4$. Once again, if c is not divisible by 3, create new values b', c' as before. Note that the new values for b, c are still even and congruent $\bmod 4$, as a is unchanged and $b^2 + 3c^2$ is still divisible by 8. $\bigcirc$

43 Another Even 1728 $= 12^3 = 64 \cdot 27$, $\Delta = 6912$

This is item number 67 in [40].

$$h = 4x^2 + 13y^2 + 37z^2 + 2yz + 4zx + 4xy.$$

From the expression

$$h = (2x+y+z)^2 + 12y^2 + 36z^2,$$

we see that we need merely arrange

$$n = a^2 + 12b^2 + 36c^2, \quad a+b+c \equiv 0 \bmod 2.$$

The eligible numbers prime to the discriminant are of the form $n \equiv 13 \bmod 24$. Given $n = a^2 + 12b^2 + 36c^2$, we need to show that $a+b+c$ is even. Since n is odd, we know that a is odd. Since $n \equiv 5 \bmod 8$, we know that one of b, c is odd and the other even, so that $b+c$ must be odd. It follows that $a+b+c$ is even. $\bigcirc$

44 Even 3136 $= 56^2 = 64 \cdot 49$, $\Delta = 12544$

This is item number 68 in [40].

$$h = 3x^2 + 19y^2 + 56z^2 + 2xy.$$

The only other form in the genus of h is

$$m = 12x^2 + 19y^2 + 19z^2 - 18yz + 4zx + 4xy.$$

We ignore the mate. We simply show how to represent all $n \equiv 3 \bmod 8$ by h, which follows readily from

$$h = (x + y - 6z)^2 + (x - 3y + 2z)^2 + (x + 3y + 4z)^2.$$

If $n \equiv 3 \bmod 8$, and either n is divisible by 7 or n is a nonresidue mod7, we name $n = a^2 + b^2 + c^2$. To find integer values for x, y, z, we need to arrange that

$$a + 2b + 4c \equiv 0 \bmod 7, \quad a + b + 2c \equiv 0 \bmod 4.$$

There is a form of discriminant 784 which is alone in its genus, therefore automatically regular. It is

$$r = 3x^2 + 19y^2 + 19z^2 - 18yz + 2zx + 2xy,$$

$$r = (x - 3y + 3z)^2 + (x + 3y + z)^2 + (x + y - 3z)^2.$$

Given an eligible number $n \equiv 3 \bmod 8$, i.e. n is divisible by 7 or n is a nonresidue mod7, we represent n by r. That is, we fix integer values for x, y, z with $n = 3x^2 + 19y^2 + 19z^2 - 18yz + 2zx + 2xy$. Then we construct the numbers

$$a = x - 3y + 3z, b = x + 3y + z, c = x + y - 3z.$$

With these values for a, b, c, we have shown how to write

$$n = a^2 + b^2 + c^2, \ \ a + 2b + 4c \equiv 0 \bmod 7.$$

So far we have completely ignored the condition mod4.It is critical that the equation $a + 2b + 4c \equiv 0 \bmod 7$ respects cyclic permutations: multiply the equation by 2, resulting in $c + 2a + 4b$; multiply by 2 again, giving $b + 2c + 4a \equiv 0 \bmod 7$. Since $n \equiv 3 \bmod 8$, we know that a, b, c are odd numbers. Odd numbers must be either $1, 3 \bmod 4$, so there must be an agreement mod4 among the three values a, b, c by the pigeonhole principle. Rename (from among the three cyclic permutations) the variables a, b, c so that $a + 2b + 4c \equiv 0 \bmod 7$ and $a \equiv b \bmod 4$. It follows that $a + b \equiv 2c \equiv 2 \bmod 4$, therefore $a + b + 2c \equiv 0 \bmod 4$. $\bigcirc$

45 Even 5184 $= 72^2 = 64 \cdot 81$, $\Delta = 20736$

This is item number 69 in [40].

$$h = 3x^2 + 16y^2 + 112z^2 + 16yz.$$

This form represents all eligible multiples of three by a reduction to the quasidiagonal (1,48,48,48,0,0). Even numbers work out by a related technique. We concentrate on $n \equiv 19 \bmod 24$. From the expression

$$h = (4y + 2z)^2 + 3(6z)^2 + 3x^2,$$

we see that we need merely arrange

$$n = a^2 + 3b^2 + 3c^2, \quad b \equiv 0 \bmod 6, \quad a \equiv b \bmod 4.$$

Notice that we may assume c odd from the beginning. Furthermore, as then $a^2 + 3b^2$ is divisible by 8, we know that a, b are even and congruent mod4. We just need to force b to be divisible by 3 without disturbing the other properties. This is easy, as a must be prime to 3. If b is also prime to 3, choose $\pm b$ so that $a \equiv b \bmod 3$, then create new values $a' = (a + 3b)/2$ and $b' = (a - b)/2$. $\bigcirc$

46 Another Even 5184 $= 72^2 = 64 \cdot 81$, $\Delta = 20736$

This is item number 70 in [40].

$$h = 7x^2 + 15y^2 + 55z^2 + 6yz - 2zx + 6xy.$$

From the expression

$$h = (x - 3y - z)^2 + 3(x + y + 3z)^2 + 3(x + y - 3z)^2,$$

we see that we need merely arrange

$$n = a^2 + 3b^2 + 3c^2, \quad b \equiv c \bmod 6, \quad a \equiv b \bmod 4.$$

for an eligible number n.

We concentrate on numbers prime to the discriminant, that is on $n \equiv 7 \bmod 24$. There is a regular form of discriminant 576 that represents all such n,

$$r = 7x^2 + 7y^2 + 15z^2 - 6yz + 6zx + 2xy.$$

In the expression

$$r = (x + y + 3z)^2 + 3(x + y - z)^2 + 3(x - y + z)^2,$$

note that the three terms are congruent $\bmod 2$. Since n is odd, all three terms must be odd, so we have $n = r^2 + 3s^2 + 3t^2$ with r, s, t odd. If both of s, t are divisible by 3 we can finish by choosing appropriate $\pm$ signs. The same is true if both of s, t are prime to 3. For the final case, assume that s is divisible by 3 but t is not. Choose $\pm s$ so that $r + 3s \equiv 2 \bmod 4$. Create new values

$$a = (r + 3s)/2, \quad b = (r - s)/2, \quad c = t.$$

These values a, b, c are again odd, but both b, c are prime to 3 and we finish by choosing appropriate $\pm$ signs throughout. ◯

47 Even 6400 $= 256 \cdot 25$, $\Delta = 25600$

This is item number 71 in [40].

$$h = 3x^2 + 27y^2 + 80z^2 + 2xy.$$

The only other form in this genus is

$$m = 12x^2 + 27y^2 + 27z^2 - 26yz + 4zx + 4xy.$$

We get three similar expressions,

$$4m(x, y, z) = h(4x + y, -y + 2z, y),$$

$$4m(x, y, z) = h(4x + z, 2y - z, z),$$

$$4m(x, y, z) = h(4x + y + z, -y - z, y - z).$$

This gives a proof: let n be represented by the mate m. At least one of the three numbers $y, z, y + z$ must be even. ◯

48 Even 6912 $= 256 \cdot 27$, $\Delta = 27648$

This is item number 72 in [40].

$$h = 9x^2 + 17y^2 + 48z^2 + 6xy.$$

The only other form in this genus is

$$m = 17x^2 + 17y^2 + 32z^2 - 8yz + 8zx + 14xy.$$

We get three similar expressions,

$$4m(x, y, z) = h(x - y - 2z, -x + y - 2z, x + y),$$

$$4m(x, y, z) = h(x - 3z, -x - 2y + z, x + z),$$
$$4m(x, y, z) = h(y + 3z, -2x - y - z, y - z).$$

This gives a proof: let n be represented by the mate m. At least one of the three numbers $x + y, x + z, y + z$ must be even. ◯

49 Another Even 6912 $= 256 \cdot 27$, $\Delta = 27648$

This is item number 73 in [40].

$$h = 5x^2 + 20y^2 + 77z^2 + 20yz + 2zx + 4xy.$$

From the expression

$$h = (x - 2y - 7z)^2 + (x - 2y + 5z)^2 + 3(x + 2y + z)^2,$$

we see that we need merely arrange

$$n = a^2 + b^2 + 3c^2, \quad a \equiv b \bmod 12, \quad b \equiv c \bmod 4.$$

for an eligible number n.

We concentrate on numbers prime to the discriminant, that is on $n \equiv 5 \bmod 24$. There is a regular form of discriminant 768 that represents all such n,

$$r = 5x^2 + 8y^2 + 20z^2 + 4zx.$$

In the expression

$$r = (x + 2y - 2z)^2 + (x - 2y - 2z)^2 + 3(x + 2z)^2,$$

note that the three terms are congruent mod 2. Since n is odd, all three terms must be odd, so we have $n = r^2 + 3s^2 + 3t^2$ with r, s, t odd. Both of r, s must be prime to 3, so the only problem occurs if $r \equiv s \bmod 4$ but $r \equiv -s \bmod 3$. In this case, choose $\pm t$ so that $r \equiv 3t \bmod 4$. Then construct the numbers a, b, c given by

$$a = (r + 3t)/2, \quad b = s, \quad c = (r - t)/2.$$

Notice that $a \equiv r \bmod 4$ but $a \equiv -r \bmod 3$. Therefore we have arranged that $a \equiv b \bmod 12$, and a solution is found by choosing $\pm c$. $\bigcirc$

50 Even 8640 $= 64 \cdot 27 \cdot 5$, $\Delta = 34560$

This is item number 74 in [40].

$$h = 13x^2 + 24y^2 + 28z^2 + 4zx.$$

The only other form in this genus is

$$m = 13x^2 + 13y^2 + 52z^2 + 4yz + 4zx + 2xy.$$

We get three similar expressions,

$$4m(x, y, z) = h(2y, x - 2z, x + 2z),$$

$$4m(x, y, z) = h(2x, y - 2z, y + 2z),$$

$$4m(x, y, z) = h(4z, x - y, x + y).$$

This gives a proof: let n be represented by the mate m. At least one of the three numbers $x, y, x + y$ must be even. $\bigcirc$

51 Even 43200 $= 64 \cdot 27 \cdot 25$, $\Delta = 172800$

This is item number 75 in [40].

$$h = 9x^2 + 41y^2 + 120z^2 + 6xy.$$

The only other form in this genus is

$$m = 36x^2 + 41y^2 + 41z^2 - 38yz + 12zx + 12xy.$$

We get three similar expressions,

$$4m(x,y,z) = h(4x + y, -y + 2z, y),$$

$$4m(x,y,z) = h(4x + z, 2y - z, z),$$

$$4m(x,y,z) = h(4x + y + z, -y - z, y - z).$$

This gives a proof: let n be represented by the mate m. At least one of the three numbers $y, z, y + z$ must be even. $\bigcirc$

52 Spinor Genus, Exceptions, Regularity

Spinor regular forms that are not regular were first discovered by Jones and Pall [36] but the current terminology and conceptual framework began with Eichler [22]. Jones and Pall [36] proved several 'diagonal' forms $f(x,y,z) = ax^2 + by^2 + cz^2$ to be regular, denoting such by the shorthand (a,b,c). Quoting from page 167:

> With the exception of $(1, 48, 144)$ which belongs to a genus of four classes, all regular forms (a,b,c) belong to genera of one or two classes. The companion class we find, in many cases, is regular except that either it fails to represent a finite number of integers represented by forms of the genus, or it fails to represent an infinite number specified by a finite number of formulas involving square factors: for example, all odd squares whose every prime factor is in some cases $\equiv 1 \pmod 4$ and in other cases $\equiv 1 \pmod 3$. These almost regular forms are new and are one of the most significant products of the method of proof.

In the first paragraph of section 5, pages 180-181, they sketch a proof for the first example: $(1, 1, 16)$ is regular, but $2x^2 + 2y^2 + 5z^2 + 2yz + 2zx \neq m^2$, where all prime factors of m are $\equiv 1 \pmod 4$. On the other hand, $2x^2+2y^2+5z^2+2yz+2zx$ does represent everything else that is represented by $x^2+y^2+16z^2$. These days we say that the two forms, while in the same genus, are in different spinor genera. Then we say that as $2x^2+2y^2+5z^2+2yz+2zx$ is alone in its spinor genus, it is spinor regular by default.

To expand on the use of language, they prove that $(1, 48, 144)$ is regular, while $9x^2 + 16y^2 + 48z^2 \neq w^2, 4w^2$ where all prime factors of w are $\equiv 1 \pmod 3$. Nowadays, we split the four classes of the genus into two spinor genera.

```
=====Discriminant  27648  ==Genus Size==   4
-------------**----------------------  27648 s. g. size---   2
1   48   144   0   0   0     {regular!}
4   48   49   48   4   0
-------------**----------------------  27648 s. g. size---   2
9   16   48   0   0   0  {spinor regular!}
16   25   25   14   16   16
---------**---------------  27648 s. genus count   2
```

The first spinor genus, taken together, represents everything represented by any form in the genus, and since $x^2 + 48y^2 + 144z^2$ does the same all by itself we call it regular. Either form in the second spinor genus fails to represent the 'spinor exceptional integers' denoted earlier by $w^2, 4w^2$. However, as $9x^2 + 16y^2+48z^2$ represents everything else represented by the genus, and, in short, dominates $16x^2+25y^2+25z^2+14yz+16zx+16xy$, we say that $9x^2+16y^2+48z^2$ is spinor regular.

A lesser-known phenomenon, anticipated by Kap in a 1995 letter to Schulze-Pillot and Hsia, is Theorem 4.3 on page 312 of [52], with example $4x^2+48y^2+49z^2+48yz+4zx \neq q^2$ for prime $q \equiv 5 \bmod 6$. Kap's examples of 1995 begin with $2x^2+2y^2+17z^2+2yz+2zx \neq s^2$, prime $s \equiv 3 \bmod 4$. In this case I had little trouble proving that $2x^2+2y^2+17z^2+2yz+2zx$ represents all other squares larger than 1, however it misses several nonsquares that are represented by its genus. Quoting the last sentence in Theorem 4.3 on page 312 of [52]:

> In particular, if there is a spinor exceptional integer a' for the genus of L that is represented by $\mathrm{spn}(L)$ but not by L (so

a' is below the bound for being sufficiently large), then there are infinitely many integers $a'p^2$ with p prime that are not represented by L.

Anyway, the 'almost regular' forms that involve square factors eventually became known as 'spinor regular,' and the sets of numbers missed by 'spinor genera,' such as m^2 became known as 'spinor exceptional integers.'

53 Kaplansky's Jones-Pall forms

In what follows we are trying to mimic the notation about numbers not represented ("missed") used in [36]. An error about the missed numbers on page 191 of [36] was corrected by Schulze-Pillot in [49], as item number 6 in Tabelle 1 on page 537 and Lemma 5 on page 538 "in Fall 6." Instructions for carrying out the computer search are taken from [8] and [20].

```
William C. Jagy; Kaplansky's``Jones-Pall Forms''
Integer coefficient positive ternary quadratic forms that are
spinor regular but are NOT regular. In each case the
coefficients of the form are Schiemann-reduced. As to the
order, the integer sextuple
{a b c d e f} refers to the quadratic form T defined by

T(x, y, z)  = a x^2 + b y^2 + c z^2 + d y z + e z x + f x y.

Below is the symmetric (Gram) Matrix for 2T:

2 a    f     e
 f    2 b    d
 e     d    2 c

The discriminant "Disc" is the absolute value of that favored
by Watson and Brandt and Intrau, half the determinant of 2T:

Disc = 4 a b c  +  d e f  -  a d^2  -  b e^2  -  c f^2.

Fact:  2 a b c <= Disc <= 4 a b c; see Watson's book, p. 29.
```

```
Some multiplicative semigroups; all primes are positive!
M :  generated by 1 and all primes congruent to 1 modulo 4.
W :  generated by 1 and all primes (not 3) 1 modulo 3.
E :  generated by 1 and all primes  1 or 3 modulo 8.
S :  generated by 1 and all primes (including 2 but not 7)
  congruent to 1 or 2 or 4 modulo 7.
=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=

=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=
Disc 64 Coef {2 2 5 2 2 0}, misses M^2.
Disc 108 Coef {3 3 4 0 0 3}, misses W^2.
Disc 108 Coef {3 4 4 4 3 3}, misses W^2.
Disc 128 Coef {1 4 9 4 0 0}, misses 2 M^2.
Disc 256 Coef {2 5 8 4 0 2}, misses M^2, 4 M^2.
Disc 256 Coef {4 4 5 0 4 0}, misses M^2.
Disc 324 Coef {1 7 12 0 0 1}, misses 3 W^2.
Disc 343 Coef {2 7 8 7 1 0}, misses S^2.
Disc 432 Coef {3 7 7 5 3 3}, misses W^2, 4 W^2.
Disc 432 Coef {4 4 9 0 0 4}, misses W^2.
Disc 432 Coef {3 4 9 0 0 0}, misses W^2.
Disc 1024 Coef {4 9 9 2 4 4}, misses M^2.
Disc 1024 Coef {4 5 13 2 0 0}, misses M^2.
Disc 1024 Coef {5 8 8 0 4 4}, misses M^2, 4 M^2.
Disc 1372 Coef {7 8 9 6 7 0}, misses S^2.
Disc 1728 Coef {4 9 12 0 0 0}, misses W^2.
Disc 2048 Coef {4 8 17 0 4 0},  misses E^2.
Disc 3888 Coef {4 9 28 0 4 0}, misses W^2.
Disc 4096 Coef {9 9 16 8 8 2}, misses M^2, 4M^2.
Disc 4096 Coef {4 9 32 0 0 4}, misses M^2.
Disc 4096 Coef {5 13 16 0 0 2}, misses 4 M^2.
Disc 5488 Coef {8 9 25 2 4 8}, misses S^2.
Disc 6912 Coef {9 16 16 16 0 0}, misses W^2, 4 W^2.
Disc 6912 Coef {13 13 16 -8 8 10}, misses W^2.
Disc 16384 Coef {9 17 32 -8 8 6}, misses M^2, 4 M^2, 16 M^2.
Disc 16384 Coef {9 16 36 16 4 8}, misses M^2, 4 M^2.
Disc 27648 Coef {9 16 48 0 0 0}, misses W^2, 4 W^2.
Disc 62208 Coef {9 16 112 16 0 0}, misses W^2, 4 W^2.
Disc 87808 Coef {29 32 36 32 12 24}, misses 4 S^2.
```

```
------------------------------------------------------------
CONJECTURED complete. Checked to Disc 1,400,000 by April 2007
```

That's it, I've been able to find only 29 of these. So it is difficult for a positive ternary quadratic form to be spinor regular unless it is also regular. See [8], [20], some history in [36], [49].

54 Recognizing Irregular Forms, Part 2

A surprising example showed up while I was searching for spinor regular forms: genera with a very large (minimal) spinor exception. Let

$$N = u^2 + v^2,$$

but require also that N be **squarefree**, so that N is not divisible by 3,7,11,19, or any prime of shape $4n + 3$. Let

$$T(x, y, z) = x^2 + y^2 + 16Nz^2.$$

Then, as has been proved for me by John S. Hsia, the genus of T splits into exactly two spinor genera, and N itself is a spinor exception. He mentioned once that the methods used are in Earnest, Hsia, and Hung [21].

In sum, using discriminant,

$$\Delta = 4abc + def - ad^2 - be^2 - cf^2$$

for any

$$T(x, y, z) = ax^2 + by^2 + cz^2 + dyz + ezx + fxy,$$

we find

$$\Delta = 64N.$$

That is, it may be necessary to check as high as

$$\frac{\Delta}{64}$$

to detect a spinor exception in the spinor genus of a form of interest, which is one way a form may fail to be regular.

55 The Standard Descent

Suppose the positive binary forms $x^2 + ky^2$ and $ax^2 + 2\beta xy + cy^2$ have the same discriminant, so that

$$k = ac - \beta^2.$$

Suppose further that

$$n = au^2 + 2\beta uv + cv^2.$$

Then we are going to "descend" from the ternary (W is also positive)

$$ax^2 + 2\beta xy + cy^2 + nWz^2$$

to n times

$$x^2 + ky^2 + Wz^2.$$

That is, letting B be the symmetric (Gram) matrix for $ax^2 + 2\beta xy + cy^2 + nWz^2$ and letting A be the symmetric (Gram) matrix for $x^2 + ky^2 + Wz^2$, we are going to display a matrix product

$$P'BP = nA.$$

with

$$\det P = n = \frac{\det B}{\det A}.$$

Here P' denotes the transpose of P. By abusing notation, we may regard B and A as lattices, and say (in the terminology of Wai Kiu Chan) that B *and* A *are* $\mathbf{Z}-$*lattices such that* $d(B) = nd(A)$, *where* n *is a positive integer, and* B *represents* $A^{(n)}$.

$$\begin{pmatrix} u & v & 0 \\ -\beta u - cv & au + \beta v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & \beta & 0 \\ \beta & c & 0 \\ 0 & 0 & nW \end{pmatrix} \begin{pmatrix} u & -\beta u - cv & 0 \\ v & au + \beta v & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

becomes

$$\begin{pmatrix} n & 0 & 0 \\ 0 & kn & 0 \\ 0 & 0 & nW \end{pmatrix} = n \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & W \end{pmatrix}.$$

To calculate this next bit, I used the method of Arndt, as reported on pages 62-63 of Buell [5]. Suppose instead that positive forms $x^2 + xy + ky^2$ and $ax^2 + (2\beta - 1)xy + cy^2$ have the same discriminant, so that

$$k = ac + \beta - \beta^2.$$

Suppose further that

$$n = au^2 + (2\beta - 1)uv + cv^2.$$

Then we are going to descend from the ternary

$$ax^2 + (2\beta - 1)xy + cy^2 + nWz^2$$

to n times

$$x^2 + xy + ky^2 + Wz^2.$$

$$\begin{pmatrix} u & v & 0 \\ (1-\beta)u - cv & au + \beta v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & \beta - \frac{1}{2} & 0 \\ \beta - \frac{1}{2} & c & 0 \\ 0 & 0 & nW \end{pmatrix} \begin{pmatrix} u & (1-\beta)u - cv & 0 \\ v & au + \beta v & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

becomes

$$\begin{pmatrix} n & \frac{n}{2} & 0 \\ \frac{n}{2} & kn & 0 \\ 0 & 0 & nW \end{pmatrix} = n \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & k & 0 \\ 0 & 0 & W \end{pmatrix}.$$

56 Recognizing Irregular Forms, Part 3: Binaries in Ternaries

Here is a curiosity, proved by combining results of Hsia and Billy Chan [7]: given $N = u^2 + v^2$, N squarefree, every form in the spinor genus containing $x^2 + y^2 + 16Nz^2$ represents N itself (primitively, by construction), meanwhile there are exactly two spinor genera and N itself is the smallest spinor exception.

A reason this is interesting is a result of Schulze-Pillot: quoting the last sentence in Theorem 4.3 on page 312 of [52]: "In particular, if there is a spinor exceptional integer a' for the genus of L that is represented by $\mathrm{spn}(L)$ but not by L (so a' is below the bound for being sufficiently large), then there are infinitely many integers $a'p^2$ with p prime that are not represented by L." My $(1, 1, 16N)$ example gives an infinite family of genera where the quoted clause of Schulze-Pillot's Theorem does not apply, as all forms in the "regular" spinor genus represent all spinor exceptional integers. If the families below work as conjectured, the cases of N squarefree would provide more families where the quoted clause of Schulze-Pillot's Theorem does not apply.

I have a list of similar items: below, I will display a binary form in variables u, v in the expression $N = g(u, v)$. Then I will display a ternary form $h(x, y, z)$. As I now see the matter, the important conjecture is that every form in the regular spinor genus of the higher discriminant descends to one or more forms in the regular spinor genus downstairs, and irregular only to irregular. That is, in these examples *spinor genera are preserved by descent*! For these examples anyway. I have many, many other examples where that fails.

For all these Gauss composition of binary forms is used. The first version is this: suppose $n = \alpha^2 + k\beta^2$. Then

$$\begin{pmatrix} \alpha & \beta \\ -k\beta & \alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} \alpha & -k\beta \\ \beta & \alpha \end{pmatrix} = \begin{pmatrix} n & 0 \\ 0 & nk \end{pmatrix}.$$

The second version is: suppose $n = \alpha^2 + \alpha\beta + k\beta^2$. Then

$$\begin{pmatrix} \alpha & \beta \\ -k\beta & \alpha+\beta \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & k \end{pmatrix} \begin{pmatrix} \alpha & -k\beta \\ \beta & \alpha+\beta \end{pmatrix} = \begin{pmatrix} n & \frac{n}{2} \\ \frac{n}{2} & nk \end{pmatrix}.$$

$\underline{N = u^2 + v^2}$, $h(x, y, z) = x^2 + y^2 + 16Nz^2$. Conjectures: $\operatorname{gen} h$ has exactly two spinor genera, every form in $\operatorname{spn} h$ has a homothety to $Nx^2 + Ny^2 + 16Nz^2$, also N is a spinor exception, while every form in $\operatorname{spn} h$ represents N. If we can take $\gcd(u, v) = 1$, then every form in $\operatorname{spn} h$ represents N primitively. If N is squarefree, N is also the smallest spinor exception for $\operatorname{gen} h$, and all the spinor exceptions are NM^2, where all prime factors of M are congruent to 1 modulo 4. **Note:** proved for squarefree N!

$\underline{N = u^2 + 2v^2}$, $h(x, y, z) = x^2 + 2y^2 + 64Nz^2$. Conjectures: $\operatorname{gen} h$ has exactly two spinor genera, every form in $\operatorname{spn} h$ has a homothety to either $Nx^2 + 2Ny^2 + 64Nz^2$ or $Nx^2 + 8Ny^2 + 8Nyz + 18Nz^2$, also N is a spinor exception, while every form in $\operatorname{spn} h$ represents N. If we can take $\gcd(u, v) = 1$, then every form in $\operatorname{spn} h$ represents N primitively. If N is squarefree, N is also the smallest spinor exception for $\operatorname{gen} h$, and all the spinor exceptions are NE^2, where all prime factors of E are congruent to 1 or 3 modulo 8.

$\underline{N = u^2 + uv + v^2}$, $h(x, y, z) = x^2 + xy + y^2 + 36Nz^2$. Conjectures: $\operatorname{gen} h$ has exactly two spinor genera, every form in $\operatorname{spn} h$ has a homothety to $Nx^2 + Nxy + Ny^2 + 36Nz^2$, also N is a spinor exception, while every form in $\operatorname{spn} h$ represents N. If we can take $\gcd(u, v) = 1$, then every form in $\operatorname{spn} h$ represents N primitively. If N is squarefree, N is also the smallest spinor exception for gen h, and all the spinor exceptions are NW^2, where all prime factors of W are congruent to 1 modulo 3.

$\underline{N = u^2 + uv + v^2}$, $h(x, y, z) = x^2 + 3y^2 + (9N + 1)z^2 + 3yz + zx$. Conjectures: gen h has exactly two spinor genera, every form in spn h has a homothety to either $Nx^2 + 3Ny^2 + 10Nz^2 + 3Nyz + Nzx$ or (when N is even) $Nx^2 + Nxy + Ny^2 + 36Nz^2$. When N is even $x^2 + 3y^2 + (9N + 1)z^2 + 3yz + zx$ and $x^2 + xy + y^2 + 36Nz^2$ are in the same genus! Also N is a spinor exception, while every form in spn h represents N. If we can take $\gcd(u, v) = 1$, then every form in spn h represents N primitively. If N is squarefree, N is also the smallest spinor exception for gen h, and all the spinor exceptions are NW^2, where all prime factors of W are congruent to 1 modulo 3.

Apparently for $N \equiv 0 \bmod 4$ these two give the same genus, e.g. $N = 4$

```
=====Discriminant  432  ==Genus Size==   4
-------------**----------------------  432 s. g. size---   2
1   1   144   0   0   1
1   3   37   3   1   0     (regular)
-------------**----------------------  432 s. g. size---   2
3   7   7   5   3   3  (spinor regular)
3   3   16   0   0   3
---------**---------------  432 s. genus count   2
```

while the spinor exceptions with $N = 4$ are the union of the W^2 and $4W^2$.

I was a little worried about descent from $\{1, 3, 9N+1, 3, 1, 0\}$. If N is odd and $N = \alpha^2 + 3\beta^2$, then $\alpha + \beta \equiv 1 \bmod 2$ and

$$\begin{pmatrix} \alpha & \beta & 0 \\ -3\beta & \alpha & 0 \\ \frac{\alpha-3\beta-1}{2} & \frac{\alpha+\beta-1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 3 & \frac{3}{2} \\ \frac{1}{2} & \frac{3}{2} & 9N+1 \end{pmatrix} \begin{pmatrix} \alpha & -3\beta & \frac{\alpha-3\beta-1}{2} \\ \beta & \alpha & \frac{\alpha+\beta-1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

becomes

$$\begin{pmatrix} N & 0 & \frac{N}{2} \\ 0 & 3N & \frac{3N}{2} \\ \frac{N}{2} & \frac{3N}{2} & 10N \end{pmatrix} = N \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 3 & \frac{3}{2} \\ \frac{1}{2} & \frac{3}{2} & 10 \end{pmatrix},$$

so $\{1, 3, 9N + 1, 3, 1, 0\}$ descends by a factor of N to $\{1, 3, 10, 3, 1, 0\}$.

If N is even and $N = \alpha^2 + 3\beta^2$, then N is divisible by 4, so let $N = 4M$. First we descend by 4 to $\{1, 1, 9N, 0, 0, 1\} = \{1, 1, 36M, 0, 0, 1\}$:

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 3 & \frac{3}{2} \\ \frac{1}{2} & \frac{3}{2} & 9N+1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

becomes

$$\begin{pmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 36N \end{pmatrix} = 4 \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 36M \end{pmatrix}.$$

Next there is the additional descent by a factor of M to $\{1, 1, 36, 0, 0, 1\}$. With $4M = N$, we have shown descent by a factor of N from $\{1, 3, 9N + 1, 3, 1, 0\}$ to $\{1, 1, 36, 0, 0, 1\}$ $\bigcirc\bigcirc$.

$\underline{N = u^2 + uv + 2v^2}$, $h(x, y, z) = x^2 + xy + 2y^2 + 49Nz^2$. Conjectures: $\operatorname{gen} h$ has exactly two spinor genera, every form in $\operatorname{spn} h$ has a homothety to $Nx^2 + Nxy + 2Ny^2 + 49Nz^2$, also N is a spinor exception, while every form in $\operatorname{spn} h$ represents N. If we can take $\gcd(u, v) = 1$, then every form in $\operatorname{spn} h$ represents N primitively. If N is squarefree, N is also the smallest spinor exception for $\operatorname{gen} h$, and all the spinor exceptions are NS^2, where all prime factors of S are congruent to 1 or 2 or 4 modulo 7.

Next we have two pair, where the primitive binary forms that drive the construction come two per discriminant:

$\underline{N = u^2 + 8v^2}$, $h(x, y, z) = x^2 + 8y^2 + 64Nz^2$. Conjectures: $\operatorname{gen} h$ has exactly two spinor genera, every form in $\operatorname{spn} h$ has a homothety to $Nx^2 + 8Ny^2 + 64Nz^2$, also N is a spinor exception, while every form in $\operatorname{spn} h$ represents N. If we can take $\gcd(u, v) = 1$, then every form in $\operatorname{spn} h$ represents N primitively. If N is squarefree, N is also the smallest spinor exception for $\operatorname{gen} h$, and all the spinor exceptions are NE^2, where all prime factors of E are congruent to 1 or 3 modulo 8.

Next we need this: suppose $N = 3\alpha^2 + 2\alpha\beta + 3\beta^2$. Then

$$\begin{pmatrix} \alpha & \beta \\ -\alpha - 3\beta & 3\alpha + \beta \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \alpha & -\alpha - 3\beta \\ \beta & 3\alpha + \beta \end{pmatrix} = \begin{pmatrix} N & 0 \\ 0 & 8N \end{pmatrix}.$$

$\underline{N = 3u^2 + 2uv + 3v^2}$, $h(x, y, z) = 3x^2 + 2xy + 3y^2 + 64Nz^2$. Conjectures: $\operatorname{gen} h$ has exactly two spinor genera, every form in $\operatorname{spn} h$ has a homothety to $Nx^2 + 8Ny^2 + 64Nz^2$, also N is a spinor exception, while every form in $\operatorname{spn} h$ represents N. If we can take $\gcd(u, v) = 1$, then every form in $\operatorname{spn} h$ represents N primitively. If N is squarefree, N is also the smallest spinor exception for $\operatorname{gen} h$, and all the spinor exceptions are NE^2, where all prime factors of E are congruent to 1 or 3 modulo 8.

$\underline{N = u^2 + uv + 4v^2}$, $h(x, y, z) = x^2 + xy + 4y^2 + 225Nz^2$. Conjectures: $\operatorname{gen} h$ has exactly two spinor genera, every form in $\operatorname{spn} h$ has a homothety to either $Nx^2 + Nxy + 4Ny^2 + 225Nz^2$ or $Nx^2 + 15Ny^2 + 15Nyz + 60Nz^2$, also

N is a spinor exception, while every form in spn h represents N. If we can take $\gcd(u, v) = 1$, then every form in spn h represents N primitively. If N is squarefree, N is also the smallest spinor exception for gen h, and all the spinor exceptions are NT^2, where all prime factors of T are congruent to 1 or 2 or 4 or 8 modulo 15.

I worked out the next one with help about "united forms" from page 57 of Buell [5]: if $X = x_1x_2 - Cy_1y_2$ and $Y = a_1x_1y_2 + a_2x_2y_1 + By_1y_2$, then

$$\left(a_1x_1^2 + Bx_1y_1 + a_2Cy_1^2\right)\left(a_2x_2^2 + Bx_2y_2 + a_1Cy_2^2\right)$$

is equal to

$$\left(a_1a_2X^2 + BXY + CY^2\right).$$

See also [13, pages 37,49].

Suppose $N = 2\alpha^2 + \alpha\beta + 2\beta^2$. Then

$$\begin{pmatrix} \alpha & \beta \\ -2\beta & 2\alpha + \beta \end{pmatrix}\begin{pmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix}\begin{pmatrix} \alpha & -2\beta \\ \beta & 2\alpha + \beta \end{pmatrix} = \begin{pmatrix} N & \frac{N}{2} \\ \frac{N}{2} & 4N \end{pmatrix}.$$

$\underline{N = 2u^2 + uv + 2v^2}$, $h(x, y, z) = 2x^2 + xy + 2y^2 + 225Nz^2$. Conjectures: gen h has exactly two spinor genera, every form in spn h has a homothety to either $Nx^2 + Nxy + 4Ny^2 + 225Nz^2$ or $Nx^2 + 15Ny^2 + 15Nyz + 60Nz^2$, also N is a spinor exception, while every form in spn h represents N. If we can take $\gcd(u, v) = 1$, then every form in spn h represents N primitively. If N is squarefree, N is also the smallest spinor exception for gen h, and all the spinor exceptions are NT^2, where all prime factors of T are congruent to 1 or 2 or 4 or 8 modulo 15.

Alright, I checked all of the listed families for spinor genus preservation during descent and $N \leq 200$, including N that cannot be primitively represented by the appropriate binary form. For any of the families mentioned and $N-$values $N_1|N_2$, there is often a descent by a factor of $\frac{N_2}{N_1}$ from the genus with parameter $N = N_2$ down to the genus with parameter $N = N_1$, not just for $N_1 = 1$. In every case, forms in the regular spinor genus with $N = N_2$ descended only to forms in the regular spinor genus with $N = N_1$, and forms in the irregular spinor genus with $N = N_2$ descended only to forms in the irregular spinor genus with $N = N_1$.

57 Once More, with Four Spinor Genera

Prof. Hsia asked me if I could produce any similar example with four spinor genera, so I continued experimenting with this material. As near as I can make out, it is difficult to have a genus with four spinor genera, two independent spinor exceptions, and even one form that represents the minimal number in the two squareclasses of spinor exceptions. Still, I suspect I found a two parameter family that does just that.

First, I have computed some extra with the smallest of the following examples, $w = 17$, so $16w^2 = 4624$. The genus of the ternary $x^2 + 17y^2 + 4624z^2$ has four spinor genera and a full plate of spinor exceptions: one family is U^2 where U is an arbitrary product of $\{1, 3, 7, 11, 13, 23, 31, 53, 71, 79, 89, 101, 107, \ldots\}$. These appear to be 1 and all the primes r with Legendre symbol $(-17|r) = 1$. Then $17V^2$, V factors $\{1, 5, 13, 17, 29, 37, 41, 53, 61, 73, 89, 97, 101, 109, \ldots\}$. These are apparently just 1 and all the primes $p \equiv 1 \bmod 4$, so Jones and Pall [36] would have called the second family $17m^2$.

The second smallest of the following examples, $w = 41$, so $16w^2 = 26896$. The genus of the ternary $x^2 + 41y^2 + 26896z^2$ has four spinor genera and a full plate of spinor exceptions: one family is U^2, U is an arbitrary product of $\{1, 3, 5, 7, 11, 19, 37, 47, 61, 67, 71, 73, 79, 113, \ldots\}$. These appear to be 1 and all the primes r with Legendre symbol $(-41|r) = 1$. Then $41V^2$, V factors $\{1, 5, 13, 17, 29, 37, 41, 53, 61, \ldots\}$. Jones and Pall would have called the second family $41m^2$.

The third smallest of the following examples, $w = 65$, so $16w^2 = 67600$. The genus of the ternary $x^2 + 65y^2 + 67600z^2$ has four spinor genera and a full plate of spinor exceptions: one family is U^2, U is an arbitrary product of $\{1, 3, 11, 19, 23, 29, 31, 37, 43, 59, 61, 71, 73, 97, 101, \ldots\}$. These appear to be 1 and all the primes r with Legendre symbol $(-65|r) = 1$. Then $65V^2$, V factors $\{1, 5, 13, 17, 29, 37, 41, 53, 61, \ldots\}$. Jones and Pall would have called the second family $65m^2$.

I did a little run with all $w \leq 100$ such that the binary form class group containing $x^2 + wy^2$ has half as many fourth powers as there are squares. For each of these 33 values of w I checked the spinor exceptions in the genus of the ternary $x^2 + wy^2 + 16w^2z^2$. Sometimes the only squareclass of exceptions is 1, often there are two independent squareclasses of exceptions, but only a few times are both 1 and w distinct squareclasses of exceptions. Anyway, here is the list of w with the minimal exceptions: 14:1,2. 17:1,17. 20:1,5. 32:1. 34:1,17. 36:1. 39:1,13. 41:1,41. 46:1,2. 49:1. 52:1,13. 55:1,5. 56:1,8.

62:1,2. 63:1. 64:1. 65:1,65. 66:1. 68:1,17. 69:1. 73:1,73. 77:1. 80:1,20. 82:1,41. 84:1. 89:1,89. 90:1. 94:1,2. 95:1,5. 96:1. 97:1,97. 98:1. 100:1.

Wai Kiu Chan provided this example about binaries added to unaries: $x^2 + 12y^2$ and $3x^2 + 4y^2$ are binaries of the same discriminant but different genera, while $x^2 + 12y^2 + 2z^2$ and $3x^2 + 4y^2 + 2z^2$ are ternaries in the same genus. Ben Kane found a simple counterexample to the analogous statement for spinor genera. His example happens quite often, as a genus of positive ternaries cannot have more than one spinor genus unless the discriminant Δ is divisible by 64 or by p^3 for some odd prime p. There is a different type of counterexample which does not depend on the factorization of N in $g(x, y) + Nz^2$: if $g(x, y)$ and $g'(x, y) = g(x, -y)$ are opposite forms in a genus without any ambiguous forms, then they are in distinct spinor genera. However, no matter what we choose for N, it follows that $g(x, y) + Nz^2$ and $g'(x, y) + Nz^2$ are actually equivalent with a determinant of $+1$. So the examples in this section denote one of the few possibilities where binaries in the same genus but distinct spinor genera map to ternaries in distinct spinor genera.

If $h_1(x, y)$ is a fourth power in the binary form class group containing $x^2 + wy^2$, and $h_2(x, y)$ is a square but not a fourth power, then the computer thinks that $h_1(x, y) + 16w^2z^2$ lies in the same spinor genus as $x^2 + wy^2 + 16w^2z^2$, while $h_2(x, y) + 16w^2z^2$ lies in a different spinor genus. That is really important, spinor genus respecting direct sum here. Direct sum is not something to be taken for granted either. $x^2 + 17y^2$ is the fourth power and $2x^2 + 2xy + 9y^2$ is the square that is not a fourth power. So Estes and Pall [24] say they are in the same (binary) genus but different spinor genera. Both forms represent 21 and 33. With 21 and discriminant 6603072, $x^2 + 17y^2 + 97104z^2$ and $2x^2 + 2xy + 9y^2 + 97104z^2$ are in the same genus and spinor genus. With 33 and discriminant 10376256 , $x^2 + 17y^2 + 152592z^2$ and $2x^2 + 2xy + 9y^2 + 152592z^2$ are in the same genus and spinor genus. It is worth noting that in both these cases the ternary genus does not have four spinor genera, only two.

Let $w \in \{17, 41, 65, 73, 89, 97, 113, 137, 145, 185, 193, 233, 241, \ldots\}$. That is, w is squarefree, $w \equiv 1 \bmod 8$, w is not divisible by any prime $q \equiv 3 \bmod 4$, and the positive binary quadratic form $g(x, y) = x^2 + wy^2$ is in a (principal) genus with exactly two ambiguous classes, so that there are half as many fourth powers as squares in the class group, and two spinor genera per genus of that binary form discriminant (I'll look for the reference, for binaries the spinor kernel is the fourth powers in the class group [24] or [12, page 366]).

I want w squarefree because I want minimal spinor exceptions to be 1 and w itself, rather than some divisor of w. And I do know that things never worked properly with even w or w divisible by some prime $q \equiv 3 \bmod 4$, and things went wrong when I tried w values 205, 221, 1513. When $w = 205$, which is 5 mod 8, the independent spinor exception squareclasses have minima 1 and 41. So the factor of 5 just disappears. That happens to prime factors that are 3 mod 4, or to 2 if all other prime factors are 1 mod 4. When $w = 221$, which is 5 mod 8, the independent spinor exception squareclasses have minima 1 and 17. So the factor of 13 just disappears. The last one, $1513 = 17 \cdot 89$, is the smallest number that fits all the other conditions for w but has four ambiguous classes in the (principal) genus containing $x^2+1513y^2$, those being (1,0,1513),(2,2,757),(17,0,89),(34,34,53). The resulting ternaries had eight spinor genera, confirmed by Andrew Earnest.

As mentioned, I asked the computer to check $w = 1513$. The binary form $x^2 + 1513y^2$ is in a principal genus of four forms, all of which are ambiguous, the list being (1,0,1513), (2,2,757), (17,0,89), (34,34,53). That is, the square of each is the identity. The only fourth power in the class group is the identity. The class number is sixteen. I wanted to know if this example works properly, I thought the principal genus stuffed with ambiguous forms might force extra spinor genera in the genus of $T(x, y, z) = x^2+1513y^2+36626704z^2$. And that did happen, there were eight spinor genera, this being confirmed by Earnest.

The squares in the class group of $x^2 + wy^2$ make up a subgroup called the principal genus. By hypothesis there are half as many fourth powers as squares. The fourth powers also make a subgroup of the subgroup of squares. So we have two cosets: the fourth powers and the squares that are not fourth powers. Now, let us define a sequence of numbers s, where each s is represented by a form in the principal genus but **not** by forms in both cosets.

For example, with $w = 17$, the numbers represented by (1,0,17) are 1, 4, 9, 16, 17, 18, 21, 25, 26, 33, 36, 42, 49, 53, 64, up to 65. The square that is not a fourth power is (2,2,9), which represents 2, 8, 9, 13, 18, 21, 32, 33, 34, 36, 42, 49, 50, 52. Striking out the numbers common to the two lists, we are taking $s \in \{1, 2, 4, 8, 13, 16, 17, 25, 26, 32, 34, 50, 52, 53, 64, \ldots\}$. I just finished a run, and a desired detail works, one that explains everything else: $x^2 + 17y^2 + 4624sz^2$ and $2x^2 + 2xy + 9y^2 + 4624sz^2$ lie in the same genus but different spinor genera, just as the binary forms $x^2 + 17y^2$ and $2x^2 + 2xy + 9y^2$ lie in the same genus but different spinor genera.

With $w = 41$, the fourth powers in the class group are $(1, 0, 41)$ and

$(2,2,21)$, while the squares that are not fourth powers are $(5,\pm 4,9)$. The list of possible s values for $(1,0,41)$ is 1 , 4, 9, 16, 25, 36, 41, 42, 45, 49, 50 up to 50. The list for $(2,2,21)$ is 2, 8, 18, 21, 25, 32, 33, 45, 50. So the combined list for the fourth powers starts 1, 2, 4, 8, 9, 16, 18, 21, 25, 32, 33, 36, 41, 42, 45, 49, 50. The list for $(5,\pm 4,9)$, the squares that are not fourth powers, is 5, 9, 10, 18, 20, 21, 33, 36, 37, 40, 42, 45, 49. Crossing out common terms we get $s \in \{1,2,4,5,8,10,16,20,25,32,37,40,41,50\ldots\}$.

Recall $g(x,y) = x^2 + wy^2$. For a pair (w,s), define a positive ternary quadratic form,

$$T(x,y,z) = x^2 + wy^2 + 16sw^2z^2 \;=\; g(x,y) + 16sw^2z^2.$$

For any other form $h(x,y) = ax^2 + bxy + cy^2$ in the principal genus, define

$$U_h(x,y,z) = ax^2 + bxy + cy^2 + 16sw^2z^2 \;=\; h(x,y) + 16sw^2z^2.$$

Then what the computer output says is this: $T(x,y,z)$ and $U_h(x,y,z)$ are in the same genus of positive ternary quadratic forms but if h is not a fourth power in the class group T and U_h lie in different spinor genera. The ternary genus has four spinor genera. There are two squareclasses of spinor exceptions, with representatives s and ws. Either $T(x,y,z)$ represents both s and ws, (as $g \circ g = g$), or some $U_h(x,y,z)$ represents s and therefore sw, as $g \circ h = h$. Thus there is at least one form in the genus that represents both these spinor exceptions. If s is not squarefree or if $\gcd(w,s) \neq 1$, perhaps there are smaller exceptions in either squareclass. There may be more than one equivalence class of forms that represent both s and sw, but the number of such classes is not large compared with the number of classes in the spinor genus. Finally, for fixed w and $s_1|s_2$, there is a descent from the genus with $s = s_2$ to one with $s = s_1$, and ***spinor genera are preserved by the descent***! That is, if a form upstairs descends to two or more forms downstairs, those downstairs forms are in the same spinor genus. If two forms upstairs are in the same spinor genus and descend to some forms downstairs, those forms downstairs are all in the same spinor genus. If two forms f_2, g_2 upstairs are in different spinor genera, f_2 descends to f_1 and g_2 descends to g_1, the results f_1, g_1 of descent lie in two different spinor genera downstairs. If some forms downstairs are in the same spinor genus and are descended to by a bunch of forms upstairs, those upstairs forms are all in the same spinor genus. And so on.

Suppose $s = \alpha u^2 + 2\beta uv + \gamma v^2$ with $\alpha\gamma - \beta^2 = w$.

$$\begin{pmatrix} u & v \\ -\beta u - \gamma v & \alpha u + \beta v \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} u & -\beta u - \gamma v \\ v & \alpha u + \beta v \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & sw \end{pmatrix}.$$

So we see how the ternary form $\alpha x^2 + 2\beta xy + \gamma y^2 + 16sw^2z^2$ descends by a factor of s to $x^2 + wy^2 + 16w^2z^2$.

But

$$\begin{pmatrix} \alpha u + \beta v & -v \\ \beta u + \gamma & u \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} \alpha u + \beta v & \beta u + \gamma v \\ -v & u \end{pmatrix} = \begin{pmatrix} \alpha s & \beta s \\ \beta s & \gamma s \end{pmatrix}.$$

So $x^2 + wy^2 + 16sw^2z^2$ also descends by a factor of s to $\alpha x^2 + 2\beta xy + \gamma y^2 + 16w^2z^2$.

This, finally, explains the strange restriction on s. If binaries h_1 and h_2 have the same discriminant as $x^2 + wy^2$, and both are squares in the class group while one is a fourth power and the other not, there would be a problem if both represented s. $x^2 + wy^2 + 16sw^2z^2$ would descend to both $h_1(x,y) + 16w^2z^2$ and $h_2(x,y) + 16w^2z^2$ downstairs, and we believe these lie in distinct spinor genera. We also believe that $h_1(x,y) + 16w^2sz^2$ and $h_2(x,y) + 16w^2sz^2$ lie in different spinor genera. Both these ternaries would descend by a factor of s to $x^2 + wy^2 + 16w^2z^2$. So, among the many things that can go wrong: a wrong choice of w could see $h_1(x,y) + 16w^2z^2$ and $h_2(x,y) + 16w^2z^2$ in the same spinor genus, horrible. A wrong choice of s could see $h_1(x,y) + 16w^2sz^2$ and $h_2(x,y) + 16w^2sz^2$ in the same spinor genus, very bad. Finally, even if all the spinor genera are as we like, we would still violate spinor genus preservation in descent if s were represented by a fourth power and a square that is not a fourth power.

The conditions on s prevent s from being divisible by some primes $q \equiv 3 \bmod 4$. Not always, with $w = 65$ it is legal to have $s = 49$. If s is divisible by $q \equiv 3 \bmod 4$ but not by q^2, the notation for this being $q \| s$, the genus of $x^2 + wy^2 + 16sw^2z^2$ collapses to just two spinor genera. It does turn out that you get four spinor genera with fairly pleasant independent exceptions for $s = 9, 18, 49, \cdots$, that is with $q \equiv 3 \bmod 4$ and $q^2 \| s$. The part I don't like is that the preservation of spinor genera is disrupted, at least when $(-w|q) = 1$. With $w = 17$ and $s = 49$, the form $x^2 + 17y^2 + 226576z^2$ descends by 49 to both $x^2 + 17y^2 + 4624z^2$ and $2x^2 + 2xy + 9y^2 + 4624z^2$, which are in different spinor genera. From a different spinor genus upstairs, the form $2x^2+2xy+9y^2+226576z^2$ descends by 49 to both $x^2+17y^2+4624z^2$

and $2x^2+2xy+9y^2+4624z^2$. The smallest discriminant $64w^3q^2$ available with prime $q \equiv 3 \bmod 4$ and Legendre symbol $(-w|q) = -1$ is with $w = 17$ and $q = 19$. So I'm trying to compute $w = 17$ and $s = 361$ now. With $w = 41$ and $s = 9$, the form $x^2+41y^2+242064z^2$ descends by 9 to both $x^2+41y^2+26896z^2$ and $5x^2 + 4xy + 9y^2 + 26896z^2$, which are in different spinor genera. From a different spinor genus upstairs, the form $5x^2+4xy+9y^2+242064z^2$ descends by 9 to both $x^2+41y^2+26896z^2$ and $5x^2+4xy+9y^2+26896z^2$. With $w = 65$ and $s = 9$, the form $x^2 + 65y^2 + 608400z^2$ descends by 9 to both $x^2 + 65y^2 + 67600z^2$ and $9x^2 + 8xy + 9y^2 + 67600z^2$, which are in different spinor genera. From a different spinor genus upstairs, the form $9x^2 + 8xy + 9y^2 + 608400z^2$ descends by 9 to both $x^2 + 65y^2 + 67600z^2$ and $9x^2 + 8xy + 9y^2 + 67600z^2$.

In case anyone ever looks at the computer outputs, the four spinor genera have labels 4,3,2,1 depending on whether the spinor genus represents s and ws. Label 4 means both are represented, label 1 means neither is represented, label 3 means ws is represented but not s, and label 2 means s is represented but not ws. So when $s = 1$, the spinor genus labelled 2 has at least one form that represents 1 but w is missed. Some edited C++ code follows, the STL set called temp is numbers represented by the full genus but NOT by the spinor genus under consideration. Confusing, of course.

```
if ( temp.count(s) &&  temp.count(  w * s)  )
{
  S.SetRegularFlag(1);
}
else if ( temp.count(s) &&  !(temp.count(  w * s))  )
{
  S.SetRegularFlag(3);
}
else if ( !(temp.count(s)) &&  temp.count(  w * s)  )
{
  S.SetRegularFlag(2);
}
else S.SetRegularFlag(4);
```

Anyway, for the moment, let all the fourth powers in the class group of x^2+wy^2 be called $g_i(x,y)$, and all the squares that are not fourth powers the $h_j(x,y)$. If one of the g_i represents s, then every $g_i(x,y) + 16sw^2z^2$ is in the spinor genus labelled 4, and every $h_j(x,y) + 16sw^2z^2$ is in the spinor genus labelled 2 (no ws). If one of the h_j represents s, then every $h_j(x,y)+16sw^2z^2$

is in the spinor genus labelled 4, and every $g_i(x,y) + 16sw^2z^2$ is in the spinor genus labelled 2 (no ws). Labels 1 and 3 don't get any binaries from the principal genus added to $16sw^2z^2$. For this output, the computer puts the discriminant in front of each form, then a colon, then six coefficients (Brandt-Intrau order), then a semicolon, then the spinor genus flag from 1,2,3,4. So the 6-tuple $a\ b\ c\ d\ e\ f$ between the colon and the semicolon refers to $T(x,y,z) = ax^2 + by^2 + cz^2 + dyz + ezx + fxy$, with discriminant before the colon given by $\Delta = 4abc + def - ad^2 - be^2 - cf^2$.

Here is a sample, $w = 17$, $s = 1$, the list of forms in each spinor genus preceded by a list of the first twenty-six numbers missed below 10,000.

```
=====Discriminant  314432  ==Genus Size==   36
  Spinor genus misses     no exceptions
  314432:  1   17    4624    0   0   0 ;4
  314432:  1  272     289    0   0   0 ;4
  314432:  2  145     272    0   0   2 ;4
  314432:  9   93     100  -36   8   4 ;4
  314432: 13   13     514   -2   2   8 ;4
  314432: 16   17     289    0   0   0 ;4
  314432: 17   32     145    8   0   0 ;4
  314432: 34   53      66   50   0  34 ;4
  314432: 42   49      50   22  40   2 ;4
--------------------------size 9
  Spinor genus misses        1     9    49    81   121   169
   441   529   729   961  1089  1521  2401  2809  3969  4761
  5041  5929  6241  6561  7921  8281  8649  9801
  314432:  2   34    1165   34   2   0 ;3
  314432:  4  137     154   70   4   4 ;3
  314432: 13   21     297  -20  12   2 ;3
  314432: 13   25     272    0   0  12 ;3
  314432: 16   34     157   34  16   0 ;3
  314432: 17   25     185    2   0   0 ;3
  314432: 17   52      89    4   0   0 ;3
  314432: 18   25     186    6   8  10 ;3
  314432: 25   25     149  -12  22  16 ;3
--------------------------size 9
  Spinor genus misses       17   425  2873  4913
  314432:  1  153     544  136   0   0 ;2
```

```
  314432:  2    9    4624    0   0   2 ;2
  314432:  9   32     289    0   0   8 ;2
  314432: 13   68     106   68   2   0 ;2
  314432: 16   34     153   34   0   0 ;2
  314432: 18   49      98   10  16  14 ;2
  314432: 26   50      81  -38  14  24 ;2
  314432: 34   49      66   38   0  34 ;2
  314432: 42   42      53   26  26  16 ;2
--------------------------size 9
  Spinor genus misses        1     9    17    49    81   121
   169   425   441   529   729   961  1089  1521  2401  2809
  2873  3969  4761  4913  5041  5929  6241  6561  7921  8281
  314432:  2  213     213  154   2   2 ;1
  314432:  4   18    1157    2   4   4 ;1
  314432:  4   69     290    2   4   4 ;1
  314432: 13   33     189  -20   6   4 ;1
  314432: 16   21     293    8  16  16 ;1
  314432: 18   34     137   34   2   0 ;1
  314432: 25   25     144   16  16  16 ;1
  314432: 25   25     149  -12  12  18 ;1
  314432: 25   36      98   28  22   8 ;1
--------------------------size 9
```

This shows how I was able to check spinor genus preservation during descent. For a fixed w but several values of s I sent all the lines with a semicolon to a separate file. The computer just checked pairs of forms: if the discriminant of one divided the discriminant of the other, we had a situation with $s_2 > s_1$ and $s_1 | s_2$. If there was a homothety from the s_2 form to $\frac{s_2}{s_1}$ times the s_1 form, we call that a descent and printed both forms on the same line to an output file, with the ratio $\frac{s_2}{s_1}$ in the middle. Finally, each of the forms also had a flag from 1,2,3,4 at the end. The program printed out "DISAGREE" to screen and to output text file and raised all kinds of hell if that happened. But with the restrictions on w and s I described that never happened. Flag 4 always matched with flag 4, 3 with 3, 2 with 2, 1 with 1. For the previous section, with just two spinor genera per genus, the same type of check was done with flags 0,1, standing for irregular spinor genus and for regular.

58 Conjectures about homotheties

Billy Wai Kiu Chan [7] of Wesleyan has proved Conjectures 1 and 2, March 2008!

After a week of writing software I have finally managed to confirm the Conjectures of this section for all pairs of genera with discriminants $\Delta, n\Delta \leq 1000$. That is, the Conjectures are true for the entirety of the 1958 tables by Heinrich Brandt and Oskar Intrau [4], see Mathematical Reviews MR0106204 (21 #4938). The tables amount to 36433 positive ternary forms gathered into 4534 genera. Forms with $\Delta = 1000$ include $\{1, 1, 250, 0, 0, 0\}$ and $\{3, 7, 13, -3, 1, 2\}$, the latter being regular (its genus has only the one equivalence class). Indeed, I have confirmed my Conjecture 3 much higher, all pairs of genera with both discriminants $\Delta \leq 11664$.

So: it appears (much computer experimenting) that one of Kaplansky's ideas is true far more generally than the setting in which he wrote it, [40, Appendices I, II] or [39]. Kap mentions in [40, Appendix II] that the facts he is discussing are in [55] but are not quite explicit. Note that what we describe here is not possible for binary forms. Also, no checking has been done for indefinite ternaries.

In [40, Appendices I, II] there are references to these three pairs of forms:

If $f(x, y, z) = x^2+2y^2+7z^2+2yz+zx$ and $g(x, y, z) = x^2+2y^2+13z^2+2yz$, then $f(2Y, X - Z, 2Z) = 2g(X, Y, Z)$ and $g(2Y + Z, X, Z) = 2f(X, Y, Z)$.

If $f(x, y, z) = x^2 + y^2 + 3z^2 + zx$ and $g(x, y, z) = x^2 + 3y^2 + 11z^2 + xy$, then $f(X+6Y, 11Z, -2X-Y) = 11g(X, Y, Z)$ and $g(X+6Z, -2X-Z, Y) = 11f(X, Y, Z)$.

If $f(x, y, z) = x^2 + 3y^2 + 4z^2 + xy$ and $g(x, y, z) = x^2 + 3y^2 + 44z^2 + xy$, then $f(X+6Y, -2X-Y, 11Z) = 11g(X, Y, Z)$ and $g(X+6Y, -2X-Y, Z) = 11f(X, Y, Z)$.

Suppose we are given two positive ternary quadratic forms, f and g, such that the discriminant of one divides the discriminant of the other. So we have 3 by 3 Gram matrices A and B, symmetric positive definite, with integer entries on the diagonal and integer or half-integers off diagonal.

Let $\det A = D$, and let the ratio of determinants be an integer n, so $\det B = nD$.

Suppose there is a "homothety" from B (which has larger determinant) to nA. That is an integer matrix P, also 3 by 3, with transpose P', such that

$$P'BP = nA.$$

By relating determinants we find

$$nD(\det P)^2 = n^3 D, \quad (\det P)^2 = n^2, \quad \det P = \pm n,$$

and by choosing $\pm P$ we may insist $\det P = n$.

Let Q be the adjoint of P, so $PQ = QP = nI$ and $\det Q = n^2$. Now

$$nQ'AQ = Q'P'BPQ = nIBnI = n^2 B,$$

so

$$Q'AQ = nB.$$

In all my experiments, if I have found just one such $\{A, B, n, P, Q\}$, then EVERY other form in the genus of B (the larger discriminant) corresponds in the same way (integer homotheties in both directions) with at least one form in the genus of A. Furthermore each form in the smaller genus is covered by something as well. My Conjectures 1 and 2 below describe this and were proved by Chan in a letter to me, March 2008. He does a great job of translating my language into the terminology of today's experts in quadratic forms, lattices and the like.

> I will adopt the language of (quadratic) lattices and spaces. If L is a $\mathbf{Z}-$lattice, then $L^{(n)}$ denotes the $\mathbf{Z}-$lattice whose underlying set is L but with the quadratic form scaled by the factor n. We say that a $\mathbf{Z}-$lattice M represents another $\mathbf{Z}-$lattice L if M has a sublattice M' which is isometric to L, or, equivalently, there exists an isometry sending L into M. Our basic assumption is:
>
> *L and M are $\mathbf{Z}-$lattices such that $d(M) = nd(L)$, where n is a positive integer, and M represents $L^{(n)}$.*
>
> (This is the same as the hypothesis in your Conjecture stated in terms of polynomials and "homotheties.")
>
> Notice that by the assumption we may assume that $L^{(n)}$ and M are $\mathbf{Z}-$lattices on the same space, say V. There are a couple of immediate consequences. First, if M' is a sublattice of M which is isometric to $L^{(n)}$, then $[M : M'] = n$. Second, we have the "dual" statement which says that L also represents $M^{(n)}$. As you

already indicated in your notes, the fact that n is an integer is crucial here.

Now, let K be a $\mathbf{Z}-$lattice in $\text{gen}(M)$. We claim that K represents $H^{(n)}$ for some $H \in \text{gen}(L)$. Let τ be an isometry of V which sends $L^{(n)}$ into M, and let $M' = \tau(L^{(n)})$. Since $K \in \text{gen}(M)$, for each prime p there exists an isometry σ_p of V_p which sends M_p to K_p. For almost all p (in fact, for those p that do not divide n), $M'_p = M_p$, and so there is a sublattice K' of K with $K'_p = \sigma_p(M'_p)$ for all p. Moreover, $L_p^{(n)}$ is isometric to K'_p at each p; $\sigma_p \tau$ is an isometry between them. Therefore, $K' \in \text{gen}(L^{(n)})$ and so K' is isometric to $H^{(n)}$ for some $H \in \text{gen}(L)$.

Now we can "reverse" the steps to show that for every $H \in \text{gen}(L)$ there will be a $K \in \text{gen}(M)$ which represents $H^{(n)}$. Since $H^{(n)}$ is in $\text{gen}(L^{(n)})$, there will be an isometry σ_p of V_p such that $\sigma_p(H_p^{(n)}) = L_p^{(n)}$ at each prime p. Then $\tau\sigma_p(H_p^{(n)}) = M'_p$ for all p. Now, for $p \not| \; n$ we have $M_p = M'_p$ and so $(\tau\sigma_p)^{-1}(M_p) = H_p^{(n)}$. Therefore there will be a $\mathbf{Z}-$lattice K on V such that $K_p = (\tau\sigma_p)^{-1}(M_p)$. Obviously, $K \in \text{gen}(M)$ and K represents $H^{(n)}$, in fact K contains $H^{(n)}$!

Conjecture 1, proved by Chan [7]: given positive ternary quadratic forms f and g with integer coefficients, integer n, such that discriminant $g = n \cdot$ discriminant f, and homotheties from g to nf and from f to ng, ANY other form g_1 in the genus of g has such a correspondence with at least one form f_1 in the genus of f.

Conjecture 2, proved by Chan [7]: The correspondence of Conjecture 1, while usually many-to-many, is surjective in both directions.

Caution: if you start with just the upwards homothety, discriminant Δ to discriminant $n\Delta$, sometimes there is no homothety in the downwards direction. The simplest example is surely this:

$$(x-z)^2 + (x+y+2z)^2 + (-y+z)^2 = 2(x^2+y^2+3z^2+yz+zx+xy),$$

but there is no homothety down to $2(x^2+y^2+z^2)$ because $x^2+y^2+3z^2+yz+zx+xy$ does not represent 2.

Caution: Chan's Theorem requires that we allow common factors of the nine entries in a homothety matrix, I suspect usually in the upwards direction. This can only happen if the discriminant ratio has square factors, of course. An example: the genus $\Delta = 135$, two classes $\{1,3,12,3,0,0\}$ and $\{1,1,45,0,0,1\}$, with the genus $\Delta = 15$, two classes $\{1,2,2,1,0,0\}$ and $\{1,1,5,0,0,1\}$. Now $\{1,1,5,0,0,1\}$ actually represents (has a homothety to) $\{1,1,45,0,0,1\}$ itself, so at least some of the homotheties from $\{1,1,5,0,0,1\}$ to $\{9,9,405,0,0,9\}$ are 3 times the previous homothety matrices, and it turns out those are the only ones. Plus there is no homothety at all from $\{1,1,45,0,0,1\}$ down to $9\cdot\{1,2,2,1,0,0\}$. Meanwhile, $\{1,3,12,3,0,0\}$ corresponds only with $\{1,2,2,1,0,0\}$, and the adjoint of any downwards homothety matrix has common matrix entry factor 3, but there are other homotheties from $\{1,2,2,1,0,0\}$, to $\{9,27,108,27,0,0\}$ with gcd 1.

Conjecture 3: given positive ternary quadratic forms f and g with integer coefficients, **squarefree** integer n, such that discriminant $g = n \cdot$ discriminant f, and homotheties from g to nf and from f to ng, such that both the genus of g and the genus of f have exactly two spinor genera and both genera have spinor exceptions, then forms in the regular spinor genus of gen g correspond only with forms in the regular spinor genus of gen f, and forms in the irregular spinor genus of gen g correspond only with forms in the irregular spinor genus of gen f.

I have checked Conjecture 3 pretty high, all pairs of genera with both discriminants $\Delta \leq 11664$. If proved, it explains everything about my $(1,1,16N)$ example for squarefree N other than Hsia's calculation of the spinor genera and spinor exceptional integers. So, taken together, Conjecture 4' and Conjecture 3' explain almost everything about $(1,1,16N)$, the related families, and my examples with four spinor genera.

Conjecture 4: given positive ternary quadratic forms f and g with integer coefficients, integer n, such that discriminant $g = n \cdot$ discriminant f, and homotheties from g to nf and from f to ng, such that gen f has two spinor genera and spinor exceptions, but we are not sure about g. If gen g has two spinor genera then it does have spinor exceptions.

> **Conjecture 4'**: Given gen(M) and gen(L) with $d(M) = nd(L)$ for squarefree n, and both gen(M) and gen(L) with the same number 2^r of spinor genera. Suppose gen(L) has a "complete system of spinor exceptional integers," indeed r of these, in the phrasing of Benham and Hsia (1982) [2]. We also require that the r independent spinor exceptional integers for gen(L) be relatively prime to n and themselves be squarefree (I can't tell

whether Benham and Hsia demand them squarefree). Finally, suppose M represents $L^{(n)}$. **Then** gen(M) also has a complete system of spinor exceptional integers. Furthermore, I would expect to simply multiply by n to find a complete system of spinor exceptional integers for gen(M).

Conjecture 3': Given gen(M) and gen(L) with $d(M) = nd(L)$ for squarefree n, and both gen(M) and gen(L) with the same number 2^r of spinor genera and both with a complete system of spinor exceptional integers, indeed r of these. We also require that the r independent spinor exceptional integers for gen(L) be relatively prime to n and themselves be squarefree. Given $M_1, M_2 \in$ gen(M) and $L_1, L_2 \in$ gen(L), while M_1 represents $L_1^{(n)}$ and M_2 represents $L_2^{(n)}$. **Then** spn(M_1) = spn(M_2) if and only if spn(L_1) = spn(L_2).

Conjecture 5 below is just my way of conjecturing, for all my other examples analogous to $(1, 1, 16N)$, that the genera produced with the parameter N do have exactly two spinor genera, one regular and one not, and the smallest spinor exceptional integer is exactly as hoped.

Conjecture 5: what I am really looking for is this: given a primitive positive binary $Ax^2 + Bxy + Cy^2$ with negative "discriminant" $B^2 - 4AC$, and given a fixed coefficient M which is a multiple of $(B^2-4AC)^2$ or at least of $(B^2 - 4AC)^2/4$, such that the genus of the ternary $Ax^2 + Bxy + Cy^2 + Mz^2$ has exactly two spinor genera and has spinor exceptional integers, the smallest of which is σ, which we take to be squarefree. Furthermore let $\sigma = Ar^2 + Brs + Cs^2$, so that $Ax^2 + Bxy + Cy^2 + Mz^2$ is in the regular spinor genus, and every form in the regular spinor genus represents all spinor exceptional integers of the genus, because each form is required to represent σ. Note that the discriminant of $Ax^2 + Bxy + Cy^2 + Mz^2$ is divisible by $(4AC-B^2)^3$ or $(4AC-B^2)^3/4$. Now, given a positive binary $A_1x^2 + B_1xy + C_1y^2$ with $B_1^2 - 4A_1C_1 = B^2 - 4AC$, and given some squarefree integer N with $\gcd(N, \sigma) = 1$ and N is represented by the ratio of the two binary forms in the class group, so that $N\sigma$ is still squarefree and $N\sigma = A_1u^2 + B_1uv + C_1v^2$. Consider $A_1x^2 + B_1xy + C_1y^2 + MNz^2$. We require that there be homotheties in both directions, with multiplier N, between $A_1x^2 + B_1xy + C_1y^2 + MNz^2$ and $Ax^2 + Bxy + Cy^2 + Mz^2$. The **conjecture** is that the genus of $A_1x^2 + B_1xy + C_1y^2 + MNz^2$ is forced to have exactly two spinor genera and to have spinor exceptions, while $N\sigma$ is forced to be a spinor exceptional integer, therefore the smallest such as we have insisted $N\sigma$ be squarefree. Therefore $A_1x^2 + B_1xy + C_1y^2 + MNz^2$ is in the regular spinor genus and, by Conjecture 3, every form in the regular spinor genus represents the smallest spinor exceptional integer $N\sigma$.

Chan's Theorem gives a proof of the curiosity in Section 56, that for $N = u^2 + v^2$, N squarefree, every form in the spinor genus containing $x^2 + y^2 + 16Nz^2$ represents N itself. This is an application of the Corollary to Theorem 3 on page 56 of [17].

Proof Chan's Theorem implies $(1, 1, 16N)$ **curiosity** : The base genus, with $\Delta = 64$, consists of two forms, the regular $\{1, 1, 16, 0, 0, 0\}$ and the "spinor regular" form $\{2, 2, 5, 2, 2, 0\}$. Now, see [36] or [15], $\{1, 1, 16, 0, 0, 0\}$ represents all numbers except $4n + 3, 8n + 6, 32n + 12, 4^k(8n + 7)$. The notation on the other is $\{2, 2, 5, 2, 2, 0\} \neq m^2$: the numbers represented by $\{1, 1, 16, 0, 0, 0\}$ but not by $\{2, 2, 5, 2, 2, 0\}$ are 1, 25, 169, 289, 625, 841, 1369, 1681, 2809, 3721, 4225, 5329, 7225, 7921, 9409, 10201, 11881, 12769, 15625 ... , squares whose prime factors are all congruent to 1 modulo 4. $\{2, 2, 5, 2, 2, 0\}$ is also item (3.1) of Theorem 1 in [1].

First we show the proof for N **odd**, later a small revision allows for twice odd.

Let $N = u^2 + v^2$ be odd and squarefree, so it is represented primitively, i.e. $\gcd(u, v) = 1$. The genus with larger disciminant is that of $\{1, 1, 16N, 0, 0, 0\}$ or

$$f(x, y, z) = x^2 + y^2 + 16Nz^2.$$

There is a homothety from f to $N\{1, 1, 16, 0, 0, 0\}$ given by

$$\begin{pmatrix} u & v & 0 \\ -v & u & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

as in

$$\begin{pmatrix} u & -v & 0 \\ v & u & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16N \end{pmatrix} \begin{pmatrix} u & v & 0 \\ -v & u & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} N & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & 16N \end{pmatrix}.$$

There is also a homothety from $\{1, 1, 16, 0, 0, 0\}$ to Nf given by the adjoint

$$\begin{pmatrix} u & -v & 0 \\ v & u & 0 \\ 0 & 0 & N \end{pmatrix},$$

as in

$$\begin{pmatrix} u & v & 0 \\ -v & u & 0 \\ 0 & 0 & N \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{pmatrix} \begin{pmatrix} u & -v & 0 \\ v & u & 0 \\ 0 & 0 & N \end{pmatrix} = \begin{pmatrix} N & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & 16N^2 \end{pmatrix}.$$

J. S. Hsia proved that the genus of f splits into exactly two spinor genera, and N itself is a spinor exception. He mentioned once that the methods used are in Earnest, Hsia, and Hung [21].

Let $h(x, y, z)$ be in the same genus and spinor genus as f. Chan's Theorems state that there is a pairing of h with at least one of $\{1, 1, 16, 0, 0, 0\}$ or $\{2, 2, 5, 2, 2, 0\}$. We will show that it must be the first choice by considering the spinor exceptions.

Lemma: if a number is the product of primes all of which are 1 mod 4, then it has a primitive representation as $a^2 + b^2$, that is $\gcd(a, b) = 1$.

There are many proofs of this, but you can prove it yourself based on the fact that any prime $p \equiv 1 \bmod 4$ is so represented and then working with gcd in the induction step

$$(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2.$$

Notice there is no need to prohibit square factors, think of Pythagorean triples.

On page 56 of Duke and Schulze-Pillot [17] we have the corollary to Theorem 3:

> **Corollary.** Let $q(x_1, x_2, x_3)$ be a positive integral ternary quadratic form. Then every large integer n represented primitively by a form in the spinor genus of q is represented by q itself and the representing vectors are asymptotically uniformly distributed on the ellipsoid $q(\mathbf{x}) = n$.

As f, h are in the same spinor genus, this says that there is some large M such that any number primitively represented by f and larger than M is also represented by h. So, take a prime $p \equiv 1 \bmod 4$ that is so large that

$Np^2 > M$. Since Np^2 is a product of primes all of which are 1 mod 4, it is primitively represented by $x^2 + y^2$ by the Lemma above. But that means Np^2 is primitively represented by $f(x, y, z) = x^2 + y^2 + 16Nz^2$ with $z = 0$. From $Np^2 > M$ it follows [17] that $h(x, y, z)$ also represents Np^2.

If there were a correspondence between h and $\{2, 2, 5, 2, 2, 0\}$, one of the homotheties would be from $\{2, 2, 5, 2, 2, 0\}$ to Nh. This would provide a representation of $N \cdot Np^2 = N^2p^2$ by $\{2, 2, 5, 2, 2, 0\}$. But this is prohibited, N^2p^2 is a square and all prime factors of N^2p^2 are congruent to 1 modulo 4.

Therefore, Chan's Theorem says that h corresponds with $\{1, 1, 16, 0, 0, 0\}$. That is, there is a homothety from h to

$$N \cdot \{1, 1, 16, 0, 0, 0\} = \{N, N, 16N, 0, 0, 0\}.$$

In particular, h represents N.

The same method gives a proof for the genus of $f(x, y, z) = x^2 + y^2 + 32Nz^2$, with $\Delta = 128N$, so that the discriminant ratio with $\{1, 1, 16, 0, 0, 0\}$ is twice odd, as $\Delta = 64$ for the latter.

Lemma: if an odd number $K = a^2 + b^2$ with $\gcd(a, b) = 1$, then $2K = (a - b)^2 + (a + b)^2$ is also a primitive representation.

There is a genus of three classes with $\Delta = 128$, that splits into two spinor genera. The first spinor genus is $\{1, 1, 32, 0, 0, 0\}$ and $\{2, 2, 9, 2, 2, 0\}$. Note that both represent 2. The other spinor genus has the single form $\{1, 4, 9, 4, 0, 0\}$ which is spinor regular, it is item (3.4) of Theorem 1 in [1], the numbers represented by the full genus but not by $\{1, 4, 9, 4, 0, 0\}$ are precisely those of shape $2m^2$, that is 2, 50, 338, 578, 1250, 1682, 2738, 3362, 5618, 7442, 8450,...

The same homotheties as before work for $x^2+y^2+32Nz^2$ and $x^2+y^2+32z^2$,

$$\begin{pmatrix} u & -v & 0 \\ v & u & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 32N \end{pmatrix} \begin{pmatrix} u & v & 0 \\ -v & u & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} N & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & 32N \end{pmatrix},$$

$$\begin{pmatrix} u & v & 0 \\ -v & u & 0 \\ 0 & 0 & N \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 32 \end{pmatrix} \begin{pmatrix} u & -v & 0 \\ v & u & 0 \\ 0 & 0 & N \end{pmatrix} = \begin{pmatrix} N & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & 32N^2 \end{pmatrix}.$$

Let $h(x, y, z)$ be in the same spinor genus as $f(x, y, z) = x^2 + y^2 + 32Nz^2$. There is some large M_2 such that any number primitively represented by f and larger than M_2 is also represented by h. So, take a prime $p \equiv 1 \bmod 4$

that is so large that $2Np^2 > M_2$. Since Np^2 is a product of primes all of which are 1 mod 4, it is primitively represented by $x^2 + y^2$ by the first Lemma above. The second Lemma says that $2Np^2$ is also primitively represented by $x^2 + y^2$. But that means $2Np^2$ is primitively represented by $f(x, y, z) = x^2 + y^2 + 16Nz^2$ with $z = 0$. From $2Np^2 > M_2$ it follows [17] that $h(x, y, z)$ also represents $2Np^2$.

If there were a correspondence between h and $\{1, 4, 9, 4, 0, 0\}$, one of the homotheties would be from $\{1, 4, 9, 4, 0, 0\}$ to Nh. This would provide a representation of $N \cdot 2Np^2 = 2N^2p^2$ by $\{1, 4, 9, 4, 0, 0\}$. But this is prohibited, $2N^2p^2$ is twice a square and all odd prime factors of $2N^2p^2$ are congruent to 1 modulo 4.

Therefore, Chan's Theorem says that h corresponds with $\{1, 1, 32, 0, 0, 0\}$ or $\{2, 2, 9, 2, 2, 0\}$. That is, there is a homothety from h to $\{N, N, 16N, 0, 0, 0\}$ or $\{2N, 2N, 9N, 2N, 2N, 0\}$. In particular, h represents $2N$.

Combining the two proofs, N and $2N$, we get the desired implication: if $W = s^2 + t^2$ is squarefree, Chan's Theorem [7] says that every form in the spinor genus of $x^2 + y^2 + 16Wz^2$ represents W.

Conjecture 3: given positive ternary quadratic forms f and g with integer coefficients, **squarefree** integer n, such that discriminant $g = n \cdot$ discriminant f, and homotheties from g to nf and from f to ng, such that both the genus of g and the genus of f have exactly two spinor genera and both genera have spinor exceptions, then forms in the regular spinor genus of $\operatorname{gen} g$ correspond only with forms in the regular spinor genus of $\operatorname{gen} f$, and forms in the irregular spinor genus of $\operatorname{gen} g$ correspond only with forms in the irregular spinor genus of $\operatorname{gen} f$.

Of course, many interesting examples occur when spinor genera are respected but the ratio of discriminants n has square factors. When n is squarefree, the set of spinor exceptions for $\operatorname{gen} f$, are just multiplied by n to get the spinor exceptions for $\operatorname{gen} g$. But when n is square or has square factors, say $n = mq^2$, it is common for the spinor exceptions for $\operatorname{gen} g$ to be the union of m times the spinor exceptions for $\operatorname{gen} f$ with mq^2 times the spinor exceptions for $\operatorname{gen} f$. For example, with $\Delta = 108$, the spinor regular $\{3, 3, 4, 0, 0, 3\} \neq w^2$, while a corresponding spinor regular with $n = 4$ and $\Delta = 432$ is $\{3, 7, 7, 5, 3, 3\} \neq w^2, 4w^2$.

Meanwhile, for any n, even allowing square factors, it appears the genus of the form with higher discriminant can either have just one spinor genus or can have two spinor genera with spinor exceptions. It can't be one of those with two spinor genera that represent all the same numbers.

Conjecture 4: given positive ternary quadratic forms f and g with integer coefficients, integer n, such that discriminant $g = n \cdot$ discriminant f, and homotheties from g to nf and from f to ng, such that $\operatorname{gen} f$ has two spinor genera and spinor exceptions, but we are not sure about g. If $\operatorname{gen} g$ has two spinor genera then it does have spinor exceptions.

59 Hanke and Schulze-Pillot

It turns out that Jonathan Hanke [26, 25] proved similar results on infinite sets of numbers not represented by forms in the regular spinor genus as Schulze-Pillot. Quoting the last sentence in Theorem 4.3 on page 312 of [52]: "In particular, if there is a spinor exceptional integer a' for the genus of L that is represented by $\operatorname{spn}(L)$ but not by L (so a' is below the bound for being sufficiently large), then there are infinitely many integers $a'p^2$ with p prime that are not represented by L." Hanke [26] summarizes this as "even a refined local-global principle based on the spinor genus fails infinitely often" and seems to be wondering how frequently genera give examples of the described behavior.

I have just come across a very satisfying example of this, related to things I knew already. Given an odd prime p it is not difficult to show that the positive ternary form $\langle 1, p, p, 0, 0, 0\rangle$ is the only form in its genus allowed to represent the number 1. The same proof generalizes to odd squarefree S, that is $\langle 1, S, S, 0, 0, 0\rangle$ is the only form in its genus allowed to represent the number 1. Some extra detail shows that for $0 \leq k \leq 4$, the form $\langle 1, 2^k S, 2^k S, 0, 0, 0\rangle$ does the same. The observation about 1 being represented by only one form is by Alexander Berkovich.

Now, let N be odd and squarefree but require that $N = u^2+v^2$ in integers, so that N is a prime congruent to 1 modulo 4 or the product of distinct such primes. My computer is convinced that the genus of $\langle 1, 16N, 16N, 0, 0, 0\rangle$ splits into exactly two spinor genera. I can actually prove this with the Watson transformations! In his original article [57], Watson gave a mapping that we will call λ that takes a positive integer m and a quadratic form and produces a new quadratic form. Taking $m = 64N$ the transformation is

$$\lambda_{64N}\langle 1, 1, 16N, 0, 0, 0\rangle = \langle 1, 16N, 16N, 0, 0, 0\rangle$$

and

$$\lambda_{64N}\langle 1, 16N, 16N, 0, 0, 0\rangle = \langle 1, 1, 16N, 0, 0, 0\rangle.$$

As the transformation λ_{64N} does not increase the number of spinor genera, while Hsia proved that the genus of $\langle 1, 1, 16N, 0, 0, 0\rangle$ splits into exactly two spinor genera, it follows that the genus of $\langle 1, 16N, 16N, 0, 0, 0\rangle$ also splits into exactly two spinor genera. My computer thinks that the spinor exceptions are the Jones-Pall m^2. Anyway, no form in the spinor genus that lacks $\langle 1, 16N, 16N, 0, 0, 0\rangle$ itself represents the number 1, so 1 is a spinor exceptional integer! Here is the punchline: all the other forms in the regular spinor genus, the good spinor genus containing $\langle 1, 16N, 16N, 0, 0, 0\rangle$, fail to represent 1 (as we knew), which is the smallest spinor exceptional integer. As a result, **all but one** of the forms in the good spinor genus fail to represent an infinite set of squares, and my computer thinks these numbers can be taken to be any q^2 where $q \equiv 3 \bmod 4$ is a prime. This is the Schulze-Pillot or Hanke result above in a fairly extreme setting. Before I forget, my computer also thinks that every form in the good spinor genus represents N^2.

In comparison, switch one coefficient to get $\langle 1, 1, 16N, 0, 0, 0\rangle$. In this case all has been proved, by Hsia, Chan, and me. There are exactly two spinor genera, N itself is the smallest spinor exceptional integer, and every form in the good spinor genus represents N and therefore **all** spinor exceptional integers, this when N is odd and squarefree and $N = u^2 + v^2$ in integers. So in this case no forms at all fall prey to the Schulze-Pillot or Hanke result.

And I believe I invented another infinite set of genera that display this behavior, although it is for others to check the exact relationship to the published Theorems. Let $P \equiv 7 \bmod 8$ be prime, let $P \geq 23$, and let $P = 8T - 1$, so that $T \geq 3$. It is known that 2 is a quadratic residue mod$P$. It is also known that the positive binary form $x^2 + xy + 2Ty^2$ is in the same genus as $2x^2 + xy + Ty^2$, as there is only one genus which has an odd number of classes. Thus the fourth-power map is one to one and surjective, which is to say (see Estes and Pall [24]) that $x^2 + xy + 2Ty^2$ and $2x^2 + xy + Ty^2$ are also in the same spinor genus. So I think $x^2 + xy + 2Ty^2 + P^2z^2$ and $2x^2 + xy + Ty^2 + P^2z^2$ are in the same genus and same spinor genus of positive ternaries.

Wai Kiu Chan provided this example about binaries added to unaries: $x^2 + 12y^2$ and $3x^2 + 4y^2$ are binaries of the same discriminant but different genera, while $x^2 + 12y^2 + 2z^2$ and $3x^2 + 4y^2 + 2z^2$ are ternaries in the same genus. Ben Kane found a simple counterexample to the analogous statement for spinor genera. His example happens quite often, as a genus of positive ternaries cannot have more than one spinor genus unless the discriminant Δ is divisible by 64 or by p^3 for some odd prime p. There is a different type of counterexample which does not depend on the factorization of N in $g(x,y)+Nz^2$: if $g(x,y)$ and $g'(x,y)=g(x,-y)$ are opposite forms in a genus without any ambiguous forms, then they are in distinct spinor genera. However, no matter what we choose for N, it follows that $g(x,y)+Nz^2$ and $g'(x,y)+Nz^2$ are actually equivalent with a determinant of $+1$.

Following Kaplansky's 1995 letter to Hsia and Schulze-Pillot, we have little trouble proving that

$$2x^2+xy+Ty^2+P^2z^2 \neq s^2$$

for prime s with Legendre symbol $(s|P)=(-P|s)=-1$. Note that we have guaranteed $s\neq 2$, and primes are always positive for us.

Assume that

$$2x^2+xy+Ty^2+P^2z^2 = s^2.$$

First, if $z=0$, we know that $s|x$ and $s|y$, giving

$$2\left(\frac{x}{s}\right)^2+\left(\frac{x}{s}\right)\left(\frac{y}{s}\right)+T\left(\frac{y}{s}\right)^2=1,$$

which is false as the nonzero "minimum" of the binary is 2.

Second, if $z\neq 0$, choose $z>0$. Then

$$2x^2+xy+Ty^2=s^2-P^2z^2=(s+Pz)(s-Pz).$$

Since s is a nonresidue mod$P$, both $s+Pz$ and $s-Pz$ are nonresidues mod$P$. Thus there is some (odd) prime q with $(q|P)=(-P|q)=-1$ that divides $s+Pz$ to an odd power, or $q^{2m+1}||(s+Pz)$. But the fact that q must divide $2x^2+xy+Ty^2$ to an even power shows that $q|(s-Pz)$, indeed $q^{2n+1}||(s-Pz)$. So q divides $2s$ and q divides $2Pz$. But $q|2s$ implies that $q=s$. Next $s|2Pz$ implies that $s|z$, where we have chosen $z>0$ so $z\geq s$. As a result, $Pz\geq Ps\geq 23s$. So $s+Pz>0$ but $s-Pz<0$, therefore $s^2-P^2z^2<0$, which contradicts $2x^2+xy+Ty^2\geq 0$ in the assumption

$$2x^2+xy+Ty^2=s^2-P^2z^2=(s+Pz)(s-Pz).$$

60 Some Involutions

It is fairly common, once a genus has exactly two spinor genera, for these to have the same number of classes. From my original examples I thought this meant something extremely special as far as forms representing square multiples of forms in the other spinor genus, so let me start out with a pessimistic example.

```
   ===Discriminant   1331 ==Genus Size==  6
   1331 = 11^3
  Spinor genus misses     no exceptions
      A :    1    11        33     11    0    0    auto 8
      B :    3     4        31      1    3    2    auto 4
      C :    4     4        25     -2    2    3    auto 4
-------------------------size 3
  Spinor genus misses     no exceptions
      D :    1     3       121      0    0    1    auto 8
      E :    1    14        25      6    1    1    auto 4
      F :    5     9         9      7    2    2    auto 4
-------------------------size 3
Disc    1331
```

In the genus above, I have deliberately renamed the forms A, B, C, D, E, F. Note that A and D have 8 integer automorphs (I allow both determinants 1 and -1 here) while the others have 4. Despite everything favorable, there is no preferred bijection: A represents $D^{(4)}, E^{(4)}, C^{(121)}, D^{(121)}, F^{(121)}$. Then B represents $D^{(4)}, F^{(4)}, E^{(121)}$. C represents $E^{(4)}, F^{(4)}, A^{(121)}, D^{(121)}, F^{(121)}$. D represents $A^{(4)}, B^{(4)}, A^{(121)}, C^{(121)}, F^{(121)}$. E represents $A^{(4)}, C^{(4)}, B^{(121)}$. Finally F represents $B^{(4)}, C^{(4)}, A^{(121)}, C^{(121)}, F^{(121)}$.

In comparison, let N be odd and squarefree and $N = u^2 + v^2$ in integers. For all such $N \leq 157$, the genus of $\langle 1, 1, 16N, 0, 0, 0 \rangle$ has exactly two spinor genera which are of equal size (number of classes). For any A in this genus, there is exactly one $B \neq A$ such that A represents $B^{(4)}$ and B represents $A^{(4)}$, and A and B always lie in distinct spinor genera! I think this is wonderful. There is a built-in involution within the genus that exchanges the spinor genera. The same thing happens for the genus of $\langle 1, 16N, 16N, 0, 0, 0 \rangle$. Note that here it is not really all that important for N to be squarefree, but as soon as N is allowed to be even the two spinor genera have different sizes.

A similar family of examples comes from combining either

$$\underline{N = u^2 + uv + 4v^2}, \quad h(x, y, z) = x^2 + xy + 4y^2 + 225Nz^2$$

or

$$\underline{N = 2u^2 + uv + 2v^2}, \quad h(x, y, z) = 2x^2 + xy + 2y^2 + 225Nz^2.$$

Let us stick with squarefree N for now. With $N \leq 141$, we get two spinor genera. If N is divisible by 15 the spinor genera are of different size. However, if N is not divisible by 3, they are the same size and there is an involution that interchanges the spinor genera given as before by $B \neq A$ while A represents $B^{(9)}$ and B represents $A^{(9)}$. If N is not divisible by 5, they are the same size and there is an involution that interchanges the spinor genera given by $B \neq A$ while A represents $B^{(25)}$ and B represents $A^{(25)}$. So, when $\gcd(N, 15) = 1$, there are two distinct involutions.

It is also fairly common for a genus with four spinor genera to have, at least, those of equal size. Benham and Hsia [2] show that this happens in the genus of $\langle 1, 20, 400, 0, 0, 0 \rangle$. In an earlier section I display this in the genus of $\langle 1, 17, 4624, 0, 0, 0 \rangle$. The same happens for $\langle 1, 17, 4624s, 0, 0, 0 \rangle$ for $s < 140$ and s odd, while s is represented by either u^2+17v^2 or $2u^2+2uv+9v^2$ but not both; that is $s \in \{1, 13, 17, 25, 53, 89, 101, 137\}$. A simple way to satisfy the peculiar looking conditions for larger s is to take s prime, while $s \equiv 1 \bmod 4$ and $(s|17) = 1$.

61 Benham and Hsia example

From the 1982 Nagoya Math. Journal article [2] by J. W. Benham and J. S. Hsia, "On Spinor Exceptional Representations." On page 252 they give the genus

$$A_1 = \langle 4, 5, 400, 0, 0, 0 \rangle , A_2 = \langle 1, 80, 100, 0, 0, 0 \rangle , A_3 = \langle 16, 20, 29, 0, 16, 0 \rangle ,$$

$$B_1 = \langle 1, 20, 400, 0, 0, 0 \rangle , B_2 = \langle 9, 9, 100, 0, 0, 2 \rangle , B_3 = \langle 4, 45, 45, 10, 0, 0 \rangle ,$$

$$C_1 = \langle 4, 25, 80, 0, 0, 0 \rangle , C_2 = \langle 5, 16, 100, 0, 0, 0 \rangle , C_3 = \langle 4, 20, 101, 0, 4, 0 \rangle ,$$

$$D_1 = \langle 16, 20, 25, 0, 0, 0 \rangle , D_2 = \langle 4, 20, 105, 20, 0, 0 \rangle , D_3 = \langle 4, 21, 100, 0, 0, 4 \rangle .$$

For me

$$\langle a, b, c, d, e, f \rangle$$

refers to the quadratic form

$$T(x, y, z) = ax^2 + by^2 + cz^2 + dyz + ezx + fxy$$

Note that the two independent sets of spinor exceptional integers are $5m^2$, where all prime factors p of m satisfy $p \equiv 1 \pmod 4$, and φ^2, where all prime factors q of φ satisfy $(-5|q) = 1$. About the spinor genera, A is regular, $B \neq 5m^2$, $C \neq \varphi^2$, $D \neq \varphi^2, 5m^2$.

Next, we take a number N which is squarefree and whose prime factors η all satisfy $\eta \equiv 1, 9 \pmod{20}$. Or, to put it another way, N is squarefree, prime to 2 and 5, and N is integrally represented by either the binary form $x^2 + 20y^2$ or by $4x^2 + 5y^2$ but **not** by both. This relates to a paper of Estes and Pall on spinor genera for binary forms.

So now we consider the genus containing both

$$\langle 1, 20, 400N, 0, 0, 0\rangle \quad \text{and} \quad \langle 4, 5, 400N, 0, 0, 0\rangle .$$

Or, put another way, the genus containing the four forms

$$\langle 1, 80, 100N, 0, 0, 0\rangle \quad \text{and} \quad \langle 4, 21, 100N, 0, 0, 4\rangle ,$$

$$\langle 5, 16, 100N, 0, 0, 0\rangle \quad \text{and} \quad \langle 9, 9, 100N, 0, 0, 2\rangle ,$$

where I believe these four forms lie in four distinct spinor genera.

The first conjectures are that this genus has exactly four spinor genera, while the independent sets of spinor exceptions are $5Nm^2$ and $N\varphi^2$. Once again we label the spinor genera so that A_N is regular, $B_N \neq 5Nm^2$, $C_N \neq N\varphi^2$, $D_N \neq N\varphi^2, 5Nm^2$.

Chan proved something for me about positive ternary forms in 2008, here it is.

Theorem (Chan) Suppose that M and L lie in genera with discriminant ratio n, where n is required to be an integer but has no other restrictions. Suppose further that M represents $L^{(n)}$ and L represents $M^{(n)}$. Then, given any M_1 in the genus of M, there is at least one L_1 in the genus of L such that M_1 represents $L_1^{(n)}$ and L_1 represents $M_1^{(n)}$. Also, given any L_2 in the genus of L, there is at least one M_2 in the genus of M such that L_2 represents $M_2^{(n)}$ and M_2 represents $L_2^{(n)}$.

Now, let M mean either $\langle 1, 20, 400N, 0, 0, 0\rangle$ or $\langle 4, 5, 400N, 0, 0, 0\rangle$, and take the letter L to mean $L = B_1 = \langle 1, 20, 400, 0, 0, 0\rangle$, then for one (and only one) of the choices for M we have M represents $L^{(N)}$ and L represents $M^{(N)}$.

From Chan's result and the fact that for any of the spinor exceptions, multiplying by N^2 gives us a spinor exception for the same forms, it follows that all forms in A_N correspond only with forms in spinor genus A of the original genus, B_N only with B, then C_N only with C, and D_N only with D.

The correspondence is generally many-to-many but here **respects spinor genus.** Most of my conjectures in this area are about situations where there is no such immediate trick to show that the correspondence respects spinor genus. Other conjectures are about how matching numbers of spinor genera and such a correspondence force the presence of spinor exceptional integers in the genus with larger discriminant. The natural setting for my work is genera with "complete systems of spinor exceptional integers."

The next conjecture is that for each such N, the four spinor genera have exactly the same number of classes. We have

$$(N = 1, 3), \ \ (N = 29, 21), \ \ (N = 41, 27), \ \ (N = 61, 36).$$

Finally, there are what I call the "involutions" in my manuscript, which match forms in the same genus but different spinor genera. For all the N I have checked, for each form M in A_N, there is exactly one form L in C_N such that the M represents $L^{(25)}$ and L represents $M^{(25)}$. Similar for B_N forms and D_N forms.

Then, for each form M in A_N, there are two forms $L_1, L_2 \in B_N$, two more forms $L_3, L_4 \in C_N$, and a single form L_5 in D_N such that M represents $L_1^{(4)}, L_2^{(4)}, L_3^{(4)}, L_4^{(4)}, L_5^{(4)}$ and L_1, L_2, L_3, L_4, L_5 all represent $M^{(4)}$.

For each form M in B_N, there are two forms $L_1, L_2 \in A_N$, two more forms $L_3, L_4 \in D_N$, and a single form L_5 in C_N such that M represents $L_1^{(4)}, L_2^{(4)}, L_3^{(4)}, L_4^{(4)}, L_5^{(4)}$ and L_1, L_2, L_3, L_4, L_5 all represent $M^{(4)}$.

For each form M in C_N, there are two forms $L_1, L_2 \in A_N$, two more forms $L_3, L_4 \in D_N$, and a single form L_5 in B_N such that M represents $L_1^{(4)}, L_2^{(4)}, L_3^{(4)}, L_4^{(4)}, L_5^{(4)}$ and L_1, L_2, L_3, L_4, L_5 all represent $M^{(4)}$.

For each form M in D_N, there are two forms $L_1, L_2 \in B_N$, two more forms $L_3, L_4 \in C_N$, and a single form L_5 in A_N such that M represents $L_1^{(4)}, L_2^{(4)}, L_3^{(4)}, L_4^{(4)}, L_5^{(4)}$ and L_1, L_2, L_3, L_4, L_5 all represent $M^{(4)}$.

Multiplication by 25 pairs A_N with C_N and then B_N with D_N. Then multiplication by 4 gives bijections between A_N and D_N, then B_N and C_N. Note that for both 4 and 25, forms never match any form in their own spinor genus except themselves. So, this gives a reason for finding all four spinor genera possessing the same number of classes.

References

[1] J.W. Benham, A. G. Earnest, J. S. Hsia, and D. C. Hung. Spinor regular positive ternary quadratic forms. *Journal of the London Mathematical Society*, 42:1–10, 1990.

[2] J.W. Benham and J. S. Hsia. On spinor exceptional representations. *Nagoya Mathematical Journal*, 87:247–260, 1982.

[3] M. Bhargava. On the Conway-Schneeberger Fifteen Theorem. In Eva Bayer-Fluckiger, David Lewis, and Andrew Ranicki, editors, *Quadratic Forms and Their Applications*, pages 27–37. American Mathematical Society, 2000. Series: Contemporary Mathematics, volume 272.

[4] H. Brandt and O. Intrau. Tabelle reduzierten positiver ternärer quadratischer Formen. *Abh. der Sächsischen Akad. der Wissenschaften zu Leipzig, Math.-Naturw.*, 45(4), 1958. This article is said (Zentralblatt) to have taken up the entire issue, total 261 pages, and been available for sale separately, almost as though it were a book. I've seen excerpts only.

[5] D. A. Buell. *Binary Quadratic Forms: Classical Theory and Modern Computations*. Springer-Verlag, 1989.

[6] J.W.S. Cassels. *Rational Quadratic Forms*. Academic Press, 1978.

[7] W. K. Chan. personal communication, 2008. One page.

[8] W. K. Chan and A. G. Earnest. Discriminant bounds for spinor regular ternary quadratic lattices. *Journal of the London Mathematical Society*, 69:545–561, 2004.

[9] W. K. Chan, D. R. Estes, and M. Jöchner. Representations of codimension ≥ 3 by definite quadratic forms. *Journal of Number Theory*, 71:81–85, 1998.

[10] J. H. Conway. *The Sensual (Quadratic) Form*. Number 26 in Carus Mathematical Monographs. The Mathematical Association of America, 1997.

[11] J. H. Conway. Universal Quadratic Forms and the Fifteen Theorem. In Eva Bayer-Fluckinger, David Lewis, and Andrew Ranicki, editors, *Quadratic Forms and Their Applications*, pages 23–26. American Mathematical Society, 2000. Series: Contemporary Mathematics, volume 272.

[12] J. H. Conway and N. J. A. Sloane. *Sphere Packings, Lattices and Groups.* Springer-Verlag, 1988.

[13] D. A. Cox. *Primes of the form* $x^2 + ny^2$. John Wiley and Sons, 1989.

[14] L. E. Dickson. Ternary quadratic forms and congruences. *Annals of Mathematics*, 28:333–341, 1927.

[15] L. E. Dickson. *Modern Elementary Theory of Numbers.* University of Chicago Press, 1939.

[16] W. Duke. Some old problems and new results about quadratic forms. *Notices of the American Mathematical Society*, 44:190–196, 1997.

[17] W. Duke and R. Schulze-Pillot. Representations of integers by positive ternary quadratic forms and equidistribution of lattice points on ellipsoids. *Inventiones Mathematicae*, 99:49–57, 1990.

[18] A. G. Earnest. The representation of binary quadratic forms by positive definite quaternary quadratic forms. *Transactions of the American Mathematical Society*, 345:853–863, 1994.

[19] A. G. Earnest. Universal and regular positive quadratic lattices over totally real number fields. In Myung-Hwan Kim, John S. Hsia, Yoshiyuki Kitaoka, and Rainer Schulze-Pillot, editors, *Integral Quadratic Forms and Lattices*, pages 17–27. American Mathematical Society, 1999. Series: Contemporary Mathematics, volume 249.

[20] A. G. Earnest. personal communication, 2004. Three pages.

[21] A. G. Earnest, J. S. Hsia, and D. C. Hung. Primitive representations by spinor genera of ternary quadratic forms. *Journal of the London Mathematical Society*, 50:222–230, 1994.

[22] M. Eichler. *Quadratische Formen und orthogonale Gruppen.* Springer-Verlag, 1952.

[23] J. Ellenberg and A. Venkatesh. Local-global principles for representations of quadratic forms. *Inventiones Mathematicae*, 171:257–279, 2007.

[24] D. R. Estes and G. Pall. Spinor genera of binary quadratic forms. *Journal of Number Theory*, 5:421–432, 1973.

[25] J. Hanke. On a local-global principle for integral quadratic forms. Unpublished preprint, 2003.

[26] J. Hanke. Some recent results about (ternary) quadratic forms. In : Hershy Kisilevsky and Eyal Z. Goren, editors, *Number Theory*, pages 147–164. American Mathematical Society and Centre de Recherches Mathématiques, 2004. Series: CRM Proceedings & Lecture Notes, volume 36.

[27] J. S. Hsia. Two theorems on integral matrices. *Linear and Multilinear Algebra*, 5:257–264, 1978.

[28] J. S. Hsia. Regular positive ternary quadratic forms. *Mathematika*, 28:231–238, 1981.

[29] J. S. Hsia, Y. Kitaoka, and M. Kneser. Representations of positive definite quadratic forms. *Journal für die reine und angewandte Mathematik*, 301:132–141, 1978.

[30] W. C. Jagy. Five regular or nearly-regular ternary quadratic forms. *Acta Arithmetica*, 77:361–367, 1996.

[31] W. C. Jagy, I. Kaplansky, and A. Schiemann. There are 913 regular ternary quadratic forms. *Mathematika*, 44:332–341, 1997.

[32] B. W. Jones. *Representation by positive ternary quadratic forms*. PhD thesis, University of Chicago, 1928.

[33] B. W. Jones. A new definition of genus for ternary quadratic forms. *Transactions of the American Mathematical Society*, 33:92–110, 1931.

[34] B. W. Jones. The regularity of a genus of positive ternary quadratic forms. *Transactions of the American Mathematical Society*, 33:111–124, 1931.

[35] B. W. Jones. *The Arithmetic Theory of Quadratic Forms.* Number 10 in Carus Mathematical Monographs. Mathematical Association of America, 1950.

[36] B. W. Jones and G. Pall. Regular and semi-regular positive ternary quadratic forms. *Acta Mathematica*, 70:165–191, 1939.

[37] I. Kaplansky. The first nontrivial genus of positive definite ternary forms. *Mathematics of Computation*, 64:341–345, 1995.

[38] I. Kaplansky. Ternary positive quadratic forms that represent all odd positive integers. *Acta Arithmetica*, 70:209–214, 1995.

[39] I. Kaplansky. Notes on the classification of regular ternary forms. Unpublished, 1996.

[40] I. Kaplansky. Proofs of regularity. Unpublished, 1997.

[41] I. Kaplansky. A unified proof of regularity for 36 forms. Unpublished, 2002.

[42] W. J. LeVeque. *Topics in Number Theory*, volume 2. Addison-Wesley, 1956.

[43] G. L. Nipp. *Quaternary Quadratic Forms: Computer Generated Tables.* Springer-Verlag, 1991.

[44] B.-K. Oh. Regular positive ternary quadratic forms. *Acta Arithmetica*, 147:233–243, 2011.

[45] K. Ono and K. Soundararajan. Ramanujan's ternary quadratic form. *Inventiones Mathematicae*, 130:415–454, 1997.

[46] S. Ramanujan. On the expression of a number in the form $ax^2 + by^2 + cz^2 + du^2$. *Proceedings of the Cambridge Philosophical Society*, 19:11–21, 1916.

[47] H. E. Rose. *A Course in Number Theory.* Oxford University Press, 1988.

[48] A. Schiemann. Ternary positive definite quadratic forms are determined by their theta series. *Mathematische Annalen*, 308:507–517, 1997.

[49] R. Schulze-Pillot. Darstellung durch Spinorgeschlechter ternäre quadratische Formen. *Journal of Number Theory*, 12:529–540, 1980.

[50] R. Schulze-Pillot. Darstellung durch definite ternäre quadratische Formen. *Journal of Number Theory*, 14:237–250, 1982.

[51] R. Schulze-Pillot. Exceptional integers for genera of integral ternary positive definite quadratic forms. *Duke Mathematical Journal*, 102:351–357, 2000.

[52] R. Schulze-Pillot. Representation by integral quadratic forms - a survey. In Ricardo Baeza, John S. Hsia, Bill Jacob, and Alexander Prestel, editors, *Algebraic and Arithmetic Theory of Quadratic Forms*, pages 303–321. American Mathematical Society, 2004. Series: Contemporary Mathematics, volume 344.

[53] R. Schulze-Pillot. Local conditions for global representations of quadratic forms. http://arxiv.org/abs/0804.2158v3, June 2008. Downloadable as pdf or postscript.

[54] V. N. Timofeev. On positive quadratic forms representing the same numbers. (Russian). *Uspekhi Matematicheskikh Nauk*, 18:191–193, 1963.

[55] G. L. Watson. *Some problems in the theory of numbers.* PhD thesis, University of London, 1953.

[56] G. L. Watson. *Integral Quadratic Forms.* Number 51 in Cambridge Tracts in Mathematics and Mathematical Physics. Cambridge University Press, 1960.

[57] G. L. Watson. Transformations of a quadratic form which do not increase the class-number. *Proceedings of the London Mathematical Society*, 12:577–587, 1962.

[58] G. L. Watson. One-class genera of positive ternary quadratic forms, II. *Mathematika*, 22:1–11, 1975.

[59] G. L. Watson. Regular positive ternary quadratic forms. *Journal of the London Mathematical Society*, 13:97–102, 1976.