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Two Theorems on Integral Matrices

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The following two results are proved: (1) For a positive definite integral symmetric matrix S of rank $(S) < 7$ or when rank $(S) = 8$, S has an odd entry in its diagonal, there is an integral matrix A satisfying $AA^t = S$ if there is a rational matrix R with $RR^t = S$. (2) Given an integral matrix A of size $r \times n$ such that $AA^t = mI_r$ there is then always an integral completion matrix B of size $n \times n$ satisfying $BB^t = mI_n$ whenever $n-r$ is less than or equal to 7. This threshold number 7 is the best possible. (Here m, n satisfy the obvious necessary conditions.)

INTRODUCTION

In recent months the following matrix problems have been brought to my attention at various times by several colleagues:

Existence question

Given a diagonal matrix D with positive integers as its entries, suppose there is a rational matrix R such that $RR^t = D$, does there then exist an integral matrix A satisfying $AA^t = D$?

Completion question

Given an integral matrix A of size $r \times n$ such that $AA^t = mI_r$, can we complete A to an integral matrix B of size $n \times n$ satisfying $BB^t = mI_n$?

On the basis of quite a few numerical examples for 3×3 matrices, they are led to believe that Question (I) might have an affirmative answer at least for 3×3 diagonal matrices. We show below (Theorem 1) that not only this is the case, but also the answer is affirmative for D an $n \times n$ matrix with $n \leqslant 7$. When $n = 8$, it is also true provided D has an odd integer as one of its

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diagonal entries. Furthermore, all these assertions remain valid when D is replaced by *any integral symmetric* (of course, positive definite) *matrix* S. In the very special situation when $D = mI_n$, necessary and sufficient conditions on m and n for an affirmative response to Question (I) can be easily determined (Corollary 1), and this has also been done by Marshall Hall, Jr. in his recent paper [H]. In this same paper, Hall also studied the Completion Question in which he showed that if $r = n-1$, $n-2$, or if $n \leqslant 4$ with r arbitrary, then the completion matrix B can be found. Some of his results overlap with those in an earlier article by Cordes–Pall [CP]. The latter paper also showed that if $n \leqslant 14$ and is even and $r = n/2$, then B can be found provided A satisfies an additional condition. We show below (Theorem 2) that the completion matrix B can always be found when $r = n-k$ for $1 \leqslant k \leqslant 7$. This theorem, of course, subsumes all the results by Hall as well as all those by Cordes–Pall on the subject. In addition, it answers affirmatively a conjecture raised in [CP, p. 293]. Also, Theorem 2 is in the best possible form as a counter-example will be given for $r = n-8$. Our approach to both the Existence and the Completion Questions is from the standpoint of representation theory of integral quadratic forms. From this viewpoint, it is quite clear as to why this mysterious number 7 serves as the threshold in both questions. The reason is because when the number of variables is less than eight there is only one genus of positive definite integral quadratic form with discriminant 1, and this genus has but one class in it. In eight variables there are two such genera (one even the other odd) each with class number one.

LATTICE-THEORETIC INTERPRETATION

We translate the two matrix questions into the language of quadratic forms. Moreover, we view quadratic forms in the modern geometric spirit of either quadratic spaces or lattices. Any unexplained notations, facts, and terminology can be found in O'Meara's fundamental book [O]. Let V be the quadratic space over the rationals $\mathbb{Q}$ corresponding to the form that is a sum of n squares. If $\{e_i\}$ is an orthonormal basis for V, let L denote the standard $\mathbb{Z}$-lattice $\perp \mathbb{Z}e_i$, which we shall write as $L \cong \langle 1, \ldots, 1\rangle$. For spaces we shall always use the bracket symbol. Thus, $V \cong [1, \ldots, 1]$. The correspondence between symmetric matrices and quadratic forms is well-known. So, if $S = (s_{ij})$ is an integral symmetric matrix we may view S as a free $\mathbb{Z}$-module with basis $\{x_i\}$ where $B(x_i, x_j) = s_{ij}$. To say that there is a rational matrix R satisfying $RR^t = S$ is clearly equivalent to saying that the quadratic space $S \otimes \mathbb{Q}$ on which the lattice S sits is isometric to V. In other words, there is a rational representation of S by L. To require an integral matrix A with $AA^t = S$ is then seen to be equivalent to requiring an *integral*

representation of S by L. This is the framework with which we shall treat the Existence Question.

For the Completion Question, we are implicitly assuming, of course, that $S = mI_n$ is integrally representable by L. To be given an integral $r \times n$ matrix A with $AA^t = mI_r$ is to be given a sublattice K of L of rank r which is isometric to $r \times \langle m\rangle$. To require the completion matrix B to exist is equivalent to requiring a full sublattice G inside the orthogonal complement $K^\perp$ of K in L such that G is isometric to $(n-r) \times \langle m\rangle$. We show below that this G exists whenever $n-r$ is less than or equal to seven. Note that if the completion process were to permit rational entrices in the remaining rows, then the conclusion follows immediately from the well-known classical theorem of E. Witt [W]. Since Witt's theorem (either the cancellation or the extension version) is generally false for rings, this is where the difficulties lie. We use the genus theory to overcome this.

EXISTENCE PROBLEM

As mentioned above the existence of the rational matrix R with $RR^t = S$ is equivalent to the space $S \otimes \mathbb{Q}$ being isometric to V. So, without loss of any generality we may assume at the outset that V supports both L and S. We shall use the letter S to denote indistinguishably for both the matrix and the lattice corresponding to it. The main result of this section is the following:

THEOREM 1 *Let S be a positive definite integral symmetric matrix for which there is a rational matrix R satisfying $RR^t = S$. If* rank $(S) \leqslant 7$, *or if* rank (S) $=8$ *and S has an odd entry in its diagonal, then there is an integral matrix A with $AA^t = S$.*

COROLLARY 1 *Suppose $S = S_1 \oplus S_2 \oplus \ldots \oplus S_t$ where each S_j satisfies the hypothesis of the theorem, then there is an integral matrix A satisfying $AA^t = S$. In particular, if $S = mI_n$ then A exists if and only if S is rationally equivalent to a sum of squares* (i.e. *m must be a square if n is odd, a sum of two squares if $n \equiv 2$* (mod 4), *and any natural number if $n \equiv 0$* (mod 4)).

Proof of Theorem 1 (All citations without references are from [0].)

1) Suppose we can show that under the given conditions the lattice S is representable by the standard lattice L p-adically at every prime spot p, then it is well-known that there is a lattice in the genus of L that will (globally) represent S. Since the class number of L is one, this means L represents S, which is what we want.

2) Since S and L both sit on the same ambient space V, we neen only to check $S_{(p)}$ is represented by $L_{(p)}$ at every finite prime p. Let us first take care of the odd primes. $S_{(p)}$ is contained in some $\mathbb{Z}_p$-maximal lattice M on $V_{(p)}$.

But, $L_{(p)}$ is also $\mathbb{Z}_p$-maximal on $V_{(p)}$, so that $L_{(p)}$ is isometric to M (Theorem 91:2) and this implies $L_{(p)}$ represents $S_{(p)}$.

3) For the remainder of the proof, we work with the 2-adic localizations: $L_{(2)}$, $S_{(2)}$, and $V_{(2)}$. Let dim $(V) = n$. We may assume $1 < n \leqslant 8$. It is not difficult to see that $\langle 1, 1\rangle$ is exactly the set of all vectors from the supporting space $[1, 1]$ having integral lengths. Hence, from the theory of maximal lattices (Theorem 91:1) $\langle 1, 1\rangle$ is $\mathbb{Z}_2$-maximal. It follows then $S_{(2)}$ is represented by (in fact, is contained in) $L_{(2)}$. A similar calculation will show that $\langle 1, 1, 1\rangle$ is $\mathbb{Z}_2$-maximal and so contains $S_{(2)}$. Thus, we may further suppose that we are in the set-up: $4 \leqslant n \leqslant 8$.

4) Let $\mathfrak{H}$ denote a hyperbolic lattice (i.e. some copies of the binary lattice $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$). From local 2-adic unimodular theory, it is clear that $L_{(2)}$ looks like: $\langle 1, 1, 1, 1\rangle$, $\mathfrak{H} \perp \langle -1, -1, -1\rangle$, $\mathfrak{H} \perp \langle -1, -1\rangle$, $\mathfrak{H} \perp \langle -1\rangle$, $\mathfrak{H} \perp \langle 1, -1\rangle$ for n from 4 to 8 respectively. When $4 \leqslant n \leqslant 7$, $L_{(2)}$ contains a $2\mathbb{Z}_2$-maximal lattice, so that if the norm $\mathfrak{n}(S_{(2)})$ is contained in $2\mathbb{Z}_2$, we shall again have $S_{(2)}$ represented by $L_{(2)}$. Thus, we may suppose that the norm of $S_{(2)}$ is $\mathbb{Z}_2$. The hypothesis on $n = 8$ also gives this condition on the norm of $S_{(2)}$. Therefore, in all cases we may decompose $S_{(2)} = X_1 \perp Y$ where X_1 is unimodular and the scale $\mathfrak{s}(Y)$ of Y lies in $2\mathbb{Z}_2$. Moreover, $\mathfrak{n}(X_1) = \mathbb{Z}_2$. If $1 \in Q(S_{(2)})$, we would be done since we shall have relegated (by considering the orthogonal complement of $\langle 1\rangle$) the problem to a lower dimensional set-up.

5) If rank $(X_1) \geqslant 4$, then $Q(X_1) = \mathbb{Z}_2$ and so, $S_{(2)}$ represents 1. If rank $(X_1) = 3$, $1 \in Q(X_1)$ if and only if 1 is not represented by the space W_1 supporting X_1. This means $W_1 \perp [-1]$ is anisotropic, and so isometric to $[-1, -1, -1, -1]$. By Witt cancellation, $W_1 \cong [-1, -1, -1]$. By unimodular theory, $X_1 \cong \langle -1, -1, -1\rangle$. Rewrite $L_{(2)}$ in the dimensions 4 to 8 respectively as: $X_1 \perp \langle -1\rangle$, $X_1 \perp \mathfrak{H}$, $X_1 \perp \mathfrak{H} \perp \langle 1\rangle$, $X_1 \perp \mathfrak{H} \perp \langle 1, 1\rangle$, $X_1 \perp \mathfrak{H} \perp \langle 1, 1, 1\rangle$. In each of these cases, the orthogonal complement of X_1 in $L_{(2)}$ contains a a $2\mathbb{Z}_2$-maximal lattice and so will represent Y. Thus, $L_{(2)}$ represents $S_{(2)}$. So, in the remainder of our proof we may further suppose that rank $(X_1) \leqslant 2$.

6) Let $X_1 \cong \langle a, b\rangle$, a, b 2-adic units. First consider when a/b is a square Since $\langle 5, 5\rangle \cong \langle 1, 1\rangle$, so $a = b \neq 1, 5$. As $\langle 3, 3\rangle \cong \langle -1, -1\rangle$, we may take $a = b = -1$. Rewrite $L_{(2)}$ in increasing dimensions as: $X_1 \perp X_1$, $X_1 \perp \mathfrak{H} \perp \langle -1\rangle$, $X_1 \perp \mathfrak{H}$, $X_1 \perp \mathfrak{H} \perp \langle 1\rangle$, $X_1 \perp \mathfrak{H} \perp \langle 1, 1\rangle$. Once again, Y is represented by the orthogonal complement of X_1 in $L_{(2)}$, and be done. Thus the only possibilities for X_1 are: $\langle 3, 5\rangle$, $\langle 3, 7\rangle$, and $\langle 5, 7\rangle$. But, $\langle 3, 5\rangle \cong \langle 1, -1\rangle$ and $\langle 5, 7\rangle \cong \langle 1, 3\rangle$. So, we need only to treat $X_1 \cong \langle 3, 7\rangle \cong \langle -1, 3\rangle$.

When $n = 4$, write $L_{(2)} = \langle -1\rangle \perp \langle -1, -1, -1\rangle$, $S_{(2)} = \langle -1\rangle \perp T$.

Now, T is contained in an $\mathbb{Z}_2$-maximal lattice on $[-1, -1, -1]$ and so is representable by $\langle -1, -1, -1\rangle$. For $n = 5$, put $L_{(2)} \cong X_1 \perp \mathfrak{H} \perp \langle 3\rangle$. As Y is contained in an $2\mathbb{Z}_2$-maximal lattice, we are done. For $n = 6$, $L_{(2)} \cong X_1 \perp H \perp \langle 1, 3\rangle$. Here, $\langle 1, 3\rangle$ although not $\mathbb{Z}_2$-maximal, contains a $2\mathbb{Z}_2$-maximal lattice. For $n = 7$, set $L_{(2)} \cong X_1 \perp H \perp \langle 5\rangle$. For $n = 8$, $L_{(2)} \cong X_1 \perp H \perp \langle 1, 5\rangle$. Here $\langle 1, 5\rangle$ in fact is even $\mathbb{Z}_2$-maximal. Therefore, we are finished with the case where rank $(X_1) = 2$.

7) Finally, let $X_1 \cong \langle a\rangle$. Rewrite $L_{(2)}$ in increasing dimensions as: $X_1 \perp \langle a, a, a\rangle$, $X_1 \perp \mathfrak{H} \perp \langle -1, 3\rangle$, $X_1 \perp \mathfrak{H} \perp \langle 3, 3, 3\rangle$, $X_1 \perp \mathfrak{H} \perp \langle -1, 5\rangle$, $X_1 \perp \mathfrak{H} \perp \langle -a\rangle$. In each instance the orthogonal complement of X_1 in $L_{(2)}$ contains a $2\mathbb{Z}_2$-maximal lattice. This finishes the proof of the theorem.

COMPLETION PROBLEM

To recall, the setting here is: $S = mI_n$ is rationally supported by the space $V \cong [1, \ldots, 1]$. A sublattice K of the standard lattice L having rank r is given and $K \cong r \times \langle m\rangle$. The task is to find a *full* sublattice of the orthogonal complement $K^{\perp}$ of K inside L that is isometric to $(n-r) \times \langle m\rangle$. The main result is the following theorem:

THEOREM 2 *The completion problem is always solvable provided $n-r$ is less or equal to 7. This threshold number 7 is the best possible.*

Proof By Corollary 1, the lattice S is representable by L so that we may view S as a sublattice of L. Clearly, S represents K and so we can view K as a sublattice of S. Let J, $K^{\perp}$ denote respectively the orthogonal complement of K inside S and L. So, $J = K^{\perp} \cap S$. We use the notations: V^c, L^c, c a non-zero scalar, for the scaled space and lattice; similarly, for local spaces and lattices. By abuse of notation, we shall denote by $\langle c, \ldots, c\rangle$ where c is a rational (non-zero) number to be either a global lattice or a localized lattice, and similarly for spaces. There should not be any confusion as the context will make them clear.

Consider the localizations at a prime spot p of the following scaled objects $S_{(p)}^{m^{-1}} \cong \langle 1, \ldots, 1\rangle$, $L_{(p)}^{m^{-1}} \cong \langle m^{-1}, \ldots, m^{-1}\rangle \cong \langle 1, \ldots, 1\rangle^{m^{-1}}$, and $K_{(p)}^{m^{-1}} \cong r \times \langle 1\rangle$. Denote for the remainder of the proof G for the localized lattice $J_{(p)}^{m^{-1}}$, and W the supporting space for $K^{\perp}$, so that $W_{(p)}^{m^{-1}} = G \otimes Q_{(p)}$. Since $K_{(p)}^{m^{-1}}$ splits $S_{(p)}^{m^{-1}}$, comparing the discriminants will give det $(G) = 1$. Witt's theorem gives $W_{(p)}^{m^{-1}} = (n-r) \times [1]$. At every odd prime it is clear that $G \cong (n-r) \times \langle 1\rangle$. Thus, we concentrate at $p = 2$ only.

When $r = n-1$, clearly $G \cong \langle 1\rangle$ and so $G^m = J_{(p)} = \langle m\rangle$. The genus of a diagonal lattice $t \times \langle m\rangle$ has but one class when $t \leqslant 7$, so that if we show $J_{(p)} \cong t \times \langle m\rangle$ then $J \cong t \times \langle m\rangle$ and we would be finished. Now, for

$r = n-2$, the binary space [1, 1] cannot support an improper unimodular (2-adic) lattice, and so G must be isometric to $\langle 1, 1\rangle$. For $r = n-3$, $n-5$, and $n-7$, clearly $G \otimes \mathbb{Q}_{(2)}$ cannot support improper unimodular lattices either, and since two unimodular (2-adic) lattices on isometric spaces are isometric if and only if their norms are equal, G must be of the type $\langle 1, \ldots, 1\rangle$. For $r = n-4$, the space [1, 1, 1, 1] is anisotropic so that it cannot support an improper unimodular lattice. For the last case $r = n-6$, a discriminant argument quickly shows $G = 6\times\langle 1\rangle$. This completes the proof of the theorem except to furnish a counter-example for $r = n-8$.

Let $m = n = 9$ and $r = 1$. Set the first row of a 9×9 matrix as $(1\,1\,1\,1\,1\,1\,1\,1\,1) = A$. We claim this submatrix can not be completed to an integral B satisfying $BB^t = 9I_9$. There are several ways to prove this claim. We do it in the continuing spirit of quadratic forms. If $\{e_i\}$ is the standard orthonormal basis for the lattice L, then the given first row corresponds to the vector $u = e_1+e_2+\ldots+e_9$. The orthogonal complement $K^\perp$ of $K = \mathbb{Z}u$ is generated by the set $\{e_1-e_9, e_2-e_9, \ldots, e_8-e_9\}$ and this gives rise to the matrix M which has 2 in every entry along the diagonal and 1 elsewhere. The determinant $\det(M) = 9$ and so is a 2-adic unit. This implies then the 2-adic localization of $K^\perp$ is an even (i.e. improper) unimodular lattice resting on the space $(\mathbb{Q}u)^\perp$ which is a sum of 8 squares. So, clearly $K^\perp$ cannot represent $8\times\langle 9\rangle$ since 2-adically this is not possible.

CONCLUDING REMARKS

With essentially the same sort of proof as that given in the proof of Theorem 1, it is possible to generalize Theorem 1 to the following (removing the rank condition on S):

THEOREM 1′ *S is represented by the genus of L (i.e. $L_{(p)}$ represents $S_{(p)}$ at every p) but for the single exception when:* rank $(S) \equiv 0$ (mod 8) *and 2-adically $S_{(2)}$ contains an even unimodular component of corank $\leqslant 2$.*

For the Completion Problem the technique employed here can be used to treat more general matrices for S instead of the restricted cases where $S = mI_n$. Also, the point of view gained from quadratic forms clearly shows how to generalize both matrix questions to integral matrices where the integers come from an algebraic number field, indeed, a global field—although in the latter case the quadratic forms theory to be applied is generally then the *indefinite* theory, which from the present standpoint is more powerful in that more complete results are possible.

Note added in proof

There seem to have been several related activities on the questions treated here at about the same general time period. Let me record here the informations which for the most part have come to my attention after this paper has been accepted. Concerning the Existence Question, I was first brought to its attention—at least for 3×3 matrices—at around April 1976 by my colleague, Dan Shapiro, who in turn was asked about it by Tony Geramita. Theorem 1 was discovered soon afterward. Also around this time another colleague, Rick Wilson, mentioned to me about Hall's Completion Question. Then at the Quadratic Forms Conference in Kingston in August I learned that Gordon Pall has independently from me and at about the same time obtained a proof for Theorem 1. His proof was, however, along the classical styles of quadratic forms and based on the notion of *c*-irreducibility. At the conference Pall referred to this existence question as the "Geramita Problem". Shortly after the conference I found a proof of Hall's Completion Question as well (Theorem 2 here). In the course of writing both results for publication I discovered a strengthened version of Theorem 1 which is Theorem 1′ here. The latter's proof is along the same line except the technical details are considerably more complicated. Early in January 1977 Olga Taussky-Todd wrote and informed me that a student of Marshall Hall, E. Verheiden, has also solved Hall's problem. His proof will appear in *Journal of Combinatorial Theory* (Ser. A) in a paper entitled: Integral and Rational Completions of Combinatorial Matrices. In mid-February 1977, responding to a letter from Geramita earlier that month, I informed him that my Theorem 1′ may be applied to strengthen Theorem 1, removing the assumption of an odd entry in the 8×8 case, and also with some perseverance to bigger matrices. For the 8×8 case the essential point is that, keeping the same notations, 2-adically $S_{(2)}$ must have one of the following two exceptional shapes (i) and (ii) below in order for $S_{(2)}$ to be *not* represented by $L_{(2)}$:

i) $S_{(2)} \cong \begin{pmatrix}0&1\\1&0\end{pmatrix} \perp \begin{pmatrix}0&1\\1&0\end{pmatrix} \perp \begin{pmatrix}2&1\\1&2\end{pmatrix} \perp X$

ii) $S_{(2)} \cong 3\times\begin{pmatrix}0&1\\1&0\end{pmatrix} \perp \begin{pmatrix}2&0\\0&-2\end{pmatrix}$

where X is not 2-adically unimodular. From this it follows that since S is *even diagonal* $S_{(2)}$ will avoid these exceptional cases. Hence, L represents such an S since the class number of I_8 still is one.

On the other hand, a letter from Jennifer Seberry in late March informed me that the Existence Question had also entered in Peter Eades' investigations with problems in orthogonal designs. In his paper, Orthogonal designs constructed from circulants, which is to appear in *Utilitas Math.*, Eades was led to conjecture that our Theorem 1 may hold at least for 4×4 diagonal matrices. Eades and Pall are writing a joint article on Integral Quadratic Forms and Orthogonal Designs, incorporating—among other things—a proof of Theorem 1 along the concept of c-irreducibility cited above.

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A Note on Matrix Equivalence

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Let R be a principal ideal ring, R_n the ring of $n \times n$ matrices over R. It is shown that if A, B, X, Y are elements of R_n such that $A = XB$, $B = YA$, then A and B are left equivalent. Some consequences are given.

The purpose of this note is to fill a small gap in the literature on integral matrices. Let R be a principal ideal ring, R_n the ring of $n \times n$ matrices over R, and R_n' the multiplicative subgroup of R_n consisting of the unit matrices of R_n. It is known that any left ideal $\mathfrak{A}$ of R_n is principal (see [1, pp. 35–36] or [2, pp. 21–22] for a proof). If A and B are generators of $\mathfrak{A}$, and if one of them is nonsingular, then it is readily shown that A and B are left equivalent; that is, $B = UA$ for some element U of R_n'. However, this question has apparently not been treated when the nonsingularity condition is removed. Nevertheless the result remains true and we shall prove the following:

THEOREM 1 *Let $\mathfrak{A}$ be a left ideal of R_n, and suppose that A, B are generators of $\mathfrak{A}$. Then A and B are left equivalent.*

To prove this theorem, it is sufficient to prove

THEOREM 2 *Let A, B be elements of R_n. Suppose that elements X, Y of R_n exist such that*

$$A = XB, \qquad B = YA. \tag{1}$$

Then A and B are left equivalent.

Proof We may assume that neither A nor B is 0. Since A is a multiple of B and B is a multiple of A, A and B have the same determinantal divisors (see [2, pp. 25–26]). Hence A and B are equivalent, so that matrices U, V of R_n' exist such that

$$B = UAV.$$

Let $S(A)$ be the Smith Normal Form of A, so that

$$A = PS(A)Q, P, Q \in R_n'.$$

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