REGULAR POSITIVE TERNARY QUADRATIC FORMS

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Unless stated otherwise all quadratic forms have rational integer coefficients and all representations are integral representations. For positive binary quadratic forms of the same discriminant it is known that two such forms are equivalent provided they represent the same integers. See, for instance, [**Ki**$_2$], and for a sharper extension [**W**$_2$]. On the other hand, in the quaternary case these value-sets are far from characterizing the forms even within a genus. It is therefore natural to ask for positive ternary forms the corresponding question, whose answer appears to be unknown.

(**A**): *Are the forms in a genus of positive ternary quadratic forms classified by the sets of integers they represent*?

A positive ternary quadratic form is *regular* if it represents all the natural numbers not excluded by congruence considerations. This notion was introduced by L. E. Dickson [**D**] in his attempt to understand the representational properties of the odd integers by the "Ramanujan form" $x^2+y^2+10z^2$, which to this date is still not fully resolved. Aside from Dickson's own investigations of some regular diagonal ternary forms, almost all of which turned out to have class number one—hence, obviously regular—and those in the thesis (Chicago, 1928) of his pupil, B. W. Jones, which was later incorporated in [**JP**], virtually nothing more was known about regular ternary quadratic forms until recent years.

In his thesis (London, 1953) Watson proved that for positive ternary quadratic forms with square-free discriminants and class number at least two there are just four regular forms, given by:

$$x^2+y^2+yz+3z^2 ,$$

$$x^2+2y^2+yz+2z^2 ,$$

$$x^2+xy+2y^2+2yz+3z^2 ,$$

$$x^2+xy+2y^2+3z^2 .$$

Indeed, he showed in [**W**$_1$] that the last three forms are primitively regular, *i.e.*, they primitively represent all integers not excluded by congruence conditions. Primitive regularity is a stronger property in that it implies regularity. Each of these four forms belongs to a genus having a single spinor genus and two classes. In [**SP**$_2$] Schulze–Pillot used a different method, one which is based on the Bruhat–Tits building of the spin group of the completion of the associated rational form at a suitable prime, to find additional regular forms which lie in genera all of which have also a single spinor genus and two classes and moreover the discriminant is $2D'$, where D' is odd and $\leqslant 1{,}000$. Besides the four forms in Watson's list, he found eleven more primitively regular forms. Amongst these, there were two pairs having the same

Research partially supported by a U.S. N.S.F. Grant.

discriminants, and they were:

$$(D' = 135) \quad x^2+3y^2+3yz+12z^2, \quad 2x^2+xy+2y^2+9z^2\,,$$

and

$$(D' = 675) \quad x^2+xy+4y^2+45z^2, \quad 5x^2+6y^2+3yz+6z^2\,.$$

These belong, however, to different genera. Thus, the following question arises naturally.

(B): *Does there exist a genus of positive ternary quadratic forms in which there are two regular forms?*

Of course, an affirmative answer to question **(B)** gives simultaneously a negative answer to question **(A)**.

I shall show here that question **(B)** has an affirmative answer, and furnish such an example. Such examples, should more of them exist, are rather rare. The example given below in Theorem 1 is, in fact, the *only* one that I know. Once the forms are given, a simple elementary argument can be provided, in this case, to show that they do represent the same integers. Our discovery, however, is based on a more systematic search based on the theory of spinor exceptional representations, aided by a very useful technical refinement (Theorem 2), which makes the verifications for regularity extremely simple in many instances. For example, in Table II of [**JP**] there are seven regular forms each of which belongs to a genus containing two spinor genera; the regularity of six of them follows at once from our Theorem 2. As a further application, we find (Theorem 3) all the regular positive ternary forms of discriminant less then 2,000 and satisfying:

$$g \ (= \text{number of spinor genera}) = 2, \quad \text{and} \quad h \ (= \text{class number}) \leqslant 3\,.$$

There are precisely eleven regular forms in this category, three of which are diagonal forms and are already contained in the list of Jones–Pall [**JP**], but the remaining eight forms are new.

§1. *Preliminaries.* Let $f(x_1, x_2, x_3)$ be a ternary integer-valued quadratic form. Then corresponding to $2f$ we have a quadratic $\mathbb{Z}$-lattice L of rank 3 which has integral scale and even norm in the sense of [**OM**]. A *spinor exceptional integer* c for the genus **G** of L is an integer which is representable by some but not by all the spinor genera in **G**. By the discriminant D of f (or of L or of **G**) we mean the determinant of the matrix

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right).$$

Thus, D is -2 times the discriminant used in the tables of Brandt–Intrau [**BI**]. Let $V = \mathbb{Q}L$, let J_L be the subgroup of the adele group on V which leaves L invariant, and let θ be the spinor norm function. Necessary and sufficient conditions for $c \neq 0$ to be a spinor exceptional integer for **G** are:

$$c \text{ is representable by } \mathbf{G}\,; \tag{1.1}$$

$$-cD \notin \mathbb{Q}^{\times 2}\,; \tag{1.2}$$

$$\theta(J_L) \subseteq N(J_E); \tag{1.3}$$

$$\theta(L_p : c) = N_p(E) \quad \text{for all finite } p, \tag{1.4}$$

where $E = \mathbb{Q}(\sqrt{-cD})$, J_E is the idele group of E, $N = N_{E/\mathbb{Q}}$, $N_p(E)$ is the p-th component of $N(J_E)$, and

$$\theta(L_p : c) = \langle \theta(\phi) \mid \phi \in 0^+(V_p) \quad \phi(u) \in L_p \quad \text{for any } u \in L_p \text{ with } Q(u) = c \rangle .$$

For a primitive spinor exceptional integer, just replace $\theta(L_p : c)$ by the obvious primitive analogue. See [**SP**$_1$].

§2. *Statement of results.*

THEOREM 1. *There exists a genus of positive ternary quadratic forms having class number two and such that both forms are regular. Two such forms are:*

$$x^2+xy+y^2+9z^2 \quad \text{and} \quad x^2+3(y^2+yz+z^2). \tag{2.1}$$

THEOREM 2. *If a genus* **G** *of ternary quadratic forms admits a spinor exceptional integer, then it already has one c satisfying:*

$$\operatorname{ord}_p(c) \leqslant \operatorname{ord}_p(D) \quad \textit{for all finite } p. \tag{2.2}$$

THEOREM 3. *Let f be a positive ternary quadratic form with discriminant $D \leqslant 2{,}000$. Suppose further that the genus of f has two spinor genera and at most three classes. Then f is regular, if, and only if, it is equivalent to one of the following forms.*

$$(D = 54) \quad x^2+xy+y^2+9z^2, \quad x^2+3(y^2+yz+z^2); \tag{2.3}$$

$$(D = 128) \quad x^2+y^2+16z^2; \tag{2.4}$$

$$(D = 162) \quad x^2+3y^2+xz+7z^2; \tag{2.5}$$

$$(D = 216) \quad x^2+xy+y^2+36z^2; \tag{2.6}$$

$$(D = 216) \quad x^2+3y^2+10z^2+3yz+xz; \tag{2.7}$$

$$(D = 486) \quad x^2+xy+7y^2+9z^2; \tag{2.8}$$

$$(D = 512) \quad x^2+4y^2+16z^2; \tag{2.9}$$

$$(D = 864) \quad x^2+12(y^2+yz+z^2); \tag{2.10}$$

$$(D = 864) \quad x^2+3y^2+36z^2; \tag{2.11}$$

$$(D = 1944) \quad x^2+xy+7y^2+36z^2. \tag{2.12}$$

None of these forms is primitively regular.

§3. *Proof of Theorem* 2. Suppose c is a spinor exceptional integer for the genus $\mathbf{G}$ of L. Then conditions (1.1) to (1.4) are satisfied. Put $D_c = -cD$. It follows from the computations in [$\mathbf{SP}_1$] (Thms. 3 and 4) that if $D_c \notin \mathbb{Q}_p^{\times 2}$ at a finite prime p then (1.3) and (1.4) will force the relation $\operatorname{ord}_p(c) \leqslant \operatorname{ord}_p(D)$. Therefore, we may assume that D_c is a square at p. This implies that the supporting quadratic space V_p at p is isotropic. Moreover, in this case the conditions (1.3), (1.4) are automatically fulfilled. Clearly, cp^{2r} are all spinor exceptional integers for $\mathbf{G}$, and so ord_p is unbounded from above. Similarly, cp^{-2} would be a spinor exception if only it were represented by $\mathbf{G}$ (*i.e.*, by L_p). Hence, the proof will be finished after we prove the following somewhat technical result.

LEMMA. *Let $(F, \mathfrak{O}, \mathfrak{p})$ be a local field in which 2 is either a unit or a prime, and L a ternary $\mathfrak{O}$-lattice on an isotropic F-space V. If L represents b with $D_b \in F^{\times 2}$, then L represents c such that $c \in bF^{\times 2}$ and $\operatorname{ord}_{\mathfrak{p}}(c) \leqslant \operatorname{ord}_{\mathfrak{p}}(D)$, where D is the discriminant of L.*

Proof. Suppose first that $\mathfrak{p} \nmid D$. Then L is unimodular and represents all the elements of $\mathfrak{O}$. That $D_b = -bD \in F^{\times 2}$ implies $\operatorname{ord}_{\mathfrak{p}}(b)$ must be even, and so L represents a unit $c \in bF^{\times 2}$.

So, let $\mathfrak{p} \mid D$. We first consider the case of $\mathfrak{p} \nmid 2$. By scaling, we may assume that

$$L \cong \langle 1 \rangle \perp \langle \pi^e \alpha \rangle \perp \langle \pi^f \beta \rangle ,$$

where $0 \leqslant e \leqslant f$, $(\pi) = \mathfrak{p}$ and α, β are units. Since V is isotropic, a computation of local symbols gives

$$1 = (\pi^e \alpha, -1)(\pi^f \beta, \pi^{e+f} \alpha\beta) . \tag{3.1}$$

Case I. Suppose $e \equiv f \pmod 2$. Then (3.1) becomes $1 = (\pi^e\alpha, -1)(\pi^f\beta, \alpha\beta)$. If e is even, L contains a sublattice K isometric to $\langle \pi^f \rangle \perp \langle \pi^f \alpha \rangle \perp \langle \pi^f \beta \rangle$ which represents all the elements of $\pi^f \mathfrak{O}$. If $\operatorname{ord}_{\mathfrak{p}}(b) > e+f$ then $b\mathfrak{p}^{-2}$ has order $\geqslant e+f-1 \geqslant f$ when $e > 0$. The same holds for $e = 0$ and $\operatorname{ord}_{\mathfrak{p}}(b) \geqslant f+2$. On the other hand, the case of $e = 0$, $\operatorname{ord}_{\mathfrak{p}}(b) = f+1$ cannot occur due to the order parity of D_b.

For e odd, (3.1) becomes $1 = (\pi, -\alpha\beta)$. Since $D_b \in F^{\times 2}$, b must also be a square, and the desired conclusion follows.

Case II. Let $e \not\equiv f \pmod 2$. Condition (3.1) becomes

$$1 = \begin{cases} \left(\dfrac{-1}{\mathfrak{p}}\right)(\beta, \pi), & \text{if } f \text{ is even} , \\[2ex] \left(\dfrac{-1}{\mathfrak{p}}\right)(\alpha, \pi), & \text{if } f \text{ is odd} , \end{cases} \tag{3.2}$$

where $(-/\mathfrak{p})$ is the generalized Legendre symbol. Now, when f is even consider the sublattice $B \cong \langle \pi^f \rangle \perp \langle \pi^f \beta \rangle$ which is (π^f)-modular. Since the supporting space $FB \cong [1, \beta]$ and $(-\beta/\mathfrak{p}) = 1$ by (3.2), B represents all elements from $\pi^f \mathfrak{O}$. If f is odd we just take $B \cong \langle \pi^e \rangle \perp \langle \pi^e \alpha \rangle$ which represents all elements from $\pi^e \mathfrak{O}$. In either case if $\operatorname{ord}_{\mathfrak{p}}(b) > e+f$ then $b\pi^{-2}$ is represented by B and hence by L.

The case where $\mathfrak{p}$ is an unramified dyadic prime is technically more involved and has many more sub-cases. We shall not present these details here.

§4. *Proofs of Theorem* 1. From reduction theory one knows that the two listed forms comprise all the classes in the genus. See [**BI**]. A simple computation shows that there are two spinor genera so that the classes and spinor genera coincide in this case and hence the theory of spinor exceptional representations will give a complete answer. Since the discriminant is $D = 54 = 2 . 3^3$ it suffices to see, by Theorem 2, that both forms represent the following integers: 1, 3, 3^2, 3^3. This is clear by inspection.

Here we briefly mention three alternative proofs. First, since the class number is two, the regularity of these forms follows from their representing the same set of positive integers. An elementary argument can be given by merely noting that the binary forms given by x^2+xy+y^2 and x^2+3y^2, though in different genera, represent the same integers. A second elementary proof can be given by using the integral automorphs of the two forms. Still a third alternative proof is provided by the graph-theoretic arguments as used in [**SP**$_2$], [**BH**] since the genus **G** under consideration is "half-regular" at $p = 2$ and $j(2) \notin P_{\mathbb{Q}^+} J_{\mathbb{Q}}^{\mathbf{G}}$ implies that the graph $Z(L, 2)$ contains both classes. Here L is a lattice corresponding to either of the two forms. The integers represented by L are contained in the union of the sets of integers represented by the neighbours of L; but, the three neighbours of L all belong to the other spinor genus and so correspond to the other class.

§5. *Proof of Theorem* 3. The first step involves the determination of those discriminants $D \leqslant 2{,}000$ (see [**BI**]) where the genera contain two spinor genera. From the computations of local integral spinor norms there are known sufficient conditions for a genus to admit only a single spinor genus. See [**K**$_1$], [**EH**]. Specializing to the case of positive ternary quadratic forms of discriminant D this translates to the conditions: (i) $\mathrm{ord}_p(D) \leqslant 2$ at all the odd primes, and (ii) $\mathrm{ord}_2(D) \leqslant 6$. Further computations, whose details we do not provide here, then yield the following:

PROPOSITION. *The only discriminants* $D \leqslant 2{,}000$ *which can admit a genus having multiple* (*in fact, two*) *spinor genera are:* 54, 128, 162, 216, 256, 378, 486, 512, 640, 648, 686, 702, 864, 1024, 1026, 1134, 1152, 1280, 1350, 1458, 1512, 1664, 1674, 1944, 1998.

Using this proposition and Theorem 2 one can now systematically find all the regular positive ternary quadratic forms with discriminant $\leqslant 2{,}000$ which belong to genera having multiple spinor genera. Here we confine ourselves only to those genera with at most three classes. This restricts us to consider the following: $D = 54$, 128, 162, 216, 486, 512, 648, 686, 864, 1944. These are treated individually. The case of $D = 54$ has already been discussed. For the remaining cases, the forms given in (2.4), (2.6), (2.7), (2.9), (2.10), (2.11) all lie in genera with two classes. In each of these genera the number 1 is clearly a spinor exceptional integer so that the potential candidate for regularity is obvious. Applications of our Theorem 2 quickly prove regularity. For example, take the case of (2.7) with $D = 216 = 2^3.3^3$. It is, therefore, sufficient to show that the numbers from $\{1, 2, 2^2, 3, 3^2, 3^3, 2 . 3, 2 . 3^2, 2 . 3^3, 2^2 . 3, 2^2 . 3^2, 2^2 . 3^3\}$, unless excluded by congruence considerations, are all represented by $f(x, y, z) = x^2+3y^2+10z^2+3yz+xz$. The numbers $\{2, 2 . 3, 2 . 3^3\}$ must be excluded. Now, $f(1, 1, 1) = 2 . 3^2$ and the others are obvious.

For the remaining three forms it is first necessary to split each genus by the theory of spinor exceptional representations, and then use Theorem 2 to prove their regularity.

Consider first the case $D = 1944$. Clearly, 1 is a spinor exception. Denote by $f(x, y, z)$ the form given in (2.12). The other two forms in the genus are:

$$g(x, y, z) = 4x^2+9(y^2+yz+z^2), \quad \text{and} \quad h(x, y, z) = 7(x^2+y^2+z^2)+5(xy+xz+yz).$$

Since there are just two spinor genera, 7 cannot be another independent spinor exception. Now, 7 is represented by f and h, but not by g. Hence, h lies in the opposite spinor genus from f. Similarly, 4 is not represented by h, but is represented by f and g. Thus, the genus is split as follows: $\mathrm{Spn}(f) = \{f\} \cup \mathrm{Spn}(g) = \{g, h\}$. The proof for regularity is as before.

In the cases of (2.5) and (2.8) both genera have no spinor exceptional integer and in each case the given form belongs to a spinor genus consisting of a single class. This proves that all the listed forms (2.3)–(2.12) are all regular. By similar reasonings, the forms in $D = 648, 686$ are eliminated.

Finally, to see that none of the listed forms is primitively regular, we just need to observe that in the case $D = 54$, 16 (resp. 4) is primitively represented by the first (resp. the second) form and not by the other. For the forms (2.4)–(2.12), the numbers 4, 300, 4, 27, 4, 9, 9, 4, 9 are respectively not primitively represented. We explain only the case when 300 is not primitively represented by (2.5). One checks by simple computations that 12 is a primitive spinor exception, and hence $12 . 5^2 = 300$ is also a primitive spinor exception by the general theory. The form (2.5) is in a spinor genus by itself. The other spinor genus has a class represented by the form $g(x, y, z) = x^2+xy+y^2+27z^2$ which primitively represents 300 *via* $g(7, 1, 9)$. Hence, (2.5) cannot primitively represent 300. This completes the proof of Theorem 3.

§6. *Some remarks and conjectures.* As seen above, our Theorem 2 is particularly effective in proving regularity when class and spinor genus coincide. This occurs in six out of the seven genera in Table II of [**JP**] where each genus contains two spinor genera; namely,

$$x^2+y^2+16z^2, \qquad \text{see (2.4)}$$

$$x^2+4y^2+16z^2, \qquad \text{see (2.9)}$$

$$x^2+3y^2+36z^2, \qquad \text{see (2.10)}$$

$$x^2+16y^2+16z^2, \quad x^2+12y^2+36z^2, \quad x^2+8y^2+64z^2.$$

The regularity of these six forms, therefore, follows virtually by inspection. In the seventh case, the class number is four and both spinor genera contain two classes.

The forms

$$f(x, y, z) = x^2+48y^2+144z^2 \quad \text{and} \quad g(x, y, z) = 4x^2+48y^2+49z^2-4xz-48yz$$

belong to the same spinor genus which is clearly (*via* Theorem 2) a regular spinor genus in the sense of [**BH**]. Hence, the regularity of f would follow after showing that the value-set of g is a subset of the value-set of f. To see this, one notes that $g(x, y, z) = f(2x-z, y-\frac{1}{2}z, -\frac{1}{2}z)$ which means that for even z such a number is already represented by f. So, let z be odd. If x is also odd then $g(x, y, z) = g(-x, y, x+z)$. Similarly, for y odd. Thus, we only consider when both x and y are even and z is odd. By another transformation, one sees that $g(x, y, z) = f(-2x+z, -\frac{1}{2}y, \frac{1}{2}y+z)$. So f represents everything g does. See also the discussion on these seven forms in [**SP**$_1$].

The study of regularity here is related to the following general question: *To what extent does representation of integers or of forms determine classification*? A beautiful result of Kitaoka's [**Ki**$_2$] says that a form in n variables is classified by the $(n-1)$-ary forms it represents. In particular, ternary forms are classified by the binary forms they represent. We have seen here that Kitaoka's co-dimension one result cannot be replaced by co-dimension two if mere representation is required. However, if we further require primitive representations (such as Kitaoka's case does in the definition of "characteristic sublattice") then it is likely to hold. From our numerical computations we are led to the following conjecture on ternary forms.

CONJECTURE A. *Two positive ternary quadratic forms in the same genus are equivalent if they primitively represent the same integers.*

It is necessary to require definiteness since for indefinite ternaries there exist genera in which every form in the genus represents and primitively represents exactly the same integers. Furthermore, the class numbers of such genera can be arbitrarily large. See Section 2.1 of [**BH**].

More generally, if we consider representations of forms by forms, our numerical evidence is admittedly much more scarce, and has been confined so far to forms in 24 or fewer variables, but not necessarily to unimodular forms. The evidence seems to point to the following rather dramatic conjectures.

CONJECTURE B. *The n-ary positive definite quadratic forms of a fixed discriminant are "strongly characterized" by the* $[\frac{1}{2}(n+1)]$*-ary forms they represent.*

Here "strongly characterized" means that for every f in the genus **G** there exists an $[\frac{1}{2}(n+1)]$-ary form ϕ_f which is represented by f but by no other form in G. If this conjecture is true then it follows that the theta series $\theta_f^{(d)}(Z)$ of degree d for $d \geqslant [\frac{1}{2}(n+1)]$, as f varies over **G**, are linearly independent so that by Siegel's theory the dimension of their span is asymptotically twice the mass of **G**. Again. Kitaoka's result mentioned above says that if $d = n-1$ then Conjecture B is true. On the other hand, the results in [**HKK**] indicate that, modulo the condition on the minima, d needs to satisfy $d \geqslant [\frac{1}{2}(n+1)]-1$, so that the bound given by $[\frac{1}{2}(n+1)]$ in Conjecture B is likely to be very near the best possible for all n. On the other hand, if one wants the theta series of degree d merely to classify the forms in **G**, instead of being linearly independent, then there is the following stronger conjecture.

CONJECTURE C. *In a genus* **G** *of positive n-ary quadratic forms, say, for* $n \equiv 0 \pmod 4$, *the theta series* $\theta_f^{(n/4)}(Z)$ *of degree* $\frac{1}{4}n$ *classify the forms in* **G**.

Here some evidences seem to indicate that sometimes the degree may be as low as $\frac{1}{6}n$; for example, such is the case for even unimodular lattices of rank 24. However, besides the case of $n = 4$, there are eight-dimensional forms which are only distinguished by theta series of degree two.

We hope these conjectures will stimulate broader investigations of the role of representations in the classification problem of positive integral quadratic forms.

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10C02: *NUMBER THEORY; Forms; Forms over global rings.*

Received on the 17*th of June*, 1981.