PRIMITIVE REPRESENTATIONS BY SPINOR GENERA OF TERNARY QUADRATIC FORMS

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Dedicated to the memory of N. C. Ankeny

ABSTRACT

Let a be primitively represented by the genus of a ternary quadratic lattice L defined over the ring of integers of an algebraic number field F. Criteria to determine whether a is primitively represented by every spinor genus in the genus of L involve certain subgroups $\theta^*(L_p, a)$ of the multiplicative groups of the localizations F_p of F with respect to the various nonarchimedean prime spots p on F. In this paper these groups $\theta^*(L_p, a)$ are determined explicitly for nondyadic and 2-adic prime spots. Examples are given which show how this information can, in some instances, be used in combination with known results, to determine all integers primitively represented by a particular positive definite ternary quadratic form.

1. *Background and statement of results*

Throughout this paper, unexplained notation and terminology will follow that of O'Meara's book **[8]**. Let F be an algebraic number field with ring of integers R, and let S be the set of all nonarchimedean prime spots on F. Suppose that $a \in F$ is represented by the genus of an R-lattice L on a regular quadratic space (V, Q) of dimension 3 over F. Fix $d \in \dot{F}$ such that the discriminant of V is $d\dot{F}^2$, let $E = F(\sqrt{(-ad)})$ and, for $p \in S$, let $N_p(E)$ denote the group of local norms from $E_{\mathfrak{P}}$ to F_p, where $\mathfrak{P}$ is an extension of p to E. In the fundamental paper **[7]**, Kneser proved that a is represented by every spinor genus in the genus of L unless the following conditions hold:

$$a \neq 0, \quad -ad \notin F^2, \quad \theta(O^+(L_p)) \subseteq N_p(E) \text{ for all } p \in S, \tag{1}$$

where θ denotes the spinor norm mapping on the orthogonal group $O(V_p)$. Moreover, he showed that when a is not represented by every spinor genus in the genus, it is represented by exactly half of these spinor genera. If this is the case, we shall say that a is spinor exceptional for the genus of L.

Schulze-Pillot **[9]** showed that one additional set of local conditions can be added to (1) to obtain necessary and sufficient conditions for a to be spinor exceptional for the genus of L; namely, that $\theta(L_p, a) = N_p(E)$ for all $p \in S$. Here the group $\theta(L_p, a)$ is defined in the following way. Let v be a vector of L_p with $Q(v) = a$. Then $\theta(L_p, a)$ is the subgroup of $\dot{F}_p$ generated by $\{c \in \dot{F}_p \mid \text{there exists } \sigma \in O^+(V_p) \text{ with } c \in \theta(\sigma) \text{ and } \sigma(v) \in L_p\}$. That this definition is independent of the choice of the vector v follows from Witt's Theorem. These groups $\theta(L_p, a)$ were determined by Schulze-Pillot for nondyadic and 2-adic prime spots p, thus effectively solving the problem of determining the spinor exceptional integers for a genus of ternary quadratic $\mathbb{Z}$-lattices. In a forthcoming paper **[4]** Hsia, Shao and Xu have completed the

Received 5 November 1992.

1991 *Mathematics Subject Classification* 11E.

The first author was partially supported by the National Security Agency and the second was partially supported by the National Science Foundation.

determination of spinor exceptional representations of $\mathbb{Z}$-lattices K by L, where K may have any codimension in L and the rank of L is arbitrary (but at least 3).

For the remainder of the paper we shall consider the analogous problem for primitive representations. A vector v in V_p (V, respectively) is said to be a primitive vector of L_p (L, respectively) if $v \in L_p$ and $\pi^{-1}v \notin L_p$, where π is a prime element of R_p (v is a primitive vector of L_p for all $p \in S$, respectively). The element $a \in R$ is then said to be primitively represented by the (spinor) genus of L if there is a lattice K in the (spinor) genus of L and a primitive vector v of K for which $Q(v) = a$. Thus, a is primitively represented by the genus of L if and only if for each $p \in S$ there exists a primitive vector v_p of L_p for which $Q(v_p) = a$. We note in passing that the local problem of determining primitive representability by L_p is quite formidable for arbitrary lattices and prime spots (for example, see recent work of James [5] and the references given there).

From now on we shall assume that a is primitively represented by the genus of L. It is again true that a is primitively represented either by all or by exactly half of the spinor genera in the genus; in the latter case, we shall say that a is primitively spinor exceptional for the genus. From [9], necessary and sufficient conditions for a to be primitively spinor exceptional are that (1) holds and

$$\theta^*(L_p, a) = N_p(E) \quad \text{for all } p \in S. \tag{2}$$

To define the group $\theta^*(L_p, a)$, first let $P(L_p, a)$ denote the (non-empty) set consisting of all primitive vectors v of L_p for which $Q(v) = a$. For a fixed $v \in P(L_p, a)$, let $\theta^*(L_p, a)$ be the subgroup of $\dot{F}_p$ generated by the set $\{c \in \dot{F}_p \mid \text{there exists } \sigma \in O^+(V_p) \text{ with } c \in \theta(\sigma) \text{ and } \sigma(v) \in P(L_p, a)\}$. As for $\theta(L_p, a)$, this definition is independent of the choice of v. Note that the space V_p splits as $F_p v \perp W_p$, where the subspace $W_p = (F_p v)^\perp$ has dimension 2 and discriminant ad. As any element of $O^+(W_p)$ can be extended to an element of $O^+(V_p)$ which fixes v, we have $\theta(O^+(W_p)) \subseteq \theta^*(L_p, a)$. Upon scaling the space W_p so that it represents 1, it is easily seen that $\theta(O^+(W_p)) = N_p(E)$. Moreover, it is clear from the definitions that $\theta^*(L_p, a) \subseteq \theta(L_p, a)$. Hence we have the following basic containments:

$$N_p(E) \subseteq \theta^*(L_p, a) \subseteq \theta(L_p, a) \subseteq \dot{F}_p. \tag{3}$$

From this it can be seen that (2) holds whenever $-ad \in \dot{F}_p^2$, since then $N_p(E) = \dot{F}_p$. Moreover, for (2) to fail it is clearly necessary that $N_p(E) \neq \theta(L_p, a)$. All conditions under which this strict containment occurs are enumerated in [9, Satz 3 and Satz 4] for nondyadic and 2-adic prime spots, respectively. We now proceed to state the corresponding results for strict containment of $N_p(E)$ in $\theta^*(L_p, a)$.

THEOREM 1. *Let p be nondyadic and $-ad \notin \dot{F}_p^2$.*

(a) *Suppose that $E_{\mathfrak{P}}/F_p$ is unramified. Then $\theta(O^+(L_p)) \subseteq N_p(E)$ if and only if $L_p \cong \langle b_1, \pi^{2r}b_2, \pi^{2s}b_3 \rangle$ with $b_i \in \mathcal{U}_p$ for $0 \leqslant r \leqslant s$. In this case $\theta^*(L_p, a) \neq N_p(E)$ if and only if $r \neq s$, $-b_1 b_2 \in \dot{F}_p^2$, and $a \in p^{2r+1}$.*

(b) *Suppose that $E_{\mathfrak{P}}/F_p$ is ramified. If $\theta(O^+(L_p)) \subseteq N_p(E)$, then*

$$L_p \cong \langle b_1, \pi^r b_2, \pi^s b_3 \rangle$$

with $b_i \in \mathcal{U}_p$ for $0 < r < s$. In this case, $\theta^*(L_p, a) \neq N_p(E)$ if and only if one of the following holds:

(i) *r is even and $a \in p^r$,*

(ii) *r is odd and $a \in p^s$, except when $a \in \pi^s b_3 \mathcal{U}_p^2$ and $|\bar{F}_p| \leqslant 5$, in which case $\theta^*(L_p, a) = N_p(E)$.*

THEOREM 2. *Let p be 2-adic and $-ad \notin \dot{F}_p^2$.*

(a) *Suppose that $E_{\mathfrak{P}}/F_p$ is unramified. Then $\theta(\mathrm{O}^+(L_p)) \subseteq N_p(E)$ if and only if L_p is not unimodular and the Jordan components of L_p have either all even or all odd orders. In this case $\theta^*(L_p, a) \neq N_p(E)$ if and only if one of the following holds:*

(i) $L_p \cong \langle b_1, 2^{2r}b_2, 2^{2s}b_3\rangle$ *with* $b_i \in \mathcal{U}_p$, *for* $0 \leqslant r \leqslant s$, *and*

(α) $\delta(-b_1 b_2) = 2R_p$, *where δ denotes the quadratic defect, and*

(1) $r \neq s$, $a \in p^{2r}$,

(2) $r = s$, $F_p \neq \mathbb{Q}_2$, $a \in p^{2r}$,

(3) $r = s$, $F_p = \mathbb{Q}_2$, $a \in p^{2(r+1)}$;

(β) $\delta(-b_1 b_2) = 4R_p$ *and*

(1) $r = 0$, $a \in 2^{2s}\mathcal{U}_p$,

(2) $0 < r \neq s$, $a \in p^{2s}$,

(3) $0 < r = s$, $F_p \neq \mathbb{Q}_2$, $a \in p^{2s}$,

(4) $0 < r = s$, $F_p = \mathbb{Q}_2$, $a \in p^{2(s+1)}$;

(γ) $-b_1 b_2 \in \dot{F}_p^2$ *and*

(1) $r = 0$, $s = 1$, $a \in 4\mathcal{U}_p$,

(2) $r \neq 0$ *or* $s > 1$, $a \in p^{1+2r+2t}$, *where* $t := \min(1, s-1)$;

(ii) L_p *cannot be decomposed into an orthogonal sum of* 1*-dimensional lattices and*

(α) $L_p \cong A(0,0) \perp \langle 2^{2r+1}b\rangle$ *with* $b \in \mathcal{U}_p$, $r \geqslant 0$, *and*

(1) $r = 0$, $a \in 2\mathcal{U}_p$,

(2) $r \geqslant 1$, $a \in p^{2r+1}$;

(β) $L_p \cong A(2, 2\rho) \perp \langle 2^{2r+1}b\rangle$ *with* $b \in \mathcal{U}_p$, $r \geqslant 0$, *and*

(1) $r = 0$, $a \in 2\mathcal{U}_p$,

(2) $r \geqslant 1$, $a \in p^{2r+1}$.

(b) *Suppose that $E_{\mathfrak{P}}/F_p$ is ramified and $\mathrm{ord}_p(-ad)$ is even. Then $\theta(\mathrm{O}^+(L_p)) \subseteq N_p(E)$ implies that $L_p \cong \langle b_1, 2^r b_2, 2^s b_3\rangle$ with $b_i \in \mathcal{U}_p$ for $0 \leqslant r \leqslant s$. When $\theta(\mathrm{O}^+(L_p)) \subseteq N_p(E)$, let $K = \langle 2^{r-2}b_1, 2^r b_2, 2^s b_3\rangle$ and $K' = \langle 2^r b_1, 2^r b_2, 2^s b_3\rangle$. Then $\theta^*(L_p, a) \neq N_p(E)$ if and only if one of the following holds:*

(i) r *is even*, $\theta(\mathrm{O}^+(K)) \nsubseteq N_p(E)$ *and*

(α) $r \neq s$, $a \in p^{r-2}$,

(β) $r = s$, $F_p \neq \mathbb{Q}_2$, $a \in p^{r-2}$,

(γ) $r = s$, $F_p = \mathbb{Q}_2$, $a \in p^r$;

(ii) r *is even*, $\theta(\mathrm{O}^+(K)) \subseteq N_p(E)$, $\theta(\mathrm{O}^+(K')) \nsubseteq N_p(E)$, *and* $a \in p^r$;

(iii) r *is even*, $\theta(\mathrm{O}^+(K)) \subseteq N_p(E)$, $\theta(\mathrm{O}^+(K')) \subseteq N_p(E)$ *and*

(α) $F_p \neq \mathbb{Q}_2$, $a \in p^{s-2}$,

(β) $F_p = \mathbb{Q}_2$, $a \in p^s$;

(iv) r *is odd*, $a \in p^{r-3}$.

(c) *Suppose that $E_{\mathfrak{P}}/F_p$ is ramified and $\mathrm{ord}_p(-ad)$ is odd. Then $\theta(\mathrm{O}^+(L_p)) \subseteq N_p(E)$ implies that $L_p \cong \langle b_1, 2^r b_2, 2^s b_3\rangle$ with $b_i \in \mathcal{U}_p$ for $0 < r < s$. When $\theta(\mathrm{O}^+(L_p)) \subseteq N_p(E)$, let $K = \langle 2^{r-3}b_1, 2^r b_2, 2^s b_3\rangle$. Then $\theta^*(L_p, a) \neq N_p(E)$ if and only if one of the following holds:*

(i) r *is even*, $a \in p^{r-4}$,

(ii) r *is odd*, $\theta(\mathrm{O}^+(K)) \nsubseteq N_p(E)$, $a \in p^{r-3}$,

(iii) r *is odd*, $\theta(\mathrm{O}^+(K)) \subseteq N_p(E)$ *and*

(α) $F_p \neq \mathbb{Q}_2$, $a \in p^{s-4}$,

(β) $F_p = \mathbb{Q}_2$, $a \in p^{s-2}$.

2. *Proofs*

A complete proof of Theorem 1 will be presented in this section. However, due to the large number of individual cases that necessarily need to be considered in the proof of Theorem 2, we shall present here only the proof of one representative case which illustrates additional features which occur in the 2-adic case. In all the proofs we assume that the scale of the lattice L_p is R_p. This may be done without loss of generality since $\theta^*(L_p, a) = \theta^*(L_p^\lambda, \lambda a)$ holds for any $\lambda \in \dot{F}_p$, where L_p^λ denotes the lattice scaled by λ. Note that $\theta^*(L_p, a) = \theta(L_p, a)$ holds whenever $\operatorname{ord}_p a \leqslant 1$. Before proceeding to the proofs, it is convenient to fix some notation for the remainder of this section. Let $\{x, y, z\}$ denote a basis for L_p with respect to which L_p has the splitting indicated in the particular case under discussion. Then for a typical vector $v \in P(L_p, a)$ we always write the coefficients of v with respect to this fixed basis as α, β and γ; that is, $v = \alpha x + \beta y + \gamma z$.

LEMMA 1. *Let p be nondyadic, $-ad \notin \dot{F}_p^2$. Let $L_p \cong \langle b_1, \pi b_2, \pi^2 b_3\rangle$ for $b_i \in \mathscr{U}_p$. Assume that $a \in p^2$ and that $\theta(\mathrm{O}^+(L_p)) \subseteq N_p(E)$. Then $\theta^*(L_p, a) = N_p(E)$ if and only if $a \cong \pi^2 b_3$ and $|\bar{F}_p| \leqslant 5$.*

Proof. Note that the assumptions $\theta(\mathrm{O}^+(L_p)) \subseteq N_p(E) \neq \dot{F}_p$ force $b_1 \cong b_3$, since $\theta(\mathrm{O}^+(L_p)) = \dot{F}_p$ otherwise. Now if $a \not\cong \pi^2 b_3$, then $P(L_p, a) = P(\hat{L}, a)$, where $\hat{L} = R_p \pi x + R_p y + R_p z$. As $\theta(\mathrm{O}^+(\hat{L})) = \dot{F}_p$, we then have

$$\theta^*(L_p, a) = \theta^*(\hat{L}, a) = \dot{F}_p \neq N_p(E).$$

So we need only consider further the case where $a \cong \pi^2 b_3 \cong \pi^2 b_1$.

Suppose first that there exists $v \in P(L_p, a)$ such that $\operatorname{ord}_p \alpha = 1$. Since $\theta(\mathrm{O}^+(L_p))$ has index 2 in $\dot{F}_p$, to show that $\theta^*(L_p, a) = \dot{F}_p$ in this case, it suffices to show that $\theta^*(L_p, a) \neq \theta(\mathrm{O}^+(L_p))$. Consider $\sigma = \tau_x \tau_{\pi x - z} \in \mathrm{O}^+(V_p)$. Then

$$\theta(\sigma) = 2\dot{F}_p^2 \quad \text{and} \quad \sigma(v) = -\gamma\pi x + \beta y + \alpha\pi^{-1} z \in P(L_p, a)$$

since $\pi^{-1}\alpha \in \mathscr{U}_p$. If $2 \notin \dot{F}_p^2$, then $2 \in \theta^*(L_p, a) \backslash \theta(\mathrm{O}^+(L_p))$ and it follows that $\theta^*(L_p, a) \neq N_p(E)$. If $2 \in \dot{F}_p^2$, then consider $\hat{L} = R_p \pi x + R_p y + R_p z$. Since $\theta(\mathrm{O}^+(\hat{L})) = \dot{F}_p$, there exists $\mu \in \mathrm{O}^+(\hat{L})$ such that $\theta(\mu) \notin N_p(E)$. If $\mu(v)$ has z-coefficient in $\mathscr{U}_p$, then $\mu(v) \in P(L_p, a)$ and it again follows that $\theta^*(L_p, a) = \dot{F}_p$. Otherwise $\mu(v)$ has πx-coefficient in $\mathscr{U}_p$. But then $\sigma\mu(v)$ has z-coefficient in $\mathscr{U}_p$; thus, $\sigma\mu(v) \in P(L_p, a)$. So $\theta(\sigma\mu) = \theta(\mu) \in \theta^*(L_p, a)$, and we again have $\theta^*(L_p, a) = \dot{F}_p$.

On the other hand, suppose there is no $v \in P(L_p, a)$ with $\operatorname{ord}_p \alpha = 1$. Then $P(L_p, a) = P(L', a)$, where $L' = R_p \pi^2 x + R_p \pi y + R_p z$. Upon scaling by π^{-2}, we see that $P(L_p, a) = P(\tilde{L}, \pi^{-2} a)$ with $\tilde{L} \cong \langle \pi^2 b_1, \pi b_2, b_3\rangle$. Since $\pi^{-2} a \in \mathscr{U}_p$, we have $\theta^*(\tilde{L}, \pi^{-2} a) = \theta(\tilde{L}, \pi^{-2} a)$, which equals $N_p(E)$ by [**9**, Satz 3]. Hence, $\theta^*(L_p, a) = N_p(E)$ in this case.

Thus, $\theta^*(L_p, a) \neq N_p(E)$ if and only if there exists $v \in P(L_p, a)$ with $\operatorname{ord}_p \alpha = 1$. This latter condition is equivalent to the existence of A, B, C in $\mathscr{U}_p$ with $A^2 + B^2 \equiv C^2 \pmod{p}$, which is in turn equivalent to the solvability of the equation $X^2 - Y^2 = 1$ in nonzero elements X, Y in $\bar{F}_p$. Writing $Z = X - Y$ and $T = X + Y$, we see that there is such a solution in $\bar{F}_p$ if and only if there exist $Z, T \in \bar{F}_p$ such that $ZT = 1$ with $Z \neq \pm T$; that is, if and only if there exists $0 \neq Z \in \bar{F}_p$ with $Z^{-1} \neq \pm Z$. The existence of such a Z is equivalent to the existence of $Z \in \bar{F}_p$ such that $Z^2 \neq 0, \pm 1$, which occurs if and only if $|\bar{F}_p| > 5$.

LEMMA 2. *Let p be nondyadic, $-ad \notin \dot{F}_p^2$. Let $L_p \cong \langle b_1, b_2, \pi^{2s}b_3\rangle$ with $b_i \in \mathscr{U}_p$ and $s > 0$. Assume that $a \in p$ and $\theta(O^+(L_p)) \subseteq N_p(E)$. Then $\theta^*(L_p, a) = N_p(E)$ if and only if $-b_1 b_2 \notin \dot{F}_p^2$.*

Proof. Suppose first that $-b_1 b_2 \in \dot{F}_p^2$. Then there is a basis $\{x', y', z\}$ in which

$$L_p \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \langle \pi^{2s} b_3 \rangle.$$

Let $v = \pi x' + (2\pi)^{-1}(a - \pi^{2s}b_3)y' + z \in P(L_p, a)$ and define $\sigma \in O^+(V_p)$ by $\sigma(x') = \pi^{-1}x'$, $\sigma(y') = \pi y'$, $\sigma(z) = z$. Then $\theta(\sigma) = \pi \dot{F}_p^2$ and $\sigma(v) \in P(L_p, a)$. Since $\mathscr{U}_p = \theta(O^+(L_p)) \subseteq \theta^*(L_p, a)$, it follows that $\theta^*(L_p, a) = \dot{F}_p$.

Now suppose that $-b_1 b_2 \notin \dot{F}_p^2$. Since $\mathscr{U}_p = \theta(O^+(L_p)) \subseteq N_p(E)$, it follows that $E_{\mathfrak{P}}/F_p$ is unramified; in particular, $\text{ord}_p a = 2t$ for some $t \in \mathbb{N}$. For any $v \in P(L_p, a)$, we claim that $\alpha, \beta \in p^m$, where $m := \min\{s, t\}$. Suppose on the contrary that $u := \min\{\text{ord}_p \alpha, \text{ord}_p \beta\} < m$. Then $\text{ord}_p \alpha = \text{ord}_p \beta = u$, since otherwise $\text{ord}_p Q(v) < \text{ord}_p a$. Writing $\alpha = \pi^u \alpha_1$ and $\beta = \pi^u \beta_1$ with $\alpha_1, \beta_1 \in \mathscr{U}_p$, we see that

$$\alpha_1^2 b_1 + \beta_1^2 b_2 \equiv 0 \pmod{p}.$$

Thus, $\langle \overline{b_1}, \overline{b_2}\rangle$ is isotropic over $\bar{F}_p$, and so $-b_1 b_2 \equiv 1 \pmod{p}$. But then it follows from the Local Square Theorem that $-b_1 b_2 \in \dot{F}_p^2$, a contradiction. So we indeed have $\alpha, \beta \in p^m$, as claimed. It then follows that

$$P(L_p, a) \subseteq P(\hat{L}, a),$$

where $\hat{L} = R_p \pi^m x + R_p \pi^m y + R_p z$, and so $\theta^*(L_p, a) \subseteq \theta^*(\hat{L}, a)$. Upon scaling, $\theta^*(\hat{L}, a) = \theta^*(L', \pi^{-2m}a)$, where $L' \cong \langle b_1, b_2, \pi^{2s-2m}b_3\rangle$. If $m = s$, then $\theta^*(L', \pi^{-2m}a) = N_p(E)$ by [**9**, Satz 5]; if $m = t$, then

$$\theta^*(L', \pi^{-2m}a) = \theta(L', \pi^{-2m}a) = N_p(E)$$

by [**9**, Satz 3]. Hence we have $\theta^*(L_p, a) \subseteq N_p(E)$ as desired.

Proof of Theorem 1. (a) If $\theta^*(L_p, a) \neq N_p(E)$, then by (3) and [**9**, Satz 3] we must have $a \in p^{2r+1}$. So $P(L_p, a) = P(\hat{L}, a)$ for $\hat{L} = R_p \pi^r x + R_p y + R_p z$. Scaling $\hat{L}$ by π^{-2r}, we see that $P(L_p, a) = P(L', \pi^{-2r}a)$ with $L' \cong \langle b_1, b_2, \pi^{2s-2r}b_3\rangle$; so

$$\theta^*(L_p, a) = \theta^*(L', \pi^{-2r}a).$$

If $r = s$, then $\theta^*(L', \pi^{-2r}a) = N_p(E)$ by [**9**, Satz 5]. If $r \neq s$, then by Lemma 2, $\theta^*(L', \pi^{-2r}a) = N_p(E)$ holds if and only if $-b_1 b_2 \notin \dot{F}_p^2$.

(b) If $\theta^*(L_p, a) \neq N_p(E)$, then by (3) and [**9**, Satz 3] we must have either r even and $a \in p^r$, or r odd and $a \in p^s$. Consider first the case with r even and $a \in p^r$; say $r = 2t$. Since $E_{\mathfrak{P}}/F_p$ is unramified, $\mathscr{U}_p \dot{F}_p^2 \nsubseteq N_p(E)$. So if we show that $\mathscr{U}_p \dot{F}_p^2 \subseteq \theta^*(L_p, a)$, then it will follow that $\theta^*(L_p, a) = \dot{F}_p$. We first claim that there exists a $v \in P(L_p, a)$ with $\gamma \in \mathscr{U}_p$. First, if $a \in p^{r+1}$ and $-b_1 b_2 \notin \dot{F}_p^2$, then arguing as in the proof of Lemma 2 we see that $\gamma \in \mathscr{U}_p$ for any $v \in P(L_p, a)$. Next, if $-b_1 b_2 \in \dot{F}_p^2$, then the sublattice $K = R_p \pi^t x + R_p y$ is isometric to

$$\begin{pmatrix} 0 & \pi^r \\ \pi^r & 0 \end{pmatrix}.$$

So there exists $u \in K$ such that $Q(u) = a - \pi^s b_3$. Then $v = u + z \in P(L_p, a)$. Finally, suppose that $a \in \pi^r \mathscr{U}_p$. Let $v \in P(L_p, a)$. If $\gamma \notin \mathscr{U}_p$, then $v - z$ has z-coefficient a unit and $Q(v-z) \cong a$. So in every case we have verified that there exists $v \in P(L_p, a)$ with $\gamma \in \mathscr{U}_p$. Now consider the sublattice $K = R_p \pi^t x + R_p y$. Then $\theta(O^+(K)) = \mathscr{U}_p \dot{F}_p^2$, so there exists $\sigma \in O^+(K)$ for which $\theta(\sigma) \notin \sigma(O^+(L_p))$. Let $\hat{\sigma}$ be the extension of σ to $O^+(V_p)$ for which $\hat{\sigma}(z) = z$. Then $\theta(\hat{\sigma}) = \theta(\sigma)$ and $\sigma(v) \in P(L_p, a)$. It follows that $\theta^*(L_p, a) = \dot{F}_p$.

Now consider the case with r odd and $a \in p^s$. Let $e = [\![s/2]\!]$ and $t = \frac{1}{2}(2e - 1 - r)$. Suppose first that s is even; so $s = 2e$. Then $P(L_p, a) = P(\hat{L}, a)$ for

$$\hat{L} = R_p \pi^{e-1} x + R_p \pi^t y + R_p z \cong \langle \pi^{s-2} b_1, \pi^{s-1} b_2, \pi^s b_3 \rangle.$$

Let L' denote $\hat{L}$ scaled by $\pi^{-(s-2)}$; so $L' \cong \langle b_1, \pi b_2, \pi^2 b_3 \rangle$. Then $\theta^*(L_p, a) = \theta^*(L', a')$ for $a' = \pi^{-(s-2)} a$. Now unless $a' \cong \pi^2 b_1$, we have $P(L', a') = P(\tilde{L}, a')$ for $\tilde{L} \cong \langle \pi^2 b_1, \pi b_2, \pi^2 b_3 \rangle$; so $\dot{F}_p = \theta(O^+(\tilde{L})) \subseteq \theta^*(L', a')$. So only the case $a' \cong \pi^2 b_1$ remains. The assumption that $\theta(O^+(L_p)) \subseteq N_p(E)$ in this case forces $b_1 \cong b_3$. So we have $a' \cong \pi^2 b_3$, and we are in the situation of Lemma 1. Thus, $\theta^*(L_p, a) = \theta^*(L', a') = N_p(E)$ holds if and only if $|\bar{F}_p| \leqslant 5$. This completes the case when s is even. Finally, suppose that s is odd; so $s = 2e + 1$. In this case, consider

$$\hat{L} = R_p \pi^e x + R_p \pi^t y + R_p z \cong \langle \pi^{s-1} b_1, \pi^{s-2} b_2, \pi^s b_3 \rangle$$

and let L' denote $\hat{L}$ scaled by $\pi^{-(s-2)}$; so $L' \cong \langle \pi b_1, b_2, \pi^2 b_3 \rangle$. For $a \in p^s$, we have $P(L_p, a) = P(\hat{L}, a)$ and $\theta^*(L_p, a) = \theta^*(L', a')$ for $a' = \pi^{-(s-2)} a$. Unless $a' \cong \pi^2 b_2$, we have that $\theta^*(L', a')$ follows as in the previous case. When $a' \cong \pi^2 b_2 \cong \pi^2 b_3$, by Lemma 1, $\theta^*(L_p, a) = \theta^*(L', a') = N_p(E)$ holds if and only if $|\bar{F}_p| \leqslant 5$.

We now describe the verification of a single 2-adic case (falling under Theorem 2(c)(iii)) which, in particular, illustrates the key role played by the size of the residue class field.

LEMMA 3. *Let p be 2-adic,* $-ad \notin \dot{F}_p^2$. *Let* $L_p \cong \langle 1, 2b_2, 2^6 b_3 \rangle$ *with* $b_i \in \mathscr{U}_p$. *Assume that* $\theta(O^+(L_p)) \subseteq N_p(E)$. *Then* $\theta^*(L_p, a) \neq N_p(E)$ *if and only if either*

(i) $F_p = \mathbb{Q}_2$, $a \in p^4$,

or

(ii) $F_p \neq \mathbb{Q}_2$, $a \in p^2$.

Proof. Let $M = \langle 1, 2b_2 \rangle$. By [3, Proposition 1.4], $\theta(O^+(M)) = Q(F_p M)$. Since $\theta(O^+(M)) \subseteq \theta(O^+(L_p)) \neq \dot{F}_p$, it follows that $b_3 \in Q(F_p M)$. As $b_3 \in \mathscr{U}_p$, it follows that $\langle b_3 \rangle$ splits M; so $L_p \cong \langle b_3, 2b_2 b_3, 2^6 b_3 \rangle$. Hence we may assume without loss of generality that $L_p \cong \langle 1, 2b, 2^6 \rangle$. Then

$$\theta(O^+(L_p)) = \{\xi \in \dot{F}_p \mid (\xi, -2b)_p = 1\},$$

by [3, Theorem 2.7], and $N_p(E) = \{\eta \in \dot{F}_p \mid (\eta, -2ab)_p = 1\}$. From the assumption that $\theta(O^+(L_p)) \subseteq N_p(E)$, it follows that $a \in \dot{F}_p^2$. So $\mathrm{ord}_p(-ad)$ is odd and we are in case [9, Satz 4(c)]; thus, $\theta(L_p, a) = \theta^*(L_p, a) = N_p(E)$ holds unless $a \in p^2$. Moreover, if $a \in p^4$, then $P(L_p, a) = P(\hat{L}, a)$ for $\hat{L} = R_p 4x + R_p 4y + R_p z$. Since $\theta(O^+(\hat{L})) = \dot{F}_p$, it follows that $\theta^*(L_p, a) = \dot{F}_p$. So it only remains to further consider the case $a = 4$.

We consider this case first when $F_p = \mathbb{Q}_2$. Let $u, v \in P(L_p, 4)$ with $u = \alpha' x + \beta' y + \gamma' z$. Note that $\alpha, \alpha' \in 2\mathscr{U}_p$, $\beta, \beta' \in 2R_p$ and $\gamma, \gamma' \in \mathscr{U}_p$. Since $F_p = \mathbb{Q}_2$, it follows that $\alpha \pm \alpha' \in 4R_p$

and $\gamma \pm \gamma' \in 2R_p$. As $Q(u-v)+Q(u+v) = 16$, not both $Q(u-v)$ and $Q(u+v)$ can lie in p^5. By changing the sign of v if necessary, we may assume that $Q(u+v) \notin p^5$. We then claim that the symmetry τ_{u+v} lies in $O(L_p)$. To verify this, it suffices to show that

$$2B(u+v, w)\, Q(u+v)^{-1}(u+v)$$

lies in L_p for $w = x$, y and z. For $w = x$, we have

$$2B(u+v, x)\, Q(u+v)^{-1} = 2(\alpha+\alpha')\, Q(u+v)^{-1} \in \tfrac{1}{2}R_p,$$

since $\alpha+\alpha' \in 4R_p$. So $2B(u+v, x)\, Q(u+v)^{-1}(u+v)$ lies in L_p since all of $\alpha+\alpha'$, $\beta+\beta'$ and $\gamma+\gamma'$ lie in $2R_p$. The verifications for y and z are similar. So $\sigma = (-1)\,\tau_{u+v} \in O^+(L_p)$ and $\sigma(u) = v$. That $\theta^*(L_p, a) = N_p(E)$ now follows as in the proof of [**9**, Lemma 2].

Finally, we consider the case $F_p \neq \mathbb{Q}_2$. Let $\Delta = 1+4\rho$, with $\rho \in \mathscr{U}_p$, be a unit of quadratic defect $4R_p$. By perfectness of the residue class field, we may assume that $\rho = \eta^2$ for some $\eta \in \mathscr{U}_p$. Let $w = 4x+\eta z$. Then $Q(w) = 2^4\Delta$. Since $F_p \neq \mathbb{Q}_2$, there exists $\lambda \in \mathscr{U}_p$ such that $\lambda - \eta\Delta^{-1} \in \mathscr{U}_p$. By the Local Square Theorem, there exists $\xi \in \mathscr{U}_p$ such that $1+2^3b+2^4\lambda^2 = \xi^2$. Then

$$v = \xi^{-1}(2x+2^2y+\lambda z) \in P(L_p, 4).$$

Now a direct calculation shows that the z-coefficient of $\tau_w(v)$ is $\xi^{-1}(\lambda - \eta\Delta^{-1} - 2^3\Delta^{-1}\rho\lambda)$, which lies in $\mathscr{U}_p$ since $\lambda - \eta\Delta^{-1} \in \mathscr{U}_p$. So $\tau_w(v) \in P(L_p, 4)$. Then $\tau_x\tau_w(v) \in P(L_p, 4)$ and $(Q(x)\,Q(w), -2b)_p = -1$. Hence, $\theta^*(L_p, 4) = \dot{F}_p$.

3. *Examples*

In this section we shall illustrate the use of the preceding results by analysing the primitive representation properties of seven specific genera of positive definite ternary quadratic forms. The integers represented by these particular forms were first determined by Jones and Pall [**6**], who showed that in each case the genus contains one form which is regular in the sense of Dickson [**1**] (that is, which represents all integers represented by the genus) and a second form which is regular except for failing to represent an infinite family of integers from a particular square class. From a modern perspective, the explanation for these interesting representation properties can be seen to lie in the theory of spinor genus representations [**9**]. In all cases, the genus splits into two spinor genera, with the regular form lying in a spinor genus containing at most two classes, and with the integers not represented by the second form being precisely the spinor exceptional integers for the genus.

With regard to primitive representations, it will be shown here that all the forms in these genera fail to primitively represent infinite families of integers from one or more square classes which are primitively represented by their genus, but are otherwise primitively regular. Lattices corresponding to the forms in the seven genera are listed in [**9**; Tabelle I]. We shall keep the same labelling as used there and refer to the genera as Examples 1 to 7 according to their numbering in that table. Note that the relevant spinor norm groups $\theta(O^+(L_p))$ are also tabulated in [**9**; Tabelle II].

LEMMA 4. *Let all notation be as in the previous sections, with R the ring of rational integers. Then $\theta(O^+(L_p)) \subseteq N_p(E)$ and $\theta^*(L_p, a) = N_p(E)$ hold for all primes p not dividing $2d$ if and only if $a = a_0m^2$ where all prime factors of a_0 divide $2d$ and* $\gcd(m, 2d) = 1$.

Proof. For $p \nmid 2d$ we have $\theta(\mathrm{O}^+(L_p)) = \mathcal{U}_p \dot{\mathbb{Q}}_p^2$. So $\theta(\mathrm{O}^+(L_p)) \subseteq N_p(E)$ forces p to be unramified in E. Hence $\mathrm{ord}_p(-ad)$ is even and it follows that a must have the stated form. The reverse implication follows from [**9**, Satz 5].

With the aid of this lemma, the following result can be established using Theorems 1 and 2.

PROPOSITION 1. *Let L be as in one of Examples* 1 *to* 7, *let m denote an arbitrary odd integer, and l an arbitrary odd integer not divisible by* 3. *Then a is primitively spinor exceptional for the genus of L if and only if a has one of the following forms:*

(i) m^2 *or* $4m^2$, *in Examples* 1, 4 *and* 6;

(ii) m^2, *in Example* 2;

(iii) l^2, $9l^2$, $4^t l^2$ *or* $4^t 9 l^2$, *where* $t > 1$, *in Example* 3;

(iv) l^2, $4l^2$, $9l^2$, *or* $36l^2$, *in Example* 5;

(v) l^2, $4l^2$, $9l^2$, $16l^2$, $36l^2$ *or* $144l^2$, *in Example* 7.

The question of which of the two spinor genera primitively represents each of these primitive spinor exceptional integers can further be settled with the use of [**2**, Theorem 1]. For instance, since L can be seen to represent 1 in every case, the spinor genus of L must primitively represent precisely those integers of the form m^2(l^2, respectively) for which the Jacobi symbol $\left(\frac{-d}{m}\right)$ equals $+1$ $\left(\left(\frac{-d}{l}\right) = +1\text{, respectively}\right)$. The primitive spinor exceptional integers occurring in other square classes can be analysed similarly. Moreover, in Examples 1 to 6 each spinor genus consists of only a single class; hence, the given lattices must in fact primitively represent all the integers primitively represented by their spinor genus. The results are tabulated in the following proposition. For convenience, in the statement we let h, k, s, t, u and v denote arbitrary integers which are congruent to 1 (mod 4), 3 (mod 4), 1 or 3 (mod 8), 5 or 7 (mod 8), 1 (mod 6), and 5 (mod 6), respectively.

PROPOSITION 2. *Let L and K be as in one of Examples* 1 *to* 6. *If a is primitively represented by the genus of L, then*

(a) *a is primitively represented by L, except for those a of the following forms:*

  (i) k^2 *or* $4h^2$, *in Examples* 1 *and* 4;

  (ii) k^2, *in Example* 2;

  (iii) v^2, $9u^2$, $4^\lambda u^2$, $4^\mu v^2$, $4^\lambda 9 v^2$, $4^\mu 9 u^2$, *where* λ *is even,* μ *odd, in Example* 3;

  (iv) v^2, $4u^2$, $9u^2$ *or* $36v^2$, *in Example* 5;

  (v) t^2 *or* $4s^2$, *in Example* 6;

(b) *a is primitively represented by K, except for those a of the following forms:*

  (i) h^2 *or* $4k^2$, *in Examples* 1 *and* 4;

  (ii) h^2, *in Example* 2;

  (iii) u^2, $9v^2$, $4^\lambda v^2$, $4^\mu u^2$, $4^\lambda 9 u^2$, $4^\mu 9 v^2$, *where* λ *is even,* μ *odd, in Example* 3;

  (iv) u^2, $4v^2$, $9v^2$ *or* $36u^2$, *in Example* 5;

  (v) s^2 *or* $4t^2$, *in Example* 6.

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