$$p_i \frac{\partial q_i}{\partial q_j^i} - (1 + \lambda_i)p_j = 0, \qquad (i\cdot 13)\ (j\cdot 12), \tag{13}$$

with the hypothesis of strict competition. Here the λ_i are not necessarily elements of cost, but the equilibrium conditions (12) are the same as if they were such elements, appearing as rates of interest. We should then define the profits by the formulas

$$\pi_i = I_i - \lambda_i K_i,$$

and as a first approximation we might assume the λ_i all equal, $\lambda_i = \lambda$.

The same formal result may be obtained by making extremal (at any time, $t = t_0$) the quantity

$$I = \sum_{i\cdot 13} p_i q_i - \sum_{j\cdot 12} p_j q_j$$

subject to the restraints

$$\sum_{j\cdot 12} p_j q_j = K = K(t_0),$$

with all the dp_i set equal to zero; in fact, there appears in this way a single λ. Between I and the value $\Pi = \sum_{i\cdot 13}\pi_i$ there holds the relation $I = \Pi + \lambda K$.

In terms of the indices of prices and quantities, the equation (9) becomes

$$\frac{V_u}{Q_u}\,\frac{\partial Q_u}{\partial Q_w^u} = (1+\lambda)\frac{V_w}{Q_w},$$

so that for the three fundamental categories, with the various V_u, V_w eliminated, we have the equations

$$\frac{\partial Q_1}{\partial Q_1^1} = 1 + \lambda,$$

$$\lambda = \frac{\dfrac{\partial Q_3}{\partial Q_1^3}\,\dfrac{\partial Q_1}{\partial Q_2^1} - \dfrac{\partial Q_3}{\partial Q_2^3}}{\dfrac{\partial Q_3}{\partial Q_2^3}}.$$

The second of these equations states that $1+\lambda$ is equal to the ratio of the marginal productivity of indirect factors of production (that is, productive of consumption goods indirectly through the production of capital goods) to the marginal productivity of direct factors of production. The equation is essentially the formula for the rate of interest given by Wicksell.* It was obtained for the simplified system already mentioned, by Lange.† The remarks in this section are in fact a generalization or justification of this latter theory.

The index relations are particularly interesting in discussing changes from one system to another consequent on the introduction or change of interest rate λ. Thus, with the index of primary factors given, that is, Q_2 given, the introduction of a small interest rate λ induces no modification of Q_3 as far as differentials of the first order. In fact,

$$\delta Q_3 = 0,$$

and

$$\delta^2 Q_3 = \lambda \frac{\partial Q_3}{\partial Q_1^3}\,\delta Q_1.$$

Equations such as these are important for economic theory.

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NOTE ON ALMOST-UNIVERSAL FORMS‡

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Ramanujan§ and Dickson‖ proved that there are 54 universal forms $ax^2+by^2+cz^2+dt^2$ with positive integral coefficients a, b, c, d. It is the purpose of this note to investigate *almost-universal forms*, that is, to exhibit sets of positive integral coefficients a, b, c, d such that $ax^2+by^2+cz^2+dt^2$ represents every positive integer with exactly one exception.

Ramanujan§ showed that a necessary and sufficient condition that a form $ax^2+by^2+cz^2+dt^2$ be universal is that it represent the first fifteen positive integers. Consequently the integer which an almost-universal form fails to represent cannot be greater than 15. Using Ramanujan's method of bounding the coefficients we can exhibit, merely by requiring that a form fail to represent exactly one of the

* Wicksell, *Lectures on Political Economy*, London, 1935 (translation), vol. 1, p. 156.

† Lange, loc. cit.

‡ Presented to the Society, December 28, 1937.

§ Proceedings of the Cambridge Philosophical Society, vol. 19 (1917), pp. 11–21; *Collected Papers*, Cambridge, 1927, pp. 169–178.

‖ This Bulletin, vol. 33 (1927), pp. 63–70.

first fifteen positive integers, a set of 135 forms which has to contain all almost-universal forms. The well known theories of special ternary quadratic forms,* or even empirical verification, will reduce this number to 88. (Empirical verification would sometimes be cumbersome; for example, the first integer, other than 10, that $x^2+2y^2+5z^2+15t^2$ fails to represent is 250.)

The following list exhibits the 88 possibilities for almost-universal forms (where (a, b, c, d) denotes the form $ax^2+by^2+cz^2+dt^2$):

Forms that do not represent 1:
(1)–(3) (2, 2, 3, 4), (2, 3, 4, 5), (2, 3, 4, 8).
Forms that do not represent 2:
(4)–(5) (1, 3, 3, 5), (1, 3, 5, 6).
Forms that do not represent 3:
(6)–(7) $(1, 1, 4, d)$, $d=5$, 6;
(8)–(11) $(1, 1, 5, d)$, $d=5$, 6, 10, 11;
(12)–(15) $(1, 1, 6, d)$, $d=7$, 8, 10, 11.
Forms that do not represent 5:
(16)–(20) $(1, 2, 6, d)$, $d=6$, 10, 11, 12, 13;
(21)–(25) $(1, 2, 7, d)$, $d=8$, 10, 11, 12, 13.
Forms that do not represent 6:
(26)–(32) $(1, 1, 3, d)$, $d=7$, 8, 10, 11, 13, 14, 15.
Forms that do not represent 7:
(33)–(37) $(1, 1, 1, d)$, $d=9$, 10, 12, 14, 15;
(38)–(42) $(1, 2, 2, d)$, $d=9$, 10, 12, 14, 15.
Forms that do not represent 10:
(43)–(55) $(1, 2, 3, d)$, $d=11$, 12, 13, 15, 17, 19, 20, 21, 22, 23, 24, 25, 26;
(56)–(59) $(1, 2, 5, d)$, $d=11$, 12, 13, 14.
Forms that do not represent 14:
(60)–(73) $(1, 1, 2, d)$, $d=15$, 17, 18, 19, 20, 21, 22, 23, 24, 25, 27, 28, 29, 30;
(74)–(87) $(1, 2, 4, d)$, $d=15$, 17, 18, 19, 20, 21, 22, 23, 24, 25, 27, 28, 29, 30.
Forms that do not represent 15:
(88) (1, 2, 5, 5).

One general method of proof serves to establish almost-universality for most of these forms. By way of illustrating this method, we prove the following typical theorem:

* The properties of every ternary form needed in this note are either discussed by Dickson, loc. cit., or else they follow from Dickson's discussion by elementary means.

THEOREM 1. *The form $2x^2+2y^2+3z^2+4t^2$ represents every positive integer with the exception of unity.*

Since (1, 1, 2) represents all odd numbers* (where (a, b, c) denotes the ternary quadratic form $ax^2+by^2+cz^2$), (2, 2, 3, 4) represents all numbers of the form $4k+2$ with $z=0$. For $k>0$ we have $4k+1=4(k-1)+2+3$, hence (2, 2, 3, 4) represents $4k+1$ with $z=1$. Since (1, 1, 6, 2) is a universal form,† for every $k\geqq 0$ we have $2k=x^2+y^2+6z^2+2t^2$, whence $4k=2x^2+2y^2+3(2z)^2+4t^2$. Finally, it may be proved, by consideration of elementary divisibility properties, that two numbers not represented by (1, 1, 2) never differ by 12. Hence, for $k>5$ we have either $2k=x^2+y^2+2t^2$ or else $2k-12=x^2+y^2+2t^2$. According as the first or the second relation holds we have $4k+3=2x^2+2y^2+3\cdot 1^2+4t^2$ or else $4k+3=2x^2+2y^2+3\cdot 3^2+4t^2$. Since it is readily verified that (2, 2, 3, 4) represents the numbers $4k+3$, $k=0, 1, 2, 3, 4, 5$, the proof of the theorem is complete.

The above treatment is not applicable to the form (2): (2, 3, 4, 5). We prove the following theorem:

THEOREM 2. *The form $2x^2+3y^2+4z^2+5t^2$ represents every positive integer with the exception of unity.*

Let A, B, C be three numbers of the form $4^\alpha(16k+10)$, where α and k are non-negative integers. Concerning these we have the following lemma:

LEMMA. *It is impossible that the two equations $A-B=40$, $A-C=120$ hold simultaneously; also, the equation $A-B=20$ is impossible.*

The proof of the lemma is elementary and is omitted.

Empirical verification yields the result that the form (2, 3, 4, 5) represents all integers n where $2\leqq n\leqq 200$.

Consider now the ternary form (1, 2, 6). It represents every positive integer not of the form $4^\alpha(8k+5)$. But if $n=x^2+6y^2+2z^2$, then $2n=2x^2+3(2y)^2+4z^2$, whence the form $2x^2+3y^2+4z^2+5t^2$ represents all even numbers not of the form $4^\alpha(16k+10)$, with $t=0$. But if A is such a number, then, by the lemma, $A-20$ is not; whence $A-20=2x^2+3y^2+4z^2$, and $A=2x^2+3y^2+4z^2+5\cdot 2^2$. Hence the form (2, 3, 4, 5) represents all even numbers. It also represents all odd numbers, excepting unity and those of the form $A+5$, with $t=1$. By the lemma, one of the even numbers $A-40$ or $A-120$ is repre-

* This Bulletin, vol. 33 (1927), pp. 63–70.
† Ramanujan, loc. cit.

sented by (2, 3, 4); whence $A+5$ is represented by (2, 3, 4, 5) with $t=3$ or $t=5$ respectively. This completes the proof of the theorem.

One of the ternary forms, namely (2, 2, 4), involved in (2, 2, 3, 4) is regular; that is, the total set of numbers which it fails to represent coincides with the total set of numbers in a certain collection of arithmetic progressions. It is this property that makes the difference between (2, 2, 3, 4) and (2, 3, 4, 5). Every ternary form involved in (2, 3, 4, 5) is irregular.* (A form f is said to be irregular if there exists a positive integer k not represented by f, but having the property that every arithmetic progression containing k contains also numbers represented by f.) We were able to prove Theorem 1, however, because two of the coefficients of the form (2, 3, 4) are not relatively prime. If the form represents a multiple of the common divisor, it becomes a multiple of the regular form (1, 2, 6).

Either the method of Theorem 1 or else that of Theorem 2 proves the almost-universality of 86 of the 88 forms. The author has not hitherto found out whether or not the two forms (23): (1, 2, 7, 11) and (25): (1, 2, 7, 13) are almost-universal in the sense of this note. Every ternary form involved in either of the two quarternary forms is irregular, and no reduction of the sort described above is possible. Each of these forms fails to represent only one positive integer $n \leqq 300$.

Professor Carmichael has recently communicated to me the following result (for the proof of which he had to employ the Dirichlet method of dealing with ternary forms): *The form* (1, 2, 11) *represents every even number not of the form* $4(16n-10)$. With the aid of this result, one may prove by the methods exhibited above that the form (1, 2, 7, 11) is almost-universal.†

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* L. E. Dickson, Annals of Mathematics, (2), vol. 28 (1927), pp. 333–341. The results of this paper are not applicable to any of the ternary forms involved in (2, 3, 4, 5), but the methods are sufficiently general to prove the assertion above.

† The last paragraph was added in proof, January 17, 1938.

METRIC SPACES WITH GEODESIC RICCI CURVES. I

JACK LEVINE

1. Introduction. The problem of determining all Riemannian spaces of three dimensions admitting geodesic Ricci curves has been solved by G. Ricci* and P. Walberer† using, however, different methods. Although they obtained all such V_3, the complete explicit determination of all such V_n for $n>3$ does not seem possible because of the increased number and complexity of the differential equations which arise.

In this paper the following two problems related to the above problem will be considered.

In the first problem we suppose given a set of linearly independent vectors‡ $\lambda^i_{a|}$ and wish to determine necessary and sufficient conditions on the $\lambda^i_{a|}$ in order that a set of scalars $\theta_a(\neq 0)$ exist which will define a metric space V_n with a metric determined by

$$g^{ij} = \sum_h e_h \bar{\lambda}^i_{h|} \bar{\lambda}^j_{h|}, \tag{1}$$

where

$$\bar{\lambda}^i_{a|} = \theta_a \lambda^i_{a|}, \tag{2}$$

and e_h $(=\pm 1)$ are arbitrary; and such that the congruences of curves defined by the $\lambda^i_{a|}$ will be geodesics in the V_n thus determined. (The vectors $\bar{\lambda}^i_{a|}$ define the same congruences as do the $\lambda^i_{a|}$, and these congruences form an orthogonal ennuple in the V_n.)

In the second problem we assume that these conditions on the $\lambda^i_{a|}$ have been determined and that the n congruences defined by a set of $\lambda^i_{a|}$ are geodesics in the V_n determined by

$$g^{ij} = \sum_h e_h \lambda^i_{h|} \lambda^j_{h|};$$

we then find necessary and sufficient conditions that, with respect to the metric (1), the congruences be geodesic Ricci curves.

* G. Ricci, *Sulle varieta a tre dimensioni dotate die terne principali di congruenze geodetiche*, Rendiconti della Reale Accademia dei Lincei, (5), vol. 27 (1918), pp. 21–28, 75–87.

† P. Walberer, *Riemannsche Raume mit geodätischen Riccikurven*, Hamburger Abhandlungen, vol. 10 (1934), pp. 152–168.

‡ All indices take the values 1, 2, · · · , n unless otherwise noted.