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# Representation of integers by positive ternary quadratic forms and equidistribution of lattice points on ellipsoids

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## Introduction

It is a classical problem to find an asymptotic formula for the number of integral points in a region on the ellipsoid $q(x_1, \ldots, x_r)=n$ as $n \to \infty$ where $q$ is a positive definite integral quadratic form. In particular, one wants to prove that the integral points on such an ellipsoid are asymptotically uniformly distributed. For $r \geqq 4$, this problem has been solved by Pommerenke [Pom] (with some necessary restrictions on the set of numbers in which $n$ tends to infinity if $r=4$). Improvements of the error term have been obtained in [Ma, Pod, Go-Fo1]. The case $r=3$ has remained open so far; in fact one could not even give an unconditional proof of such a formula for the number of integral points on the whole ellipsoid (i.e., the number of representations of $n$ by the quadratic form $q$). Linnik's ergodic method has been applied to this problem by several authors (see [Pe, Te] and references given there). However, the method requires imposing a condition on the quadratic residue character of $n$ modulo some fixed prime. In order to remove it one has to assume certain unproved hypotheses concerning zeros of Dirichlet $L$-functions.

Recent advances in the theory of modular forms of half integral weight [I, Du] as well as in the theory of quadratic forms [SP1, SP2, SP3] make it now possible to obtain the desired asymptotic formula unconditionally for $r=3$ following the well known approach that has been succesful for $r \geqq 4$. The only restrictions still present resemble those for $r=4$. How this may be done for a 3-dimensional sphere has been shown in [Du] and [Go-Fo2].

Although each single step of this proof is in the literature, it might be of some interest to collect them in one place and thus make this beautiful result more easily accessible. Since it does not involve much extra work, we also allow congruence conditions on the integral points.

## 1. A first asymptotic formula

In this section we state the problem and show how, with the help of Iwaniec's result [I], it can be reduced to that of proving an asymptotic formula for the number of representations of an integer by a positive definite integral ternary quadratic form. The argument used is essentially the same as in the special case of a 3-dimensional sphere. It should be noted, however, that we can omit most of the restrictions that are usually imposed on the square part of $n$. Also, our exponent in the error term is slightly better than that obtained in [Go-Fo2].

Let $q(\mathbf{x})=\frac{1}{2}(\mathbf{x}^t A\mathbf{x})(\mathbf{x}\in\mathbb{Z}^3,\ A=(a_{ij})\in M_3^{\mathrm{sym}}(\mathbb{Z}),\ a_{ii}\in 2\mathbb{Z})$ be a positive definite integral quadratic form in 3 variables, $N$ the level of $q$ (i.e., $N=\min\{N'\in\mathbb{N}\,|\,N'A^{-1}$ integral with even diagonal$\}$), $\mathscr{F}$ a convex region with piecewise smooth boundary on the ellipsoid $\mathscr{E}=\{\mathbf{x}\in\mathbb{R}^3\,|\,q(\mathbf{x})=1\}$, $\mathbf{h}\in(\mathbb{Z}^3)^\#=\{\mathbf{x}\in\mathbb{Q}^3\,|\,A\mathbf{x}\in\mathbb{Z}^3\}$ a fixed vector with $q(\mathbf{h})\in\mathbb{Z}$.

Our goal is the determination of

$$r(q,\mathscr{F},\mathbf{h},n):=\#\left\{\mathbf{x}\in\mathbb{Q}^3\,|\,\mathbf{x}\equiv\mathbf{h}\bmod\mathbb{Z}^3,\ q(\mathbf{x})=n,\ \frac{\mathbf{x}}{\sqrt{n}}\in\mathscr{F}\right\}.$$

Let $r(q,\mathbf{h},n):=r(q,\mathscr{E},\mathbf{h},n)$.

**Lemma 1.** [Go-Fo2] *Let $1_{\mathscr{F}}$ be the characteristic function of $\mathscr{F}$, $\mu(\mathscr{F})=\int_{\mathscr{E}} 1_{\mathscr{F}}\,d\mu$ the area of $\mathscr{F}$ ($d\mu$ normalized to $\mu(\mathscr{E})=1$), and $\delta>0$ be given.*

*Then there exist homogeneous q-harmonic polynomials $P_\nu^\pm$ of degree $\nu$ such that*

(i) $\sum_{\nu=0}^{\infty} P_\nu^-(\mathbf{x})\leqq 1_{\mathscr{F}}\leqq\sum_{\nu=0}^{\infty} P_\nu^+(\mathbf{x})$ *for all* $\mathbf{x}\in\mathscr{E}$

(ii) $|P_0^\pm-\mu(\mathscr{F})|<\delta$

(iii) $|P_\nu^\pm(\mathbf{x})|\ll_s\delta^{-s}\nu^{-s-1/2}$ *for all* $\mathbf{x}\in\mathscr{E}$ $(s\in\mathbb{N}$ *arbitrary).*

*Proof.* The assertion is proved for the sphere in Lemma 3 of [Go-Fo2]. Obviously it carries over to the case of an arbitrary ellipsoid by a suitable linear transformation.

**Lemma 2.** *Let $f(z)=\sum_{n=1}^{\infty} a_n\exp(2\pi i n z)$ be a cusp form for*

$$\Gamma_1(N)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})\,|\,c\equiv 0\bmod N,\ a\equiv 1\bmod N\right\}$$

*of weight $3/2+\nu$ where $\nu\in\mathbb{Z}$, $\nu\geqq 0$. If $\nu=0$, assume further that the Shimura lifting of $f$* [Shi] *is a cusp form. Let $\varepsilon>0$ be given and let $n\to\infty$ under the restriction that $n=tn_0^2$ with $(n_0,N)=1$ and $t=t'n_1^2$ with $t'$ square-free and $n_1$ in some fixed finite set $S$ of integers.*

*Then*

$$|a_n|\ll_{\varepsilon,S} c_\nu(f,f)^{1/2}\,n^{\nu/2+1/2-1/28+\varepsilon}$$

*where*

$$c_\nu=\frac{(4\pi)^{\nu/2}(\nu+1)^{\eta_\nu}}{\Gamma(\nu+3/2)^{1/2}},$$

*and*

$$\eta_\nu = \begin{cases} 7/2, & \nu \text{ odd} \\ 9/2, & \nu \text{ even} \end{cases}.$$

*Proof.* We may suppose $f \in S_{3/2+\nu}(\Gamma_0(N), \chi)$ for some $\chi (\bmod N)$.

Suppose first that $t$ is square-free. Then the required estimate for $|a_t|$ is an extension of that in [I] to $\Gamma_1(N)$ which includes the case $\nu = 0$ and which is uniform in $\nu$.

It is proved using Proskurin's generalization of the Kuznetsov sum formula in a way similar to the proof of Theorem 5 in [Du] and is based ultimately on the estimate for sums of Kloostermann sums given in Theorem 3 of [I].

In the notation of [Du], for $k = 1/2$ or $k = 3/2$, $\alpha, \beta \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ with $\alpha > 3/2$ and $\beta - \alpha - 1 \in 2\mathbb{Z}^+$ let

$$\varphi_{\alpha,\beta}(x) = \varphi(x) = c\, x^{-\alpha} J_\beta(x)$$

where $c = (-1)^{(\beta-\alpha-1)/2}\, 2^{\alpha+1}\, \pi e(-k/4)\, \Gamma(\alpha+1)^{-1}$ and $J_\beta(x)$ is the usual Bessel function.

This $\varphi$ satisfies the conditions of the sum formula and a straightforward calculation shows that $\hat{\varphi}(t) > 0$ for $t \in \mathbb{R}$ or $it \in (-1/2, 1/2)$. Also, for $j \geqq 1$

$$e\left(\frac{k}{4} + \frac{j}{2}\right) \tilde{\varphi}(k+2j) = (-1)^{\frac{\beta-\alpha-1}{2}} \left(\frac{k}{2} + j\right)\left(\frac{k}{2} + j - 1\right)$$
$$\sin\left(\pi \frac{\alpha+\beta+3-k}{2}\right) \prod_{\lambda=0}^{\alpha+\beta} \left(\frac{k}{2} + j - \frac{\alpha+\beta+1}{2} + \lambda\right)^{-1}.$$

Choosing $\alpha = 5/2$, $\beta = 11/2$ if $k = 1/2$ and $\alpha = 7/2$, $\beta = 13/2$ if $k = 3/2$ we get that for $j \geqq 5$

$$e\left(\frac{k}{4} + \frac{j}{2}\right) \tilde{\varphi}(k+2j) > c_0\, j^{1-\alpha-\beta}$$

for an absolute constant $c_0 > 0$.

Now one proceeds as in §5 of [Du], using that the sum formula given there holds for all $\chi$, not only for real $\chi$.

The generalization to $t = t' n_1^2$ as above is straightforward.

If $f$ is an eigenfunction of all the Hecke operators $T(n_0^2)$ with $(n_0, N) = 1$ with eigenvalues $\lambda(n_0)$, then by Shimura's correspondence [Shi, Ni, Ci]

$$a_{tn_0^2} = a_t \sum_{m \mid n_0} \chi(m) \left(\frac{-1}{m}\right)^{1+\nu} \mu(m)\, A\left(\frac{n_0}{m}\right),$$

where $A(n_0/m) = \lambda(n_0/m)$ is the $T(n_0/m)$-eigenvalue of the cusp form of weight $2\nu+2$ associated to $f$ by this correspondence. Consequently, $|a_{tn_0^2}| \leqq |a_t|\, \tau(n_0)^2\, n_0^{1/2+\nu} \ll |a_t|\, n_0^{1/2+\nu+\varepsilon}$ (with $\tau(n) = \sum_{m \mid n} 1$) by the Ramanujan-Petersson bound [De], and our estimate for arbitrary $f$ follows.

The first named author would like to point out that the above choices of $\alpha$, $\beta$ should have been made instead of $\alpha=3/2$, $\beta=9/2$ in the proof of Theorem 5 in [Du], justifying the estimate (5.3). The value of $A$ given in Theorem 5 should be accordingly modified.

**Lemma 3.** *Let $\varepsilon>0$ be given. Then, with the notation of Lemma* 1, *one has for $n\to\infty$ (restricted to $n=tn_0^2n_1^2$ with $t$ squarefree,* $\gcd(n_0,p)=1$ *for all primes $p$ with $q$ isotropic over $\mathbb{Q}_p$ and $\mathbf{h}\notin\mathbb{Z}_p^3$, $n_1$ in some fixed finite set $S$ of integers)*

$$\left|\sum_{v=1}^{\infty}\sum_{\substack{\mathbf{x}\in\mathbf{h}+\mathbb{Z}^3\\ q(\mathbf{x})=n}} P_v^{\pm}(\mathbf{x})\right| \ll_{\varepsilon,S,q} r(q,\mathbf{h},n)\,\delta+\delta^{-21/4-\varepsilon}n^{1/2-1/28+\varepsilon}$$

*Proof.* A similar estimate is proved for the sphere in Sect. 4 of [Go-Fo2] restricting to $n=tn_0^2$ with $t$ squere free, $(n_0,N)=1$ (and without the congruence condition), using Lemma 1 and a slightly different version of Lemma 2 above.

The proof proceeds by using the fact that for $v>0$ the theta series

$$\vartheta(P_v^{\pm},q,\mathbf{h},z)=\sum_{\mathbf{x}\in\mathbf{h}+\mathbb{Z}^3} P_v^{\pm}(\mathbf{x})\exp(2\pi i q(\mathbf{x})z)=\sum_{n=1}^{\infty} a_v^{\pm}(n)\exp(2\pi i n z)$$

is a cusp form of weight $3/2+v$ for $\Gamma_1(N)$ [Pf, Shi] and its Fourier coefficients $a_v^{\pm}(n)$ can therefore be bounded with the help of Lemma 2 (using the bound on $|P_v^{\pm}(\mathbf{x})|$ from Lemma 1 to estimate the Petersson norm of $\vartheta(P_v^{\pm},q,\mathbf{h},z)$).

The sum over $v$ is then divided into $\sum_{v=[\delta^{-1-\varepsilon}]+1}^{\infty}$ and $\sum_{v=1}^{[\delta^{-1-\varepsilon}]}$, where the first sum is bounded by $r(q,\mathbf{h},n)\,\delta$ using Lemma 1 for $s$ large enough and the second sum is estimated by using the bound on $|a_v^{\pm}(n)|$ from Lemma 2 and $(\vartheta,\vartheta)\ll_s(v\delta)^{-2s}(4\pi)^{-v}\Gamma(v+1)$ for $\vartheta=\vartheta(P_v^{\pm},q,\mathbf{h},z)$. The generalization to the ellipsoid is immediate as well as that to theta series with congruence conditions.

To deal with $n$ whose square part involves primes dividing $N$ (under the additional condition on $\mathbf{h}$), consider first $p\mid N$ with $q$ isotropic over $\mathbb{Q}_p$ (i.e., there exists $y\in\mathbb{Q}_p^3$, $y\neq 0$ with $q(y)=0$).

Then by Lemma 3–5 of [SP3] there are lattices $K_i$, $M_j$ on $\mathbb{Q}^3$ and $r_0=r_0(p,q)$ such that:

(i) $(K_i)_l=(M_j)_l=\mathbb{Z}_l^3$ for all primes $l\neq p$

(ii) $\vartheta(P_v^{\pm},p^{-r_0}q,\mathbf{h}',K_i,z):=\sum_{\mathbf{x}\in\mathbf{h}'+K_i} P_v^{\pm}(\mathbf{x})\exp(2\pi i p^{-r_0}q(\mathbf{x})z)\in S_{3/2+v}(\Gamma_1(N'))$,

$$\vartheta(P_v^{\pm},p^{-r_0}q,\mathbf{h}',M_j,z)\in S_{3/2+v}(\Gamma_1(N'p))\qquad\text{with } N'=N/\gcd(N,p^{\infty})$$

(where $\mathbf{h}'\in\mathbf{h}+\mathbb{Z}_l^3$ for all $l\neq p$, $\mathbf{h}'\in(K_i)_p$, $\mathbf{h}'\in(M_j)_p$ for all $i,j$).

(iii) if $2r\geqq r_0$ and $p^2\nmid m$ one has

$$a_v^{\pm}(mp^{2r})=\sum_i\sum_{s\leqq r}\gamma_{is}p^{v(r-s)}b_{v,i}^{\pm}(mp^{2s})+\sum_j\tilde{\gamma}_j p^{v(r-r')}c_{v,j}^{\pm}(mp^{2r'}),$$

where $r' = \left[\frac{r_0 + 1 - \mathrm{ord}_p m}{2}\right]$, $b^{\pm}_{v,i}(n)$, $c^{\pm}_{v,j}(n)$ are the Fourier coefficients of $\vartheta(P_v^{\pm}, q, \mathbf{h}', K_i, z)$ and $\vartheta(P_v^{\pm}, q, \mathbf{h}', M_j, z)$ respectively and where the constants $\gamma_{is}, \tilde{\gamma}_j$ satisfy $\sum_{i,s} |\gamma_{is}| + \sum_j |\tilde{\gamma}_j| = O((\log n \log N))$
(the argument given in [SP3] for $v=0$ generalizes to arbitrary $v$). Both sums occurring here can be estimated with the help of Lemma 2.

If $p|N$ is such that $q$ is anisotropic over $\mathbb{Q}_p$, there is $r_0 = r_0(p, N)$ such that all $x \in \mathbb{Q}_p^3$ with $q(\mathbf{x}) \in p^{r_0 + 2s}\mathbb{Z}_p$ are in $p^s \mathbb{Z}_p^3 (s \neq 0)$ ([E], Satz 9.4), i.e., for $2r > r_0$

$$a_v^{\pm}(mp^{2r}) = \begin{cases} 0 & \text{if } \mathbf{h} \notin \mathbb{Z}_p^3 \\ p^{\left[\frac{2r - r_0 + 1}{2}\right] v} a_v^{\pm}\left(mp^{2\left[\frac{r_0+1}{2}\right]}\right) & \text{if } \mathbf{h} \in \mathbb{Z}_p^3. \end{cases}$$

Putting together the results for the $p|N$ proves the assertion.

*Remark.* It should be noted that with the same argument as above the estimate of Lemma 2 can be generalized to arbitrary $n$ if the cusp form $f$ considered is restricted to the subspace of $S_{3/2+v}(\Gamma_1(N))$ generated by theta series of quadratic forms in the genus of $q$ with spherical coefficients and congruence conditions at most at those primes $p$ for which $q$ is anisotropic over $\mathbb{Q}_p$).

An immediate conclusion from Lemma 1–3 is (putting $\delta = n^{-1/175}$).

**Theorem 1.** *Let $\varepsilon > 0$. Then for $n \to \infty$ restricted as in Lemma 3 one has:*

$$|r(q, \mathscr{F}, \mathbf{h}, n) - r(q, \mathbf{h}, n)\, \mu(\mathscr{F})| \ll_{\varepsilon, S, N} r(q, \mathbf{h}, n)\, n^{-1/175} + n^{1/2 - 1/175 + \varepsilon},$$

*where the implied constant does not depend on n.*

*The restriction on n is empty if* $\mathbf{h} \in \mathbb{Z}^3$.

This is an improvement over the error term $n^{1/2 - 1/336 + \varepsilon}$ obtained in [Go-Fo2].

## 2. Representation numbers of ternary quadratic forms and the final asymptotic formula

Theorem 1 leaves us with the problem of giving an asymptotic formula for $r(q, \mathbf{h}, n)$. The classical method to attack this problem is to use Siegel's mass formula. Here we need van der Blij's generalization including congruence conditions as well as a modification of the mass formula for computing the average of the representation numbers by forms in a spinor genus.

Recall that rationally equivalent nondegenerate quadratic forms $q, q'$ with matrices $A$, $A' = V^t A V \in M_r^{sym}(\mathbb{Q})(V \in \mathrm{GL}_r(\mathbb{Q}))$ are said to belong to the same class ($q' \in \mathrm{cls}\, q$) if $V \in O_{\mathbb{Q}}(A)\, \mathrm{GL}_r(\mathbb{Z})(O_{\mathbb{Q}}(A) = \{U \in \mathrm{GL}_r(\mathbb{Q}) \mid U^t A U = A\})$, to the same genus ($q' \in \mathrm{gen}\, q$) if

$$V \in \bigcap_p O_{\mathbb{Q}_p}(A)\, \mathrm{GL}_r(\mathbb{Z}_p),$$

to the same spinor genus ($q' \in \operatorname{spn} q$) if

$$V \in O_{\mathbb{Q}}(A) \bigcap_p O'_{\mathbb{Q}_p}(A)\, \mathrm{GL}_r(\mathbb{Z}_p),$$

where $O'_{\mathbb{Q}_p}(A)$ is the set of $U \in O_{\mathbb{Q}_p}(A)$ of determinant and spinor norm 1 ([OM], § 55) (for $r \neq 4$, this is just the commutator subgroup of $O_{\mathbb{Q}_p}(A)$).

We generalize these notions to congruence class (genus, spinor genus) mod $N$ ($\mathrm{cls}_N q$ etc.) by replacing $\mathrm{GL}_r(\mathbb{Z})$ resp. $\mathrm{GL}_r(\mathbb{Z}_p)$ by $\mathrm{GL}_r(\mathbb{Z}, N) := \{U \in \mathrm{GL}_r(\mathbb{Z}) \mid U \equiv 1 \bmod N\}$ (resp. $\mathrm{GL}_r(\mathbb{Z}_p, N)$).

Finally, let $o(q, N) = \#(\mathrm{GL}_r(\mathbb{Z}, N) \cap O_{\mathbb{Q}}(A))$.

**Theorem 2.** (i) (*Siegel*) *For each prime p let*

$$\alpha_p(q, \mathbf{h}, n) = \lim_{r \to \infty} p^{-2r} \# \{\mathbf{x} \in \mathbb{Z}^3 / p^r \mathbb{Z}^3, \mathbf{x} \equiv \mathbf{h} \bmod \mathbb{Z}^3, q(\mathbf{x}) \equiv n \bmod p^r\},$$

*Let*

$$r(\mathrm{gen}_N q, \mathbf{h}, n) = \left( \sum_{(q') \in \mathrm{gen}_N q} \frac{1}{o(q', N)} \right)^{-1} \sum_{(q') \in \mathrm{gen}_N q} \frac{r(q', \mathbf{h}, n)}{o(q', N)},$$

*where the summation is over a set of representatives of the congruence classes* mod $N$ *in the congruence genus* mod $N$.

*Then*

$$r(\mathrm{gen}_N q, \mathbf{h}, n) = 2\pi \sqrt{\frac{2n}{\det A}} \prod_p \alpha_p(q, \mathbf{h}, n)$$

(ii) *Let*

$$r(\mathrm{spn}_N q, \mathbf{h}, n) = \left( \sum_{(q') \in \mathrm{spn}_N q} \frac{1}{o(q', N)} \right)^{-1} \sum_{(q') \in \mathrm{spn}_N q} \frac{r(q', \mathbf{h}, n)}{o(q', N)},$$

*Then there are finitely many explicitly computable numbers $t_i$ such that for $n \notin \bigcup t_i \mathbb{Z}^2$*

$$r(\mathrm{spn}_N q, \mathbf{h}, n) = r(\mathrm{gen}_N q, \mathbf{h}, n).$$

*Moreover, one can define local densities $\alpha^{\pm}_{p,i}(q, \mathbf{h}, n)$ (which are computable in terms of numbers of solutions of congruences) such that for $n = t_i n_0^2 \in t_i \mathbb{Z}^2$ on has*

$$r(\mathrm{spn}_N q, \mathbf{h}, n) - r(\mathrm{gen}_N q, \mathbf{h}, n) = 2\pi \sqrt{\frac{2n}{\det A}} \prod_p (\alpha^{+}_{p,i}(q, \mathbf{h}, n) - \alpha^{-}_{p,i}(q, \mathbf{h}, n)).$$

*Proof.* For i), see [Si,vB] (or [Kn] for an adelic version), for (ii) in case $\mathbf{h} \in \mathbb{Z}^3$ see [Kn] and [SP2] (where also the precise definition of the $\alpha^{\pm}_p$ can be found). To modify the proof of [SP2] in order to include congruence conditions, replace $O(L_p)$ by $\{u \in O(L_p) \mid u\mathbf{x} \equiv \mathbf{x} \bmod N \cdot L_p$ for all $\mathbf{x} \in L_p\}$ and the condition $n_0 \mathbf{x} \in u_i L_p$ by $u_i^{-1} n_0 \mathbf{x} \in \mathbf{h} + L_p$. Note that explicit calculations of the $\alpha^{\pm}_{p,i}$ can be found in [SP2] for $p$ with $\mathbf{h} \in \mathbb{Z}_p^3$.

**Lemma 4.** *Let* $\vartheta(q, \mathbf{h}, z) = \sum_{n=0}^{\infty} r(q, \mathbf{h}, n) \exp(2\pi i n z)$,

$$\vartheta(\mathrm{gen}_N q, \mathbf{h}) = \sum_{n=0}^{\infty} r(\mathrm{gen}_N q, \mathbf{h}, n) \exp(2\pi i n z).$$

(i) $\vartheta(\mathrm{gen}_N q, \mathbf{h}) - \vartheta(q, \mathbf{h})$ *is a cusp form of weight* 3/2 *for the group* $\Gamma_1(N)$

(ii) *Let* $U$ *be the space of cusp forms of weight* 3/2 *for* $\Gamma_1(N)$ *generated by* 1-*dimensional theta series*

$$\sum \chi(n)\, n \exp(2\pi i t n^2 z)$$

*and* $U^{\perp}$ *its orthogonal complement with respect to the Petersson inner product. Then*

$$\vartheta(q, \mathbf{h}) - \vartheta(\mathrm{spn}_N q, \mathbf{h}) \in U^{\perp}$$

$$\vartheta(\mathrm{spn}_N q, \mathbf{h}) - \vartheta(\mathrm{gen}_N q, \mathbf{h}) \in U.$$

*Proof.* In case $\mathbf{h} \in \mathbb{Z}_p^3$ see [Si], p. 376 for i), [SP1] for ii). The proofs in the general case are analogous (using again the decomposition $S_k(\Gamma_1(N)) = \bigoplus_{\chi} S_k(\Gamma_0(N), \chi)$.

Note that by [Ci, Stu] $U^{\perp}$ consists of those cusp forms of weight 3/2 whose Shimura lifting is cuspidal.

Note also that the role played by the spinor genus in ii) is special to forms in 3 variables; for forms in more than 3 variables $\vartheta(\mathrm{spn}_N q, \mathbf{h}) = \vartheta(\mathrm{gen}_N q, \mathbf{h})$ by [Kn].

**Lemma 5.** (i) *Let* $n$ *tend to infinity in the set* $R_\nu(\mathrm{gen}_N q, \mathbf{h}) := \{n \in \mathbb{N} \mid \alpha_p(q, \mathbf{h}, n) \neq 0$ *for all primes* $p$, $p^\nu \nmid n$ *if* $q$ *is anisotropic over* $\mathbb{Q}_p\}$.

*Then for all* $\varepsilon > 0$

$$n^{1/2-\varepsilon} \ll_{\varepsilon, \nu, N} r(\mathrm{gen}_N q, \mathbf{h}, n) \ll_{\varepsilon, \nu, N} n^{1/2+\varepsilon}$$

(ii) *If* $n$ *is further restricted to the set* $R^*(\mathrm{spn}\, q, \mathbf{h}) = \{n \mid \exists q' \in \mathrm{spn}_N q$ *and* $\mathbf{x} \in \mathbf{h} + \mathbb{Z}^3$ *primitive with* $q'(\mathbf{x}) = n\}$ *then for all* $\varepsilon > 0$

$$n^{1/2-\varepsilon} \ll_{\varepsilon, N} r(\mathrm{spn}_N q, \mathbf{h}, n) \ll_{\varepsilon, N} n^{1/2+\varepsilon}.$$

(*Note that* $R^*(\mathrm{spn}\, q, \mathbf{h}) \subseteq R_\nu(\mathrm{gen}_N q, \mathbf{h})$ *for some* $\nu$ *depending on* $N$).

*Proof.* For a proof of i) in case $\mathbf{h} \in \mathbb{Z}^3$ see [Jo, Pe], where $r(\mathrm{gen}\, q, n)$ is bounded from above and below by a constant times the number of classes of primitive positive binary forms of determinant $2n \det A$ (which is estimated in [Si2]). Alternatively one can compute the local densities for almost all primes and find that $r(\mathrm{gen}_N q, \mathbf{h}, n)$ agrees up to a finite number of factors with $n^{1/2} L(1, \chi)$, where $\chi$ is the quadratic character associated with $\mathbb{Q}(\sqrt{-2n \det A})$, and again estimate $L(1, \chi)$ by [Si2]. Note that the lower bound from Siegel's estimate is not effective.

ii) has been shown in [SP2], Korollar 2 for $\mathbf{h} \in \mathbb{Z}_p^3$, again, the argument is unchanged for general $\mathbf{h}$.

Note also that the additional restriction on $n$ in ii) is satisfied for all $n$ not in one of finitely many square classes by the analogue of Theorem 2ii) for primitive representations ([Kn,SP2]), and that for $n$ in one of those exceptional square classes and $p \nmid N$ one has $n \in R^*(\mathrm{spn}_N q, \mathbf{h})$ if and only if $np^4 \in R^*(\mathrm{spn}_N q, \mathbf{h})$ [Ea].

Note finally that without the above restrictions, the lower bound given becomes wrong, since $r(\mathrm{gen}_N q, \mathbf{h}, np^{2r})$ becomes constant for $p$ with $q$ anisotropic over $\mathbb{Q}_p$ and $r$ large and $r(\mathrm{spn}_N q, \mathbf{h}, np^2)$ becomes constant for large $p$ (which are inert in a certain quadratic extension of $\mathbb{Q}$) if $n$ is representend primitively by $\mathrm{spn}_N q$ but not by all $\mathrm{spn}_N q'$ in $gen_N q$ [SP2, p. 131]. Putting together the above results we arrive at

**Theorem 3.** *Let $\varepsilon > 0$ be given, let $n \to \infty$ under the following restriction: If $\mathbf{h} \notin \mathbb{Z}^3$ and $N_2$ denotes the product of all $p \mid N$ for which $q$ is isotropic over $\mathbb{Q}_p$ and $\mathbf{h} \notin \mathbb{Z}_p^3$, and if $n = tm^2$ with $t$ squarefree, then $\gcd(m, N_2^\infty) \in S$ for some fixed finite set of integers $S$. Then*

(i) $$r(q, \mathbf{h}, n) = r(\mathrm{spn}_N q, \mathbf{h}, n) + O(n^{1/2 - 1/28 + \varepsilon}),$$

*where the main term satisfies*

$$r(\mathrm{spn}_N q, \mathbf{h}, n) \gg_{\varepsilon, N} n^{1/2 - \varepsilon} \ \text{ if } \ n \in R^*(\mathrm{spn}_N q, \mathbf{h})$$

*and can be explicitly computed by evaluating local densities.*

(ii) $$r(q, \mathscr{F}, \mathbf{h}, n) = r(\mathrm{spn}_N q, \mathbf{h}, n)\, \mu(\mathscr{F}) + O(n^{1/2 - 1/175 + \varepsilon}).$$

*The implied constants depend only on $\varepsilon$ and $N$.*

*Proof.* The second part of i) is Lemma 5 ii), the first part follows from Lemma 4 ii) and Lemma 2 i) if $n = tn_0^2 n_1^2$ with $(n_0, N) = 1$. The more general formulation given above can be deduced from Lemma 3–5 of [SP3] in the same way as in the proof of Lemma 3 of this article. The assertion of ii) finally follows from Theorem 1, using Lemma 5 ii).

If there are no congruence conditions, we can give the desired result about uniform distribution in a simple formulation:

**Corollary.** *Let $q(x_1, x_2, x_3)$ be a positive definite integral ternary quadratic form. Then every large integer $n$ represented primitively by a form in the spinor genus of $q$ is represented by $q$ itself and the representing vectors are asymptotically uniformly distributed on the ellipsoid $q(\mathbf{x}) = n$.*

## References


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