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Universal Quadratic Forms and the Fifteen Theorem

J. H. Conway

ABSTRACT. This paper is an extended foreword to the paper of Manjul Bhargava [1] in these proceedings, which gives a short and elegant proof of the Conway-Schneeberger Fifteen Theorem on the representation of integers by quadratic forms.

The representation theory of quadratic forms has a long history, starting in the seventeenth century with Fermat's assertions of 1640 about the numbers represented by $x^2 + y^2$. In the next century, Euler gave proofs of these and some similar assertions about other simple binary quadratics, and although these proofs had some gaps, they contributed greatly to setting the theory on a firm foundation.

Lagrange started the theory of universal quadratic forms in 1770 by proving his celebrated Four Squares Theorem, which in current language is expressed by saying that the form $x^2 + y^2 + z^2 + t^2$ is universal. The eighteenth century was closed by a considerably deeper statement -- Legendre's Three Squares Theorem of 1798; this found exactly which numbers needed all four squares. In his *Theorie des Nombres* of 1830, Legendre also created a very general theory of binary quadratics.

The new century was opened by Gauss's *Disquisitiones Arithmeticae* of 1801, which brought that theory to essentially its modern state. Indeed, when Neil Sloane and I wanted to summarize the classification theory of binary forms for one of our books [3], we found that the only Number Theory textbook in the Cambridge Mathematical Library that handled every case was still the *Disquisitiones*! Gauss's initial exploration of ternary quadratics was continued by his great disciple Eisenstein, while Dirichlet started the analytic theory by his class number formula of 1839.

As the nineteenth century wore on, other investigators, notably H. J. S. Smith and Hermann Minkowski, explored the application of Gauss's concept of the genus to higher-dimensional forms, and introduced some invariants for the genus from which in this century Hasse was able to obtain a complete

and very simple classification of rational quadratic forms based on Hensel's notion of " p -adic number", which has dominated the theory ever since.

In 1916, Ramanujan started the byway that concerns us here by asserting that

$$\begin{array}{cccccccc} [1, 1, 1, 1], & [1, 1, 1, 2], & [1, 1, 1, 3], & [1, 1, 1, 4], & [1, 1, 1, 5], & [1, 1, 1, 6], & [1, 1, 1, 7], & \\ [1, 1, 2, 2], & [1, 1, 2, 3], & [1, 1, 2, 4], & [1, 1, 2, 5], & [1, 1, 2, 6], & [1, 1, 2, 7], & [1, 1, 2, 8], & \\ [1, 1, 2, 9], & [1, 1, 2, 10], & [1, 1, 2, 11], & [1, 1, 2, 12], & [1, 1, 2, 13], & [1, 1, 2, 14], & [1, 1, 3, 3], & \\ [1, 1, 3, 4], & [1, 1, 3, 5], & [1, 1, 3, 6], & [1, 2, 2, 2], & [1, 2, 2, 3], & [1, 2, 2, 4], & [1, 2, 2, 5], & \\ [1, 2, 2, 6], & [1, 2, 2, 7], & [1, 2, 3, 3], & [1, 2, 3, 4], & [1, 2, 3, 5], & [1, 2, 3, 6], & [1, 2, 3, 7], & \\ [1, 2, 3, 8], & [1, 2, 3, 9], & [1, 2, 3, 10], & [1, 2, 4, 4], & [1, 2, 4, 5], & [1, 2, 4, 6], & [1, 2, 4, 7], & \\ [1, 2, 4, 8], & [1, 2, 4, 9], & [1, 2, 4, 10], & [1, 2, 4, 11], & [1, 2, 4, 12], & [1, 2, 4, 13], & [1, 2, 4, 14], & \\ [1, 2, 5, 5], & [1, 2, 5, 6], & [1, 2, 5, 7], & [1, 2, 5, 8], & [1, 2, 5, 9], & [1, 2, 5, 10] & & \end{array}$$

were all the diagonal quaternary forms that were universal in the sense appropriate to positive-definite forms, that is, represented every positive integer. In the rest of this paper, "form" will mean "positive-definite quadratic form", and "universal" will mean "universal in the above sense".

Although Ramanujan's assertion later had to be corrected slightly by the elision of the diagonal form $[1, 2, 5, 5]$, it aroused great interest in the problem of enumerating all the universal quaternary forms, which was eagerly taken up, by Gordon Pall and his students in particular. In 1940, Pall also gave a complete system of invariants for the genus, while simultaneously Burton Jones found a system of canonical forms for it, so giving two equally definitive solutions for a problem raised by Smith in 1851.

There are actually two universal quadratic form problems, according to the definition of "integral" that one adopts. The easier one is that for Gauss's notion, according to which a form is integral only if not only are all its coefficients integers, but the off-diagonal ones are even. This is sometimes called "classically integral", but we prefer to use the more illuminating term "integer-matrix", since what is required is that the matrix of the form be comprised of integers. The difficult universality problem is that for the alternative notion introduced by Legendre, under which a form is integral merely if all its coefficients are. We describe such a form as "integer-valued", since the condition is precisely that all the values taken by the form are integers, and remark that this kind of integrality is the one most appropriate for the universality problem, since that is about the values of forms.

For nearly 50 years it has been supposed that the universality problem for quaternary integer-matrix forms had been solved by M. Willerding, who purported to list all such forms in 1948. However, the 15-theorem, which I proved with William Schneeberger in 1993, made it clear that Willerding's work had been unusually defective. In his paper in these proceedings, Manjul Bhargava [1] gives a very simple proof of the 15-theorem, and derives the complete list of universal quaternaries. As he remarks, of the 204 such forms, Willerding's purportedly complete list of 178 contains in fact only 168, because she missed 36 forms, listed 1 form twice, and listed 9 non-universal forms!

The 15-theorem closes the universality problem for integer-matrix forms by providing an extremely simple criterion. We no longer need a list of universal quaternaries, because a form is universal provided only that it represent the numbers up to 15. Moreover, this criterion works for larger numbers of variables, where the number of universal forms is no longer finite. (It is known that no form in three or fewer variables can be universal.)

I shall now briefly describe the history of the 15-theorem. In a 1993 Princeton graduate course on quadratic forms, I remarked that a reworking of Willerding's enumeration was very desirable, and could probably be achieved very easily in view of recent advances in the representation theory of quadratic forms, most particularly the work of Duke and Schultze-Pillot. Moreover, it was an easy consequence of this work that there must be a constant c with the property that if a matrix-integral form represented every positive integer up to c , then it was universal, and a similar but probably larger constant C for integer-valued forms. At that time, I feared that perhaps these constants would be very large indeed, but fortunately it appeared that they are quite small.

I started the next lecture by saying that we might try to find c , and wrote on the board a putative

THEOREM 0.1. If an integer-matrix form represents every positive integer up to c (to be found!) then it is universal.

We started to prove that theorem, and by the end of the lecture had found the 9 ternary "escalator" forms (see Bhargava's article [1] for their definition) and realised that we could almost as easily find the quaternary ones, and made it seem likely that c was much smaller than we had expected.

In the afternoon that followed, several class members, notably William Schneeberger and Christopher Simons, took the problem further by producing these forms and exploring their universality by machine. These calculations strongly suggested that c was in fact 15.

In subsequent lectures we proved that most of the 200+ quaternaries we had found were universal, so that when I had to leave for a meeting in Boston only nine particularly recalcitrant ones remained. In Boston I tackled seven of these, and when I returned to Princeton, Schneeberger and I managed to polish the remaining two off, and then complete this to a proof of the 15-theorem, modulo some computer calculations that were later done by Simons.

The arguments made heavy use of the notion of genus, which had enabled the nineteenth-century workers to extend Legendre's Three Squares theorem to other ternary forms. In fact the 15-theorem largely reduces to proving a number of such analogues of Legendre's theorem. Expressing the arguments was greatly simplified by my own symbol for the genus, which was originally derived by comparing Pall's invariants with Jones's canonical forms, although it has since been established more simply; see for instance my recent little book [2].

Our calculations also made it clear that the larger constant C for the integer-valued problem would almost certainly be 290, though obtaining a proof of the resulting “290-conjecture” would be very much harder indeed. Last year, in one of our semi-regular conversations I tempted Manjul Bhargava into trying his hand at the difficult job of proving the 290-conjecture.

Manjul started the task by reproving the 15-theorem, and now he has discovered the particularly simple proof he gives in the following paper, which has made it unnecessary for us to publish our rather more complicated proof. Manjul has also proved the “33-theorem” – much more difficult than the 15-theorem – which asserts that an integer-matrix form will represent all odd numbers provided only that it represents 1, 3, 5, 7, 11, 15, and 33. This result required the use of some very clever and subtle arithmetic arguments.

Finally, using these arithmetic arguments, as well as new analytic techniques, Manjul has made significant progress on the 290-conjecture, and I would not be surprised if the conjecture were to be finished off in the near future! He intends to publish these and other related results in a subsequent paper.

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On the Conway-Schneeberger Fifteen Theorem

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ABSTRACT. This paper gives a proof of the Conway-Schneeberger Fifteen Theorem on the representation of integers by quadratic forms, to which the paper of Conway [1] in these proceedings is an extended foreword.

1. Introduction. In 1993, Conway and Schneeberger announced the following remarkable result:

THEOREM 1 (“The Fifteen Theorem”). *If a positive-definite quadratic form having integer matrix represents every positive integer up to 15 then it represents every positive integer.*

The original proof of this theorem was never published, perhaps because several of the cases involved rather intricate arguments. A sketch of this original proof was given by Schneeberger in [4]; for further background and a brief history of the Fifteen Theorem, see Professor Conway’s article [1] in these proceedings.

The purpose of this paper is to give a short and direct proof of the Fifteen Theorem. Our proof is in spirit much the same as that of the original unpublished arguments of Conway and Schneeberger; however, we are able to treat the various cases more uniformly, thereby obtaining a significantly simplified proof.

2. Preliminaries. The Fifteen Theorem deals with quadratic forms that are positive-definite and have integer matrix. As is well-known, there is a natural bijection between classes of such forms and lattices having integer inner products; precisely, a quadratic form f can be regarded as the inner product form for a corresponding lattice $L(f)$. Hence we shall oscillate freely between the language of forms and the language of lattices. For brevity, by a “form” we shall always mean a positive-definite quadratic form having integer matrix, and by a “lattice” we shall always mean a lattice having integer inner products.

A form (or its corresponding lattice) is said to be *universal* if it represents every positive integer. If a form f happens not to be universal, define the

truant of f (or of its corresponding lattice $L(f)$) to be the smallest positive integer not represented by f .

Important in the proof of the Fifteen Theorem is the notion of “escalator lattice.” An *escalation* of a nonuniversal lattice L is defined to be any lattice which is generated by L and a vector whose norm is equal to the truant of L . An *escalator lattice* is a lattice which can be obtained as the result of a sequence of successive escalations of the zero-dimensional lattice.

3. Small-dimensional Escalators. The unique escalation of the zero-dimensional lattice is the lattice generated by a single vector of norm 1. This lattice corresponds to the form x^2 (or, in matrix form, $[1]$) which fails to represent the number 2. Hence an escalation of $[1]$ has inner product matrix of the form

$$\begin{bmatrix} 1 & a \\ a & 2 \end{bmatrix}.$$

By the Cauchy-Schwartz inequality, $a^2 \leq 2$, so a equals either 0 or ± 1 . The choices $a = \pm 1$ lead to isometric lattices, so we obtain only two nonisometric two-dimensional escalators, namely those lattices having Minkowski-reduced Gram matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

If we escalate each of these two-dimensional escalators in the same manner, we find that we obtain exactly 9 new nonisometric escalator lattices, namely those having Minkowski-reduced Gram matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 5 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Escalating now each of these nine three-dimensional escalators, we find exactly 207 nonisomorphic four-dimensional escalator lattices. All such lattices are of the form $[1] \oplus L$, and the 207 values of L are listed in Table 3.

When attempting to carry out the escalation process just once more, however, we find that many of the 207 four-dimensional lattices do not escalate (i.e., they are universal). For instance, one of the four-dimensional escalators turns out to be the lattice corresponding to the famous four squares form, $a^2 + b^2 + c^2 + d^2$, which is classically known to represent all integers. The question arises: how many of the four-dimensional escalators are universal?

4. Four-dimensional Escalators. In this section, we prove that in fact 201 of the 207 four-dimensional escalator lattices are universal; that is to say, only 6 of the four-dimensional escalators can be escalated once again.

The proof of universality of these 201 lattices proceeds as follows. In each such four-dimensional lattice L_4 , we locate a 3-dimensional sublattice

L_3 which is known to represent some large set of integers. Typically, we simply choose L_3 to be unique in its genus; in that case, L_3 represents all integers that it represents locally (i.e., over each p -adic ring \mathbb{Z}_p). Armed with this knowledge of L_3 , we then show that the direct sum of L_3 with its orthogonal complement in L_4 represents all sufficiently large integers $n \geq N$. A check of representability of n for all $n < N$ finally reveals that L_4 is indeed universal.

To see this argument in practice, we consider in detail the escalations L_4 of the escalator lattice $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (labelled (4) in Table 1). The latter 3-dimensional lattice L_3 is unique in its genus, so a quick local calculation shows that it represents all positive integers not of the form $2^e(8k+7)$, where e is even. Let the orthogonal complement of L_3 in L_4 have Gram matrix $[m]$. We wish to show that $L_3 \oplus [m]$ represents all sufficiently large integers.

To this end, suppose L_4 is not universal, and let u be the first integer not represented by L_4 . Then, in particular, u is not represented by L_3 , so u must be of the form $2^e(8k+7)$. Moreover, u must be squarefree; for if $u = rt^2$ with $t > 1$, then $r = u/t^2$ is also not represented by L_4 , contradicting the minimality of u . Therefore $e = 0$, and we have $u \equiv 7 \pmod{8}$.

Now if $m \not\equiv 0, 3$ or $7 \pmod{8}$, then clearly $u - m$ is not of the form $2^e(8k+7)$. Similarly, if $m \equiv 3$ or $7 \pmod{8}$, then $u - 4m$ cannot be of the form $2^e(8k+7)$. Thus if $m \not\equiv 0 \pmod{8}$, and given that $u \geq 4m$, then either $u - m$ or $u - 4m$ is represented by L_3 ; that is, u is represented by $L_3 \oplus [m]$ (a sublattice of L_4) for $u \geq 4m$. An explicit calculation shows that m never exceeds 28, and a computer check verifies that every escalation L_4 of L_3 represents all integers less than $4 \times 28 = 112$. It follows that any escalator L_4 arising from L_3 , for which the value of m is not a multiple of 8, is universal.

Of course, the argument fails for those L_4 for which m is a multiple of 8. We call such an escalation "exceptional". Fortunately, such exceptional escalations are few and far between, and are easily handled. For instance, an explicit calculation shows that only two escalations of $L_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ are exceptional (while the other 24 are not); these exceptional cases are listed in Table 2.1. As is also indicated in the table, although these lattices did escape our initial attempt at proof, the universality of these four-dimensional lattices L_4 is still not any more difficult to prove; we simply change the sublattice L_3 from the escalator lattice $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ to the ones listed in the table, and apply the same argument!

It turns out that all of the 3-dimensional escalator lattices listed in Table 1, except for the one labeled (6), are unique in their genus, so the universality of their escalations can be proved by essentially identical arguments, with just a few exceptions. As for escalator (6), although not unique in its

genus, it does represent all numbers locally represented by it except possibly those which are 7 or 10 (mod 12). Indeed, this escalator contains the lattice

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 8 \end{bmatrix}, \text{ which is unique in its genus, and the lattices } \begin{bmatrix} 2 & -2 & 2 \\ -2 & 5 & 2 \\ 2 & 2 & 8 \end{bmatrix} \text{ and } \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 4 \\ 0 & 4 & 5 \end{bmatrix}, \text{ which together form a genus; a local check shows that the first}$$

genus represents all numbers locally represented by escalator (6) which are not congruent to 2 or 3 (mod 4), while the second represents all such numbers not congruent to 1 (mod 3). The desired conclusion follows. (This fact has been independently proven by Kaplansky [3] using different methods.)

Knowing this, we may now proceed with essentially the same arguments on the escalations of $L_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 4 \end{bmatrix}$. The relevant portions of the proofs for all nonexceptional cases are summarized in Table 1.

“Exceptional” cases arise only for escalators (4) (as we have already seen), (6), and (7). Two arise for escalator (4). Although four arise for escalator (6), two of them turn out to be nonexceptional escalations of (1) and (8) respectively, and hence have already been handled. Similarly, two arise for escalator (7), but one is a nonexceptional escalation of (9). Thus only five truly exceptional four-dimensional escalators remain, and these are listed in Table 2. In these five exceptional cases, other three-dimensional sublattices unique in their genus are given for which essentially identical arguments work in proving universality. Again, all the relevant information is provided in Table 2.

5. Five-dimensional Escalators. As mentioned earlier, there are 6 four-dimensional escalators which escalate again; they have been italicized in Table 3 and are listed again in the first column of Table 4. A rather large calculation shows that these 6 four-dimensional lattices escalate to an additional 1630 five-dimensional escalators! With a bit of fear we may ask again whether any of these five-dimensional escalators escalate.

Fortunately, the answer is no; all five-dimensional escalators are universal. The proof is much the same as the proof of universality of the four-dimensional escalators, but easier. We simply observe that, for the 6 four-dimensional nonuniversal escalators, all parts of the proof of universality outlined in the second paragraph of Section 4 go through—except for the final check. The final check then reveals that each of these 6 lattices represent every positive integer *except for one single number* n . Hence once a single vector of norm n is inserted in such a lattice, the lattice must automatically become universal. Therefore all five-dimensional escalators are universal. A list of the 6 nonuniversal four-dimensional lattices, together with the single numbers they fail to represent, is given in Table 4.

Since no five-dimensional escalator can be escalated, it follows that there are only finitely many escalator lattices: 1 of dimension zero, 1 of dimension

one, 2 of dimension two, 9 of dimension three, 207 of dimension four, and 1630 of dimension five, for a total of 1850.

6. Remarks on the Fifteen Theorem. It is now obvious that

(i) *Any universal lattice L contains a universal sublattice of dimension at most five.*

For we can construct an escalator sequence $0 = L_0 \subseteq L_1 \subseteq \dots$ within L , and then from Sections 4 and 5, we see that either L_4 or (when defined) L_5 gives a universal escalator sublattice of L .

Our next remark includes the Fifteen Theorem.

(ii) *If a positive-definite quadratic form having integer matrix represents the nine critical numbers 1, 2, 3, 5, 6, 7, 10, 14, and 15, then it represents every positive integer.*

(Equivalently, the truant of any nonuniversal form must be one of these nine numbers.)

This is because examination of the proof shows that only these numbers arise as truant of escalator lattices.

We note that Remark (ii) is the best possible statement of the Fifteen Theorem, in the following sense.

(iii) *If t is any one of the above critical numbers, then there is a quaternary diagonal form that fails to represent t , but represents every other positive integer.*

Nine such forms of minimal determinant are $[2, 2, 3, 4]$ with truant 1, $[1, 3, 3, 5]$ with truant 2, $[1, 1, 4, 6]$ with truant 3, $[1, 2, 6, 6]$ with truant 5, $[1, 1, 3, 7]$ with truant 6, $[1, 1, 1, 9]$ with truant 7, $[1, 2, 3, 11]$ with truant 10, $[1, 1, 2, 15]$ with truant 14, and $[1, 2, 5, 5]$ with truant 15.

However, there is another slight strengthening of the Fifteen Theorem, which shows that the number 15 is rather special:

(iv) *If a positive-definite quadratic form having integer matrix represents every number below 15, then it represents every number above 15.*

This is because there are only four escalator lattices having truant 15, and as was shown in Section 5, each of these four escalators represents every number greater than 15.

Fifteen is the smallest number for which Remark (iv) holds. In fact:

(v) *There are forms which miss infinitely many integers starting from any of the eight critical numbers not equal to 15.*

Indeed, in each case one may simply take an appropriate escalator lattice of dimension one, two, or three.

(vi) *There are exactly 204 universal quaternary forms.*

An upper bound for the discriminant of such a form is easily determined; a systematic use of the Fifteen Theorem then yields the desired result. We note that the enumeration of universal quaternary forms was announced previously in the well-known work of Willerding [5], who found that there are exactly 178 universal quaternary forms; however, a comparison with our tables shows that she missed 36 universal forms, listed one universal form twice, and listed 9 non-universal forms. A list of all 204 universal quaternary forms is given in Table 5; the three entries not appearing among the list of escalators in Table 3 have been italicized.

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	Three-dimensional escalator lattice	Represents nos.			Subtract	Check up to
		Truant	not of the form*	If m		
(1)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	7	$2^e u_7$	$\not\equiv 0 \pmod{8}$ $\equiv 0 \pmod{8}$	m or $4m$ does not arise	112 -
(2)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	14	$2^d u_7$	$\not\equiv 0 \pmod{16}$ $\equiv 0 \pmod{16}$	m or $4m$ does not arise	224 -
(3)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$	6	$3^d u_-$	$\not\equiv 0 \pmod{9}$ $\equiv 0 \pmod{9}$	m , $4m$, or $16m$ does not arise	864 -
(4)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	7	$2^e u_7$	$\not\equiv 0 \pmod{8}$ $\equiv 0 \pmod{8}$	m or $4m$ [See Table 2]	112 -
(5)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$	10	$2^d u_5$	$\not\equiv 0 \pmod{16}$ $\equiv 0 \pmod{16}$	m or $4m$ does not arise	1440 -
(6)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 4 \end{bmatrix}$	7	$7^d u_-$ or $7, 10 \pmod{12}$	$\not\equiv 0, 3, 9 \pmod{12}$ $\not\equiv 0 \pmod{49}$ $\equiv 0 \pmod{49}$ $\equiv 0, 3, 9 \pmod{12}$	m , $4m$, or $9m$ does not arise [See Table 2]	3087 - -
(7)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$	14	$2^d u_7$	$\not\equiv 0 \pmod{16}$ $\equiv 0 \pmod{8}$	m or $4m$ [See Table 2]	224 -
(8)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 5 \end{bmatrix}$	7	$2^e u_7$	$\not\equiv 0 \pmod{8}$ $\equiv 0 \pmod{8}$	m or $4m$ does not arise	252 -
(9)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$	10	$5^d u_-$	$\not\equiv 0 \pmod{25}$ $\equiv 0 \pmod{25}$	m or $4m$ does not arise	4000 -

Table 1. Proof of universality of four-dimensional escalators
(nonexceptional cases)

*We follow the notation of Conway-Sloane [2]: p^d (resp. p^e) denotes an odd (resp. even) power of p ; if $p = 2$, u_k denotes a number of the form $8n + k$ ($k = 1, 3, 5, 7$), and if p is odd, u_+ (resp. u_-) denotes a number which is a quadratic residue (resp. non-residue) modulo p .

<u>"Exceptional"</u> <u>Lattice</u>	<u>New unique in</u> <u>genus sublattice</u>	<u>Unrepresented</u> <u>numbers</u>	<u>m</u>	<u>Subtract</u>	<u>Check</u> <u>up to</u>
$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 2 & 1 & 1 & 7 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$	$5^d u_+$	40	m or $4m$	160
$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 7 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 7 \end{bmatrix}$	$2^e u_1, 2^e u_5,$ $2^d u_3, 2^d u_7, 3^d u_+$	1	m	14
$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 4 & 3 \\ 1 & 0 & 3 & 7 \end{bmatrix}$	$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 4 \end{bmatrix}$	$2^d u_7$	1	$m, 4m,$ or $9m$	9
$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 1 & 0 & 7 \end{bmatrix}^\dagger$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 10 \end{bmatrix}$	$2^d u_7$	90	m or $4m$	504
$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 1 & 0 & 2 & 14 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 13 \end{bmatrix}$	$2^d u_5, 2^e u_3$	2	m or $4m$	8

Table 2. Proof of universality of four-dimensional escalators (exceptional cases)

[†]In this exceptional case, the sublattice given here shows only that all even numbers are represented. However, the original argument of Table 1 (using escalator (6) as sublattice, with $m = 315$) shows that all odd numbers are represented, so the desired universality follows. [It turns out there is *no* sublattice unique in its genus that single-handedly proves universality in this case!]

1: 1 1 1 0 0 0	16: 2 3 3 2 0 0	30: 2 4 4 2 0 0	49: 2 3 9 2 2 0	72: 2 5 8 4 0 0
2: 1 1 2 0 0 0	17: 1 2 9 2 0 0	31: 2 3 6 2 2 0	49: 2 4 7 0 0 2	74: 2 4 10 2 2 0
3: 1 1 3 0 0 0	17: 1 3 6 2 0 0	31: 2 4 5 0 2 2	49: 2 5 6 0 2 2	76: 2 4 10 0 2 0
3: 1 2 2 2 0 0	17: 2 3 4 0 2 2	32: 2 4 4 0 0 0	50: 2 4 7 2 2 0	77: 2 5 9 4 2 0
4: 1 1 4 0 0 0	18: 1 2 9 0 0 0	32: 2 4 5 4 0 0	<i>50: 2 5 5 0 0 0</i>	78: 2 4 10 2 0 0
4: 1 2 2 0 0 0	18: 1 3 6 0 0 0	33: 2 3 6 0 2 0	51: 2 3 9 0 2 0	78: 2 5 8 2 0 0
4: 2 2 2 2 0 0	18: 2 2 5 2 0 0	<i>33: 2 4 5 2 0 2</i>	52: 2 3 9 2 0 0	80: 2 4 10 0 0 0
5: 1 1 5 0 0 0	18: 2 3 3 0 0 0	34: 2 3 6 2 0 0	52: 2 5 6 2 0 2	80: 2 4 11 4 0 0
5: 1 2 3 2 0 0	18: 2 3 4 2 0 2	34: 2 4 5 2 2 0	52: 2 5 6 4 0 0	80: 2 5 8 0 0 0
6: 1 1 6 0 0 0	19: 1 2 10 2 0 0	34: 2 4 6 4 0 2	53: 2 5 6 2 2 0	82: 2 4 11 2 2 0
6: 1 2 3 0 0 0	19: 2 3 4 2 2 0	35: 2 4 5 0 0 2	54: 2 3 9 0 0 0	82: 2 5 9 4 0 0
6: 2 2 2 2 0 0	20: 1 2 10 0 0 0	36: 2 3 6 0 0 0	54: 2 4 7 2 0 0	<i>83: 2 5 9 2 2 0</i>
7: 1 1 7 0 0 0	20: 2 2 5 0 0 0	36: 2 4 5 0 2 0	54: 2 5 6 0 0 2	85: 2 5 9 0 2 0
7: 1 2 4 2 0 0	20: 2 2 6 2 2 0	36: 2 4 6 4 2 0	54: 2 5 7 4 2 2	86: 2 4 11 2 0 0
7: 2 2 3 2 0 2	20: 2 4 4 4 2 0	36: 2 5 5 4 2 2	55: 2 3 10 2 2 0	87: 2 5 10 4 2 0
8: 1 2 4 0 0 0	<i>21: 2 3 4 0 2 0</i>	37: 2 5 5 4 2 0	55: 2 5 6 0 2 0	88: 2 4 11 0 0 0
8: 1 3 3 2 0 0	22: 1 2 11 0 0 0	38: 2 4 5 2 0 0	55: 2 5 7 4 0 2	88: 2 4 12 4 0 0
8: 2 2 2 0 0 0	22: 2 2 6 2 0 0	38: 2 4 6 0 2 2	56: 2 4 7 0 0 0	88: 2 5 9 2 0 0
8: 2 2 3 2 2 0	22: 2 3 4 2 0 0	39: 2 3 7 0 2 0	56: 2 4 8 4 0 0	90: 2 4 12 2 2 0
9: 1 2 5 2 0 0	22: 2 3 5 0 2 2	40: 2 3 7 2 0 0	57: 2 3 10 0 2 0	90: 2 5 9 0 0 0
9: 1 3 3 0 0 0	23: 1 2 12 2 0 0	40: 2 4 5 0 0 0	58: 2 3 10 2 0 0	92: 2 4 13 4 2 0
9: 2 2 3 0 0 2	23: 2 3 5 2 0 2	40: 2 4 6 2 0 2	58: 2 4 8 2 2 0	92: 2 5 10 4 0 0
10: 1 2 5 0 0 0	24: 1 2 12 0 0 0	40: 2 4 6 4 0 0	58: 2 5 6 2 0 0	93: 2 5 10 2 2 0
10: 2 2 3 2 0 0	24: 2 2 6 0 0 0	41: 2 4 7 4 0 2	58: 2 5 7 0 2 2	94: 2 4 12 2 0 0
10: 2 2 4 2 0 2	24: 2 2 7 2 2 0	42: 2 3 7 0 0 0	60: 2 3 10 0 0 0	95: 2 5 10 0 2 0
11: 1 2 6 2 0 0	24: 2 3 4 0 0 0	42: 2 4 6 0 0 2	60: 2 4 9 4 2 0	96: 2 4 12 0 0 0
11: 1 3 4 2 0 0	24: 2 4 4 0 2 2	42: 2 4 6 2 2 0	60: 2 5 6 0 0 0	96: 2 4 13 4 0 0
12: 1 2 6 0 0 0	24: 2 4 4 4 0 0	42: 2 5 5 4 0 0	61: 2 5 7 2 0 2	98: 2 4 13 2 2 0
12: 1 3 4 0 0 0	25: 1 2 13 2 0 0	43: 2 3 8 2 2 0	62: 2 4 8 2 0 0	98: 2 5 10 2 0 0
12: 2 2 3 0 0 0	25: 2 3 5 2 2 0	<i>43: 2 5 5 2 0 2</i>	62: 2 5 7 4 0 0	100: 2 4 13 0 2 0
12: 2 2 4 0 0 2	26: 1 2 13 0 0 0	44: 2 4 6 0 2 0	63: 2 5 7 0 0 2	100: 2 4 14 4 2 0
13: 2 2 5 2 0 2	26: 2 2 7 2 0 0	45: 2 4 7 0 2 2	63: 2 5 7 2 2 0	100: 2 5 10 0 0 0
13: 2 3 3 2 2 0	26: 2 4 4 2 2 0	45: 2 5 5 0 2 0	64: 2 4 8 0 0 0	102: 2 4 13 2 0 0
14: 1 2 7 0 0 0	27: 1 2 14 2 0 0	45: 2 5 6 4 2 2	66: 2 4 9 2 2 0	104: 2 4 13 0 0 0
14: 1 3 5 2 0 0	27: 2 3 5 0 2 0	46: 2 3 8 2 0 0	<i>67: 2 5 8 4 2 0</i>	104: 2 4 14 4 0 0
14: 2 2 4 2 0 0	27: 2 4 5 4 0 2	46: 2 4 6 2 0 0	68: 2 4 9 0 2 0	106: 2 4 14 2 2 0
15: 1 2 8 2 0 0	28: 1 2 14 0 0 0	46: 2 5 6 4 0 2	68: 2 4 10 4 2 0	108: 2 4 14 0 2 0
15: 1 3 5 0 0 0	28: 2 2 7 0 0 0	47: 2 4 7 2 0 2	68: 2 5 7 2 0 0	110: 2 4 14 2 0 0
15: 2 2 5 0 0 2	28: 2 3 5 2 0 0	47: 2 5 6 4 2 0	70: 2 4 9 2 0 0	112: 2 4 14 0 0 0
15: 2 3 3 0 2 0	28: 2 4 4 0 2 0	48: 2 3 8 0 0 0	70: 2 5 7 0 0 0	
16: 1 2 8 0 0 0	28: 2 4 5 4 2 0	48: 2 4 6 0 0 0	72: 2 4 9 0 0 0	
16: 2 2 4 0 0 0	30: 2 3 5 0 0 0	48: 2 5 5 2 0 0	72: 2 4 10 4 0 0	

Table 3. Ternary forms[†] L such that $[1] \oplus L$ is an escalator.

(The six entries not appearing in Table 5 have been italicized.)

[†]We use the customary shorthand “ $D: a b c d e f$ ” to represent the three-dimensional lattice $\begin{bmatrix} a & f/2 & e/2 \\ f/2 & b & d/2 \\ e/2 & d/2 & c \end{bmatrix}$ of determinant D .

Nonuniversal four-dimensional escalator Unique number not represented

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 4 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

10

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

15

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 5 & 2 \\ 0 & 1 & 2 & 8 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 5 & 1 \\ 0 & 1 & 1 & 9 \end{bmatrix}$$

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Table 4. Nonuniversal four-dimensional escalator lattices.
(The 1630 five-dimensional escalators are obtained from these.)

1: 1 1 1 0 0 0	16: 1 2 8 0 0 0	28: 2 3 5 2 0 0	48: 2 3 8 0 0 0	72: 2 4 9 0 0 0
2: 1 1 2 0 0 0	16: 2 2 4 0 0 0	28: 2 4 4 0 2 0	48: 2 4 6 0 0 0	72: 2 4 10 4 0 0
3: 1 1 3 0 0 0	16: 2 3 3 2 0 0	28: 2 4 5 4 2 0	48: 2 5 5 2 0 0	72: 2 5 8 4 0 0
3: 1 2 2 2 0 0	17: 1 2 9 2 0 0	30: 2 3 5 0 0 0	49: 2 3 9 2 2 0	74: 2 4 10 2 2 0
4: 1 1 4 0 0 0	17: 1 3 6 2 0 0	30: 2 4 4 2 0 0	49: 2 4 7 0 0 2	76: 2 4 10 0 2 0
4: 1 2 2 0 0 0	17: 2 3 4 0 2 2	31: 2 3 6 2 2 0	49: 2 5 6 0 2 2	77: 2 5 9 4 2 0
4: 2 2 2 2 2 0	18: 1 2 9 0 0 0	31: 2 4 5 0 2 2	50: 2 4 7 2 2 0	78: 2 4 10 2 0 0
5: 1 1 5 0 0 0	18: 1 3 6 0 0 0	32: 2 4 4 0 0 0	51: 2 3 9 0 2 0	78: 2 5 8 2 0 0
5: 1 2 3 2 0 0	18: 2 2 5 2 0 0	32: 2 4 5 4 0 0	52: 2 3 9 2 0 0	80: 2 4 10 0 0 0
6: 1 1 6 0 0 0	18: 2 3 3 0 0 0	33: 2 3 6 0 2 0	52: 2 5 6 2 0 2	80: 2 4 11 4 0 0
6: 1 2 3 0 0 0	18: 2 3 4 2 0 2	34: 2 3 6 2 0 0	52: 2 5 6 4 0 0	80: 2 5 8 0 0 0
6: 2 2 2 2 0 0	19: 1 2 10 2 0 0	34: 2 4 5 2 2 0	53: 2 5 6 2 2 0	82: 2 4 11 2 2 0
7: 1 1 7 0 0 0	19: 2 3 4 2 2 0	34: 2 4 6 4 0 2	54: 2 3 9 0 0 0	82: 2 5 9 4 0 0
7: 1 2 4 2 0 0	20: 1 2 10 0 0 0	35: 2 4 5 0 0 2	54: 2 4 7 2 0 0	85: 2 5 9 0 2 0
7: 2 2 3 2 0 2	20: 2 2 5 0 0 0	36: 2 3 6 0 0 0	54: 2 5 6 0 0 2	86: 2 4 11 2 0 0
8: 1 2 4 0 0 0	20: 2 2 6 2 2 0	36: 2 4 5 0 2 0	54: 2 5 7 4 2 2	87: 2 5 10 4 2 0
8: 1 3 3 2 0 0	<i>20: 2 3 4 0 0 2</i>	36: 2 4 6 4 2 0	55: 2 3 10 2 2 0	88: 2 4 11 0 0 0
8: 2 2 2 0 0 0	20: 2 4 4 4 2 0	36: 2 5 5 4 2 2	55: 2 5 6 0 2 0	88: 2 4 12 4 0 0
8: 2 2 3 2 2 0	22: 1 2 11 0 0 0	37: 2 5 5 4 2 0	55: 2 5 7 4 0 2	88: 2 5 9 2 0 0
9: 1 2 5 2 0 0	22: 2 2 6 2 0 0	38: 2 4 5 2 0 0	56: 2 4 7 0 0 0	90: 2 4 12 2 2 0
9: 1 3 3 0 0 0	22: 2 3 4 2 0 0	38: 2 4 6 0 2 2	56: 2 4 8 4 0 0	90: 2 5 9 0 0 0
9: 2 2 3 0 0 2	22: 2 3 5 0 2 2	39: 2 3 7 0 2 0	57: 2 3 10 0 2 0	92: 2 4 13 4 2 0
10: 1 2 5 0 0 0	23: 1 2 12 2 0 0	40: 2 3 7 2 0 0	58: 2 3 10 2 0 0	92: 2 5 10 4 0 0
10: 2 2 3 2 0 0	23: 2 3 5 2 0 2	40: 2 4 5 0 0 0	58: 2 4 8 2 2 0	93: 2 5 10 2 2 0
10: 2 2 4 2 0 2	24: 1 2 12 0 0 0	40: 2 4 6 2 0 2	58: 2 5 6 2 0 0	94: 2 4 12 2 0 0
11: 1 2 6 2 0 0	24: 2 2 6 0 0 0	40: 2 4 6 4 0 0	58: 2 5 7 0 2 2	95: 2 5 10 0 2 0
11: 1 3 4 2 0 0	24: 2 2 7 2 2 0	41: 2 4 7 4 0 2	60: 2 3 10 0 0 0	96: 2 4 12 0 0 0
12: 1 2 6 0 0 0	24: 2 3 4 0 0 0	42: 2 3 7 0 0 0	60: 2 4 9 4 2 0	96: 2 4 13 4 0 0
12: 1 3 4 0 0 0	24: 2 4 4 0 2 2	42: 2 4 6 0 0 2	60: 2 5 6 0 0 0	98: 2 4 13 2 2 0
12: 2 2 3 0 0 0	24: 2 4 4 4 0 0	42: 2 4 6 2 2 0	61: 2 5 7 2 0 2	98: 2 5 10 2 0 0
12: 2 2 4 0 0 2	25: 1 2 13 2 0 0	42: 2 5 5 4 0 0	62: 2 4 8 2 0 0	100: 2 4 13 0 2 0
<i>12: 2 3 3 0 2 2</i>	<i>25: 2 3 5 0 0 2</i>	43: 2 3 8 2 2 0	62: 2 5 7 4 0 0	100: 2 4 14 4 2 0
13: 2 2 5 2 0 2	25: 2 3 5 2 2 0	44: 2 4 6 0 2 0	63: 2 5 7 0 0 2	100: 2 5 10 0 0 0
13: 2 3 3 2 2 0	26: 1 2 13 0 0 0	45: 2 4 7 0 2 2	63: 2 5 7 2 2 0	102: 2 4 13 2 0 0
14: 1 2 7 0 0 0	26: 2 2 7 2 0 0	45: 2 5 5 0 2 0	64: 2 4 8 0 0 0	104: 2 4 13 0 0 0
14: 1 3 5 2 0 0	26: 2 4 4 2 2 0	45: 2 5 6 4 2 2	66: 2 4 9 2 2 0	104: 2 4 14 4 0 0
14: 2 2 4 2 0 0	27: 1 2 14 2 0 0	46: 2 3 8 2 0 0	68: 2 4 9 0 2 0	106: 2 4 14 2 2 0
15: 1 2 8 2 0 0	27: 2 3 5 0 2 0	46: 2 4 6 2 0 0	68: 2 4 10 4 2 0	108: 2 4 14 0 2 0
15: 1 3 5 0 0 0	27: 2 4 5 4 0 2	46: 2 5 6 4 0 2	68: 2 5 7 2 0 0	110: 2 4 14 2 0 0
15: 2 2 5 0 0 2	28: 1 2 14 0 0 0	47: 2 4 7 2 0 2	70: 2 4 9 2 0 0	112: 2 4 14 0 0 0
15: 2 3 3 0 2 0	28: 2 2 7 0 0 0	47: 2 5 6 4 2 0	70: 2 5 7 0 0 0	

Table 5. Ternary forms L such that $[1] \oplus L$ is universal.
(The three entries not appearing in Table 3 have been italicized.)

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