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ON SPINOR EXCEPTIONAL REPRESENTATIONS

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Let $f(x_1, \cdots, x_m)$ be a quadratic form with integer coefficients and $c \in \boldsymbol{Z}$. If $f(x) = c$ has a solution over the real numbers and if $f(x) \equiv c \pmod N$ is soluble for every modulus N, then at least some form h in the genus of f represents c. If $m \geqq 4$ one may further conclude that h belongs to the spinor genus of f. This does not hold when $m = 3$. However, in that situation there is a so-called "75% Theorem" which asserts that either every spinor genus in the genus of f represents c (i.e., there is a form in each spinor genus representing c) or else precisely half of all the spinor genera do. See [JW], [K], [H]. The theory of spinor exceptional representations is concerned with resolving the remaining 25% ambivalence. This we discuss in §§ 3, 4. We show in § 1 a field-theoretic interpretation for the various partitions of the genus into half-genera by certain "splitting integers", and in § 2 how this splitting feature can be exploited in certain cases to provide an invariant classification of forms up to spinor-equivalence, which may be viewed as a kind of a partial "spinor character theory", yielding in these instances an alternative to the algorithmic process of determining spinor-equivalence expounded recently by Cassels in [C], [C_1].

§ 0. Preliminaries

Unexplained terminology and notations are generally those from [OM]. Let F be an algebraic number field with R as its ring of algebraic integers, V a regular quadratic F-space of dimension m, and L an R-lattice on V with integral scale. Finite prime spots will be denoted by $\mathfrak{p} < \infty$ while infinite ones by $\mathfrak{p} \in \infty$. Let $\boldsymbol{G} = \boldsymbol{G}_L$ be the genus of L and $\boldsymbol{S}_X$ be the spinor genus of X. Suppose K is a lattice which is representable by $\boldsymbol{G}$ and such that its rank $\mathrm{rk}(K) = \dim(FK) = m - 2$. Then, by Witt's theorem we may assume that FK is a subspace of V with orthogonal complement U. Put

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$E_K = F(\sqrt{-\delta_K})$ where δ_K is the discriminant of U. Whenever E_K is used in this context, we shall always assume that $E_K \neq F$. Let J_F^G be the subgroup of the idele group J_F of F consisting of those ideles $(i_\mathfrak{p})$ such that $i_\mathfrak{p} \in \theta(O^+(L_\mathfrak{p}))$ for all $\mathfrak{p} < \infty$, where θ is the spinor norm function. Set $N_K = N_{E_K/F}(J_{E_K})$ and $H_K = N_K \cdot P_D \cdot J_F^G$ where $D = \theta(O^+(V))$. Now, the subgroup $J_U \cdot P_V \cdot J_V' \cdot J_L$ of the adele group J_V (split rotations) on V is independent of the choice of $L \in \boldsymbol{G}$ as J_V' contains the commutator subgroup of J_V. We denote this subgroup by $J(V, K)$. The second entry depends only on the isometry class of the ambient space FK. For, if K is replaced by $\bar{K}$ with $F\bar{K}$ isometric to FK, then putting $F\bar{K} = \bar{U}^\perp$ we see that $\phi(J_U)\phi^{-1} = J_{\bar{U}}$ for some $\phi \in O^+(V)$. This gives $J(V, K) = J(V, \bar{K})$. A similar assertion holds for H_K. From the general theory of spinor exceptional representations, one knows that θ induces an isomorphism from $J_V/J(V, K)$ onto J_F/H_K, and moreover, the group index $[J_F : H_K] \leq 2$. This leads us to call a regular R-lattice K a *splitting lattice* for $\boldsymbol{G}$ if (i) $\mathrm{rk}(K) = m - 2$, (ii) $\boldsymbol{G}$ represents K, and (iii) $[J_F : H_K] = 2$. When $K \cong \langle c \rangle$ we call c a *splitting integer* for $\boldsymbol{G}$ and E_K, δ_K, N_K, H_K, $J(V, K)$ are denoted by E_c, δ_c, N_c, H_c, $J(V, c)$ respectively. From the inequality $[J_F : H_K] \leq 2$ it follows that either every spinor genus in $\boldsymbol{G}$ represents K or else exactly half of all of them do. The latter can occur if and only if the following two conditions are fulfilled:

$$\text{(I)}: \qquad \theta(J_L) \subseteq N_K$$
$$\text{(II)}: \qquad \theta(L_\mathfrak{p} : K_\mathfrak{p}) = N_K(\mathfrak{p}) \qquad \text{for all } \mathfrak{p} < \infty\,.$$

Here $N_K(\mathfrak{p})$ denotes the $\mathfrak{p}$-th component of N_K and $\theta(L_\mathfrak{p} : K_\mathfrak{p})$ is a certain relative integral spinor norm group defined in [SP], [H_1]. We say K is a *spinor exceptional lattice* for $\boldsymbol{G}$ if its rank is $m - 2$ and if it is representable by some, but not by every, spinor genus in $\boldsymbol{G}$. The general theory also shows that K is splitting for $\boldsymbol{G}$ if and only if condition (I) is satisfied, and it is spinor exceptional for $\boldsymbol{G}$ precisely when both conditions (I), (II) hold. When K is splitting, $\boldsymbol{G}$ is split into two so-called *half-genera* (Halbgeschlecter—a term introduced in [K]) and we say two lattices M_1, M_2 in $\boldsymbol{G}$ belong to the same half-genus w.r.t. K iff $M_2 = \Lambda M_1$ for some $\Lambda \in J(V, K)$. Equivalently, $\theta(\Lambda) \in H_K$. When, in addition, K is also spinor exceptional then these half-genera take on added significance in that two lattices in the same half-genus either have both of their associated spinor genera represent K or both don't. Naturally, if V is indefinite the meaning is even sharper. If K is spinor exceptional, and $X \in \boldsymbol{G}$, we say X belongs to the *bad* (resp. *good*) *half-genus* if $\boldsymbol{S}_X$ doesn't (resp. does) represent K. Similarly, one may consider primitive representations and all the definitions and assertions carry over excepting only that condition (II) needs to be replaced by the obvious primitive analog:

$$\text{(II)*:} \qquad \theta^*(L_{\mathfrak{p}} : K_{\mathfrak{p}}) = N_K(\mathfrak{p}) \qquad \text{for all } \mathfrak{p} < \infty .$$

For more details of some of the assertions here, see [JW], [K], [H], [SP].

A relation between exceptionality and primitive exceptionality is the following:

LEMMA. *Every spinor exceptional lattice K for $\boldsymbol{G}$ induces a primitive spinor exceptional lattice.*

Proof. There is an $X \in \boldsymbol{G}$ which represents K. Let Y be a sublattice of X isometric to K. If T is the set of primes $\mathfrak{p}$ at which $Y_{\mathfrak{p}}$ is imprimitive in $X_{\mathfrak{p}}$, then T is a finite set. For each $\mathfrak{p} \in T$ embed $Y_{\mathfrak{p}}$ in a primitive sublattice $\overline{Y}_{\mathfrak{p}}$ of the same rank. Construct the sublattice $\tilde{Y}$ of X satisfying: $\tilde{Y}_{\mathfrak{p}} = Y_{\mathfrak{p}}$ for $\mathfrak{p} \notin T$, and $\tilde{Y}_{\mathfrak{p}} = \overline{Y}_{\mathfrak{p}}$ for $\mathfrak{p} \in T$. Hence, $\tilde{Y} \supseteq Y$ and both span the same space. So, $E_Y = E_{\tilde{Y}}$ and $\tilde{Y}$ is also splitting. Now, $\theta(X_{\mathfrak{p}} : Y_{\mathfrak{p}}) = N_Y(\mathfrak{p})$ by hypothesis, and $\theta^*(X_{\mathfrak{p}} : \tilde{Y}_{\mathfrak{p}}) \leq \theta(X_{\mathfrak{p}} : Y_{\mathfrak{p}})$ by construction. Since also $N_{\tilde{Y}}(\mathfrak{p}) = N_Y(\mathfrak{p})$ and $N_{\tilde{Y}}(\mathfrak{p}) \leq \theta^*(X_{\mathfrak{p}} : \tilde{Y}_{\mathfrak{p}})$ we conclude that $\theta^*(X_{\mathfrak{p}} : \tilde{Y}_{\mathfrak{p}}) = N_{\tilde{Y}}(\mathfrak{p})$ for all $\mathfrak{p} < \infty$, i.e., $\tilde{Y}$ is the induced primitive spinor exceptional lattice for $\boldsymbol{G}$.

§ 1. Splitting lattices and field extensions

We give a field-theoretic interpretation of the partitions of the genus into various half-genera. In particular, two spinor exceptional lattices are represented by the same set of spinor genera if and only if the induced relative quadratic extensions are identical. More precisely, we have:

PROPOSITION 1.1. *Let A, B be two splitting lattices for $\boldsymbol{G}$. Then, $H_A = H_B$ if and only if $E_A = E_B$.*

Proof. By the existence theorem of global class field theory, it is sufficient to show that $H_A = H_B$ precisely when $P_F \cdot N_A = P_F \cdot N_B$.

Suppose first that $H_A \leq H_B$, and $a \in N_A$. Write $a = d \cdot b \cdot r$, where $d \in D$, $b \in N_B$, $r \in J_F^G$. Since B is a splitting lattice, we may suppose that $r_{\mathfrak{p}} = 1$ at all $\mathfrak{p} < \infty$. If T is the set of real spots where $V_{\mathfrak{p}}$ is anisotropic, then

d is positive on T. Also, at every spot $\mathfrak{p}$ we have the Hilbert symbols satisfying: $(a_\mathfrak{p}, -\delta_A)_\mathfrak{p} = (b_\mathfrak{p}, -\delta_B)_\mathfrak{p} = 1$. Since δ_A and δ_B are both positive on T, $a_\mathfrak{p}$ and $b_\mathfrak{p}$ must also be positive. Therefore, we may, in fact, assume that $r = 1$ since $\theta(0^+(V_\mathfrak{p})) = F_\mathfrak{p}^\times$ at each infinite spot $\mathfrak{p}$ outside of T. This means that $P_D \cdot N_A \leq P_D \cdot N_B$, whence $P_F \cdot N_A \leq P_F \cdot N_B$.

Conversely, suppose $H_A \neq H_B$. Put $F^\times = \bigcup_{j=1}^t h_j \cdot D$, $h_1 = 1$. Then, $P_F \cdot N_A = P_D \cdot N_A \cup \cdots \cup h_t \cdot P_D \cdot N_A$ and $P_F \cdot N_B = P_D \cdot N_B \cup \cdots \cup h_t \cdot P_D \cdot N_B$. These unions are disjoint since $[J_F : P_D \cdot N_A] = [J_F : P_D \cdot N_B] = 2[F^\times : D]$. But, $P_D \cdot N_A$ is also disjoint from $h_j \cdot P_D \cdot N_B$ for $j \geq 2$. For, if not, some $a \in N_A$ is expressible as $h_j \cdot d \cdot b$, with $d \in D$, $b \in N_B$. But, there must be a prime $\mathfrak{p} \in T$ where $h_j <_\mathfrak{p} 0$. Again, $(a_\mathfrak{p}, -\delta_A)_\mathfrak{p} = 1 = (b_\mathfrak{p}, -\delta_B)_\mathfrak{p}$ forces h_j to be positive at $\mathfrak{p}$. Therefore, we have $P_D \cdot N_A = P_D \cdot N_B$, yielding the contradiction that $H_A = H_B$.

Using the same sort of argument and noting further that

$$(P_D \cdot N_{K_1} \cap \cdots \cap P_D \cdot N_{K_r}) P_F = P_F \cdot N_{K_1} \cap \cdots \cap P_F \cdot N_{K_r}$$

and

$$(P_D \cdot N_{K_1} \cap \cdots \cap P_D \cdot N_{K_r}) \cdot J_F^G = H_{K_1} \cap \cdots \cap H_{K_r},$$

we have a slight generalization in

PROPOSITION 1.2. *Let $K_1 \cdots K_r, B$ be splitting lattices for G. Then, $H_B \geq H_{K_1} \cap \cdots \cap H_{K_r}$ if and only if $E_B \subseteq$ compositum $E_{K_1} \cdots E_{K_r}$.*

1.3. Put $H_i = H_{K_i}$ and $E_i = E_{K_i}$, and $E = E_1, \cdots, E_r$. Let $K_1, \cdots, K_r$ be splitting lattices for $\boldsymbol{G}$ which are *independent* in the sense that E/F has degree 2^r. Using Proposition 1.2, an easy induction argument shows that $[J_F : H_1 \cap \cdots \cap H_r] = 2^r$. For each i there is an adele $\Lambda_i \in J_V$ such that $\Lambda_i \notin J(V, K_i)$ and $\Lambda_i \in J(V, K_j)$ for all $j \neq i$. Equivalently, $\theta(\Lambda_i) \in H_j$ for $j \neq i$ and $\theta(\Lambda_i) \notin H_i$. This, too, follows from independence and Prop. 1.2. From this it follows that for any $1 \leq s \leq r$ and any permutation $\{i_1, \cdots, i_r\}$ of $\{1, \cdots, r\}$, $H_{i_1} \cap \cdots \cap H_{i_s} \backslash H_{i_{s+1}} \cup \cdots \cup H_{i_r}$ is non-empty. This fact is useful, and is the basis for the (partial) spinor character theory discussed below.

§ 2. A partial spinor character theory

We show how the splittings induced by the spinor exceptional integers of a genus of ternary forms can be efficiently adapted to solve the spinor equivalence problem in certain cases. This provides an alternative approach to the effective algorithmic process treated recently by Cassels in [C], [C_1]. In principle, what we consider here should also apply to forms in any number of variables. However, because the local relative spinor norm groups $\theta(L_p : K_p)$ have as yet not been fully calculated, we confine our discussion only to the case of $\mathrm{rk}(K) = 1$ where these groups are determined in [SP].

2.1. We begin with a negative observation showing the limitation of our method. Consider the ternary positive $\boldsymbol{Z}$-lattice $L \cong \langle 1\rangle \perp \langle (q_0q_1\cdots q_r)^2\rangle \perp \langle (q_0q_1\cdots q_r)^4\rangle$ where $q_i \equiv 5 \pmod 8$. Then, $\boldsymbol{G}_L$ contains precisely 2^r spinor genera. See [EH]. This genus has no primitive spinor exceptional integers. In fact, it does not have any splitting integer. For, if c were one such integer, then 2 must ramify in $E_c = \boldsymbol{Q}(\sqrt{-c})$ since $c \equiv 1 \pmod 4$. But, $N_c(2)$ does not contain $\theta(O^+(L_2)) = \boldsymbol{Q}_2^\times$. Similarly, we consider an indefinite example with $\bar{L} = \langle -1\rangle \perp \langle (p_1\cdots p_r)^2\rangle \perp \langle (p_1\cdots p_r)^4\rangle$ where $p_j \equiv 1 \pmod 8$. Then $\boldsymbol{G}_{\bar{L}}$ also has 2^r spinor genera ($=$ classes). Any splitting integer for $\boldsymbol{G}_{\bar{L}}$ must have the properties: $c > 0$, $c \notin \boldsymbol{Q}^{\times 2}$, c is representable by $\boldsymbol{G}_{\bar{L}}$ (e.g., $c \in (p_1\cdots p_r)^4\boldsymbol{Z}$). Also, we may assume that the only prime divisors of c are from the p_j's. This example is significant for two reasons: first, $\boldsymbol{G}_{\bar{L}}$ does have splitting integers but that a simple computation shows none of them is primitively (spinor) exceptional; secondly, it shows that in the case of an *indefinite* ternary genus it is possible that *every* form in the genus represents (and primitively represents) *all* the integers allowed by congruential considerations, and furthermore, the number of classes in such a genus can be arbitrarily large. This feature is not known to hold for definite ternaries. Indeed, it has been conjectured by the second author that the classes in a definite ternary genus are characterized by their sets of primitively represented integers.

2.2. A set of splitting integers $\{c_1, \cdots, c_r\}$ for $\boldsymbol{G} = \boldsymbol{G}_L$ is *independent* if the multi-quadratic extension $F(\sqrt{-\delta_1}, \cdots, \sqrt{-\delta_r})/F$ with $\delta_j \in c_j\,\mathrm{disc}\,(FL)$ has field degree 2^r. This set is called *complete* if $[J_F : P_D J_F^{\boldsymbol{G}}] = 2^r$. In general, if a genus $\boldsymbol{G}$ has r independent splitting integers then it has at least 2^r spinor genera.

On the other side of the extreme, from the examples discussed in 2.1 there exist ternary genera (in both the definite and indefinite cases) which possess complete systems of spinor exceptional integers. We present here an example for the definite case. Consider

$$L \cong \langle 1\rangle \perp \langle (2^2 p_1 \cdots p_n)\rangle \perp \langle 2^4 (p_1 \cdots p_n)^2\rangle ,$$

where $p_i \equiv 5 \pmod 8$ and $(p_i/p_j) = 1$ for all $i \neq j$. A direct calculation will show that G has precisely 2^{n+1} spinor genera, and that $\{1, p_1, \cdots, p_n\}$ is a complete system of (clearly primitive) spinor exceptional integers for G. Therefore, these integers form a complete "*spinor character theory*" for G in the following sense. Let S be a spinor genus in G. Put $\chi_i(S) = \pm 1$ with $+1$ if and only if S represents p_i, $i = 0, 1, \cdots, n$, and $p_0 = 1$, and put

$$\chi(S) = (\chi_0(S), \cdots, \chi_n(S)) \in \{\pm 1\}^{n+1} .$$

Next, choose a spinor genus S_0 which is "*regular*" in the sense that it represents a complete set of spinor exceptional integers of G. Hence, $\chi(S_0)$ is trivial. Suppose $\Lambda \in J_V$ and S_X a spinor genus in G we put $\Lambda \cdot S_X = S_{\Lambda X}$. One sees that χ induces a homomorphism

$$\chi_G : J_V \longrightarrow \{\pm 1\}^{n+1}$$

given by $\Lambda \mapsto \chi(\Lambda \cdot S_0)$. To see this, one needs only to observe that $\Lambda \cdot S_0$ represents c_j if and only if $\Lambda \in J(V, c_j)$, i.e., iff $\theta(\Lambda) \in H_{c_j} = P_D \cdot N_{c_j} \cdot J_F^G$.

2.3. In this subsection we show by means of an explicit numerical example how the presence of sufficiently many independent spinor exceptional integers, together with Eisenstein reduction and certain graph-theoretic considerations, provide a rather efficient classification up to spinor-equivalence of positive ternary quadratic forms. See §§ 4, 5 below for more details, as well as [SP$_1$].

Denote by $\langle a, b, c, e, f, g\rangle$ for the ternary Z-lattice with inner product matrix

$$\begin{pmatrix} a & g & f \\ g & b & e \\ f & e & c \end{pmatrix}.$$

Consider the lattice $A^1 = \langle 4, 5, 400\rangle$. Its genus contains four spinor genera and twelve classes. More explicitly, let

$A^2 = \langle 1, 80, 100\rangle$, $A^3 = \langle 16, 20, 29, 0, -8, 0\rangle$,
$B^1 = \langle 1, 20, 400\rangle$, $B^2 = \langle 9, 9, 100, 0, 0, -1\rangle$, $B^3 = \langle 4, 45, 45, -5, 0, 0\rangle$,
$C^1 = \langle 4, 25, 80\rangle$, $C^2 = \langle 5, 16, 100\rangle$, $C^3 = \langle 4, 20, 101, 0, -2, 0\rangle$,
$D^1 = \langle 16, 20, 25\rangle$, $D^2 = \langle 4, 20, 105, -10, 0, 0\rangle$, $D^3 = \langle 4, 21, 100, 0, 0, -2\rangle$.

The clases of the A-lattices form a single spinor genus $\boldsymbol{S}^A$, and similarly for $\boldsymbol{S}^B$, $\boldsymbol{S}^C$, $\boldsymbol{S}^D$. Also by a direct calculation, we obtain that $\{1, 5\}$ form a complete system of spinor exceptional integers. With respect to the integer 1, $\boldsymbol{S}^A \cup \boldsymbol{S}^B$ constitute the good half-genus while $\boldsymbol{S}^A \cup \boldsymbol{S}^C$ form the good half-genus w.r.t. the integer 5. Now, if one is given two lattices X and Y which are, say, in the genus under consideration, then one simply find the reduce d forms $\overline{X}$, $\overline{Y}$ and see to which spinor genus they belong. For instance, if $X = \langle 16, 136, 9, 2, 8, -16\rangle$, and $Y = \langle 4, 180, 49, 60, 4, 20\rangle$ then by reduction theory, one sees that $\overline{X} = B^2$, $\overline{Y} = B^3$ so that X *is* spinor-equivalent to Y.

2.4. Remarks. Suppose the absolute discriminant of the base field F is an odd integer, then the local relative spinor norm groups $\theta(L_{\mathfrak{p}} : c)$ are determined in [SP]. In particular, one sees that if c is a spinor exceptional integer for $\boldsymbol{G}_L$ then the orders $\mathrm{ord}_{\mathfrak{p}}(c)$ are bounded above by $\mathrm{ord}_{\mathfrak{p}}\mathrm{disc}(L_{\mathfrak{p}})$ for all $\mathfrak{p} < \infty$ satisfying $-\delta_c \notin F_p^{\times 2}$. This is because if $\mathfrak{p} \nmid \mathrm{disc}(L_{\mathfrak{p}})$ then condition (I) property of c forces $\mathrm{ord}_{\mathfrak{p}}(c)$ to be even, and condition (II) forces $\mathrm{ord}_{\mathfrak{p}}(c) \leq 1$.

When a genus $\boldsymbol{G}$ admits some, but not a complete system of, spinor exceptional integers, then the homomorphism χ_G defined in 2.2 serves only a partial spinor character theory for $\boldsymbol{G}$. Namely, if $\{c_1, \cdots, c_r\}$ is a maximal set of independent spinor exceptional integers for $\boldsymbol{G}$ with $r < t$, $2^t =$ the number of spinor genera in $\boldsymbol{G}$. One first observes that *there is always at least one regular spinor genus.* To see this, consider any spinor genus $\boldsymbol{S}_X$ in $\boldsymbol{G}$. Suppose $\boldsymbol{S}_X$ does not represent $\{c_{i_h}, \cdots, c_{i_r}\}$ and represents the others. Then the discussions in 1.3 show that for each j there exists an adele $\Lambda(i_j) \in \bigcap_{k\neq j} J(V, c_{i_k}) \backslash J(V, c_{i_j})$. Therefore, upon putting $\Lambda = \prod_{j=h}^{r} \Lambda(i_j)$, we see that $\boldsymbol{S}_{\Lambda X}$ is regular. Furthermore, since $H_{c_1} \cap \cdots \cap H_{c_r}$ has index 2^r in $\boldsymbol{J}_F$ there are exactly 2^{t-r} spinor genera in $\boldsymbol{G}$ which are regular with respect to $\{c_1, \cdots, c_r\}$. Now, if we choose $\boldsymbol{S}_0$ to be any one of them, and define the "character homomorphism"

$$\chi_G : \boldsymbol{J}_V \longrightarrow \{\pm 1\}^r$$

as before. Then, χ_G is surjective with kernel $J(V, c_1) \cap \cdots \cap J(V, c_r)$. Hence, χ_G characterizes only up to this kernel.

§ 3. Relation between representations by L and by $\boldsymbol{S}_L$

The main result in [E] is the following: Let c be a primitive spinor exceptional integer for $\boldsymbol{G}_L$ where L is a ternary $\boldsymbol{Z}$-lattice. Then c is primitively represented by $\boldsymbol{S}_L$ if and only if L primitively represents ct^2 for some $t > 0$, $(t, 2 \operatorname{disc}(L)) = 1$, and the Jacobi symbol $(-c \operatorname{disc}(L)/t) = 1$. Here we generalize this result to an arbitrary number field which we need in the next section.

First, we set some notations straight. Let $\boldsymbol{G} = \boldsymbol{G}_L$ be a fixed ternary genus defined over a number field F, and c a splitting integer for $\boldsymbol{G}$, and $\mathfrak{v}$ the volume ideal of $\boldsymbol{G}$. Following [E], for each $t \in R = \operatorname{int}(F)$ satisfying $(t, 2\mathfrak{v}) = 1$ we define the idele $\boldsymbol{j}(t) = (j_{\mathfrak{p}}(t))$ by

$$j_{\mathfrak{p}}(t) = \begin{cases} \pi_{\mathfrak{p}} & \text{if } \operatorname{ord}_{\mathfrak{p}}(t) \text{ is odd} \\ 1 & \text{if } \operatorname{ord}_{\mathfrak{p}}(t) \text{ is even, or } \mathfrak{p} \in \infty \,, \end{cases}$$

where $\pi_{\mathfrak{p}}$ is any uniformizer at $\mathfrak{p}$. Since $U_{\mathfrak{p}}F_{\mathfrak{p}}^{\times 2} \subseteq \theta(0^+(L_{\mathfrak{p}}))$ for $\mathfrak{p} \nmid \mathfrak{v}$, $\boldsymbol{j}(t)$ is well defined in $J_F/\theta(J_L)$.

Recall that for any finite abelian extension E/F there is a canonical homomorphism

$$J_F \ni a = (a_{\mathfrak{p}}) \longmapsto \left(\frac{E/F}{a}\right) = \prod_{\mathfrak{p}} \left(\frac{a_{\mathfrak{p}}, E_{\mathfrak{P}}/F_{\mathfrak{p}}}{\mathfrak{p}}\right) \in \operatorname{Gal}(E/F)$$

inducing the Artin isomorphism from $J_F/P_F\, N_{E/F}(J_E)$ onto $\operatorname{Gal}(E/F)$. The infinite product of local norm residue symbols is only a finite product since $a_{\mathfrak{p}} \in U_{\mathfrak{p}}$ and $E_{\mathfrak{P}}/F_{\mathfrak{p}}$ is unramified almost everywhere. Applying this to our situation with $a = \boldsymbol{j}(t)$, $E = E_c$, $N_{E/F}(J_E) = N_c$, we see that

$$\left(\frac{E_c/F}{\boldsymbol{j}(t)}\right) = \prod_{\operatorname{ord}\mathfrak{p}(t)\text{odd}} \left(\frac{j_{\mathfrak{p}}(t), E_{c\mathfrak{P}}/F_{\mathfrak{p}}}{\mathfrak{p}}\right) = \mathbf{1} \in \operatorname{Gal}(E_c/F)$$

if and only if $\boldsymbol{j}(t) \in P_F \cdot N_c$, i.e., exactly when $\prod_{\operatorname{ord}\mathfrak{p}(t)\text{odd}} (-\delta_c/\mathfrak{p}) = 1$, where $(-\delta_c/\mathfrak{p}) = 1$ means that $-\delta_c \in F_{\mathfrak{p}}^{\times 2}$. On the other hand, using the kind of argument in § 1, one sees that $\boldsymbol{j}(t) \in P_F \cdot N_c$ iff $\boldsymbol{j}(t) \in P_D \cdot N_c \cdot J_F^G = H_c$. Summarizing, we have:

LEMMA 3.1. *Let c be a splitting integer for $\boldsymbol{G}$, and $E_c = F(\sqrt{-\delta_c})$. For any $t \in R$ satisfying $(t, 2\mathfrak{v}) = 1$, the idele $\boldsymbol{j}(t)$ is defined* $\operatorname{mod} \theta(J_L)$, *and*

$$\boldsymbol{j}(t) \in H_c \quad \textit{iff} \quad \left(\frac{E_c/F}{\boldsymbol{j}(t)}\right) = \mathbf{1} \quad \textit{iff} \prod_{\operatorname{ord}\mathfrak{p}(t)\text{odd}} \left(\frac{-\delta_c}{\mathfrak{p}}\right) = 1\,.$$

LEMMA 3.2. *If L primitively represents ct^2 where $t \in R$, $(t, 2\mathfrak{v}) = 1$, then c is primitively represented by ΛL for some $\Lambda \in J_V$ with $\theta(\Lambda) \equiv \boldsymbol{j}(t) \bmod \theta(J_L)$.*

Proof. The proof is similar to Prop. 1.2, [E] except we use a lemma from [HKK]. Let $v \in L$ be a primitive vector (i.e., L/Rv is torsion-free) with $Q(v) = ct^2$. For each $\mathfrak{p}|t$, $L_\mathfrak{p}$ is unimodular, and there is a local basis $\{x_{1,\mathfrak{p}}, x_{2,\mathfrak{p}}\} \perp \{x_{3,\mathfrak{p}}\}$ with matrix form$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \langle -d_\mathfrak{p} \rangle$, $d_\mathfrak{p} \in U_\mathfrak{p}$ for which v is expressible as $x_{1,\mathfrak{p}} + a_{2,\mathfrak{p}} x_{2,\mathfrak{p}}$, $a_{2,\mathfrak{p}} \in R_\mathfrak{p}$. For each $\mathfrak{p}|2\mathfrak{d}$ one may also embed v in a binary sublattice with a similar expression of v in terms of the basis vectors. Let S be the finite set of prime spots dividing $2t\mathfrak{d}$. By Lemma 1.6, [HKK] (its generalization to number fields), there exist global vectors $x_1, x_2 \in L$ such that at each $\mathfrak{p} \in S$ x_i approximates $x_{i,\mathfrak{p}}$ $(i=1, 2)$, and $d(x_1, x_2)_\mathfrak{p} \in U_\mathfrak{p}$ at all $\mathfrak{p} \notin S$ save but one spot $\mathfrak{p}_0$. Therefore, $B := Rx_1 + Rx_2$ is a direct summand of L, and we write $L = B \oplus \mathfrak{A} x_3$ for some fractional ideal $\mathfrak{A}$. Put $M = R(t^{-1}x_1) + R(tx_2) + \mathfrak{A} x_3$. If the approximation above is good enough then $B_\mathfrak{p}$ is isometric to $R_\mathfrak{p} x_{1,\mathfrak{p}} + R_\mathfrak{p} x_{2,\mathfrak{p}}$ (see e.g., [C] p. 123). One sees easily that for each $\mathfrak{p} \in S$ there is a rotation $\phi_\mathfrak{p}$ on $V_\mathfrak{p}$ satisfying: $\phi_\mathfrak{p}(L_\mathfrak{p}) = M_\mathfrak{p}$, and $\theta(\phi_\mathfrak{p}) \equiv j_\mathfrak{p}(t) \bmod U_\mathfrak{p} F_\mathfrak{p}^{\times 2}$. For $\mathfrak{p} \notin S$, $L_\mathfrak{p} = M_\mathfrak{p}$. So, putting $M = \Lambda L$, we see that $t^{-1}v$ is primitive in ΛL and $\theta(\Lambda) \equiv \boldsymbol{j}(t) \bmod \theta(J_L)$.

PROPOSITION 3.3. *Let c be a spinor exceptional integer for $\boldsymbol{G}_L$. Then, $\boldsymbol{S}_L$ primitively represents c if and only if L represents ct^2 primitively for some $t \in R$, $(t, 2\mathfrak{d}) = 1$, and $((E_c/F)/\boldsymbol{j}(t)) = \mathbf{1}$.*

Proof. Suppose $M \in \boldsymbol{S}_L$ primitively represents c. Replace M by $\overline{M} \in \mathrm{Cls}(M)$ such that $\overline{M}_\mathfrak{q} = L_\mathfrak{q}$ for all $\mathfrak{q} \neq \mathfrak{p}_0$ where $\mathfrak{p}_0$ may be chosen to be any finite prime spot where $V_{\mathfrak{p}_0}$ is isotropic. In particular, we may suppose that $\mathfrak{p}_0 = (\pi_0)$ is principal and relatively prime to $2\mathfrak{d}$. If $x \in \overline{M}$ is a primitive vector with $Q(x) = c$, choose k so that $y = \pi_0^k \cdot x$ is primitive in L. So, L primitively represents ct^2 with $t = \pi_0^k$. Lemma 3.2 implies c is primitively represented by ΛL for some $\Lambda \in J_V$ with $\theta(\Lambda) \equiv \boldsymbol{j}(t) \bmod \theta(J_L)$. Hence, $\boldsymbol{S}_L$ and $\boldsymbol{S}_{\Lambda L}$ belong to the same good half-genus w.r.t. c; in other words, $\theta(\Lambda) \in H_c$ and so $((E_c/F)/\boldsymbol{j}(t)) = \mathbf{1}$ by Lemma 3.1. The converse is now clear.

§4. Representations and graphs

In this section we adopt the notations and terminology of [K_1], and we use $q(-)$, $b(-, -)$ instead of our usual $Q(-)$, $B(-, -)$. Thus, $q(x + y) - q(x) - q(y) = b(x, y)$, and a ternary free lattice is regular when $d(e_1, e_2, e_3) := \det(b(e_i, e_j))$ is a unit for any lattice basis $\{e_1, e_2, e_3\}$, and is called *half-regular* if its *half-discriminant* $-d'(e_1, e_2, e_3) = -d(e_1, e_2, e_3)/2$ is a unit. Let L be a ternary $\boldsymbol{Z}$-lattice and p a prime when L_p is half-regular. Using

Kneser's neighborhood theory approach, a graph $Z(L, p)$ is constructed in [SP$_1$] whose vertices consist of lattices $X \in G_L$ such that $X_q = L_q$ for all $q \neq p$, and two vertices X, Y are *adjacent* (neighboring) if and only if their distance $d(X, Y, p) = 1$, i.e., $[X : X \cap Y] = [Y : X \cap Y] = p$. This graph is a tree (i.e., a connected graph without any loop). The corresponding local graph at p is identifiable with the Bruhat-Tits building for the spin group of V_p. This setting may be broadened to arbitrary number fields as well as to lattices of rank greater than three. We simply state here that if L is a ternary R-lattice which is half-regular at a prime spot $\mathfrak{p}$ on F, then there is a graph $R(L, \mathfrak{p})$ which is constructed in a similar way, and it too is a tree. A basic result about the graph $Z(L, p)$ is that it contains representative classes from at most two spinor genera; but, if a spinor genus is represented in the graph then every class in this spinor genus is represented. This property remains true for the graph $R(L, \mathfrak{p})$. We denote by $g(L, \mathfrak{p})$ the number of spinor genera in G_L that is represented in $R(L, \mathfrak{p})$. Two questions naturally arise in this context:

(A): For which prime spot $\mathfrak{p}$ is $g(L, \mathfrak{p}) = 2$?

(B): When $g(L, \mathfrak{p}) = 2$, how are the two spinor genera related?

We address to these two questions here.

4.1. We first fix some notations. Let π be a fixed uniformizer in $F_\mathfrak{p}$. A half-regular lattice is easily seen to be isotropic and $R_\mathfrak{p}$-maximal on $V_\mathfrak{p}$. Hence, $L_\mathfrak{p}$ admits a basis $\{e_1, e_2, e_3\}$ where $q(e_1) = 0 = q(e_2)$, $b(e_1, e_2) = 1$. Let $\sigma_\mathfrak{p} = S_{e_1 - e_2} \cdot S_{e_1 - \pi e_2} \in O^+(V_\mathfrak{p})$. Then, $\sigma_\mathfrak{p}(L_\mathfrak{p}) = R_\mathfrak{p}(\pi e_1) + R_\mathfrak{p}(\pi^{-1} e_2) + R_\mathfrak{p} e_3$, and has spinor norm $\theta(\sigma_\mathfrak{p}) = \pi \cdot F_\mathfrak{p}^{\times 2}$. Define the adele $\Sigma(\mathfrak{p}) = (\Sigma_\mathfrak{q}(\mathfrak{p}))$ by $\Sigma_\mathfrak{q}(\mathfrak{p}) = 1$ for $\mathfrak{q} \neq \mathfrak{p}$, and $\Sigma_\mathfrak{p}(\mathfrak{p}) = \sigma_\mathfrak{p}$. For each prime spot $\mathfrak{p}$ where $L_\mathfrak{p}$ is half-regular, we define the idele $\boldsymbol{j}(\mathfrak{p}) = (j_\mathfrak{q}(\mathfrak{p}))$ by $j_\mathfrak{q}(\mathfrak{p}) = 1$ and $j_\mathfrak{p}(\mathfrak{p}) = \pi$. Since $L_\mathfrak{p}$ is half-regular, it is not difficult to see that $\theta(O^+(L_\mathfrak{p})) = U_\mathfrak{p} \cdot F_\mathfrak{p}^{\times 2}$ and therefore, $\boldsymbol{j}(\mathfrak{p})$ is also well defined modulo $\theta(J_L)$. For $\phi \in O^+(V_\mathfrak{p})$ we denote the distance $d(L_\mathfrak{p}, \phi(L_\mathfrak{p})) = n$ by $[L_\mathfrak{p} : L_\mathfrak{p} \cap \phi(L_\mathfrak{p})] = N_{F/Q}\mathfrak{p}^n$.

LEMMA 4.2. *$\theta(\phi) \in U_\mathfrak{p} F_\mathfrak{p}^{\times 2}$ if and only if $d(L_\mathfrak{p}, \phi(L_\mathfrak{p}))$ is even.*

Proof. By the invariant factor theorem, we may assume that the basis $\{e_1, e_2, e_3\}$ for $L_\mathfrak{p}$ is also such that $\phi(L_\mathfrak{p}) = R_\mathfrak{p}(\pi^n e_1) + R_\mathfrak{p}(\pi^{-n} e_2) + R_\mathfrak{p} e_3$ where e_3 is orthogonal to e_1, e_2. Here, $n = d(L_\mathfrak{p}, \phi(L_\mathfrak{p}))$. There is an isometry $f \in O^+(\phi(L_\mathfrak{p}))$ such that $f(\pi^n e_1) = \phi(e_1)$, $f(\pi^{-n} e_2) = \phi(e_2)$, and $f(e_3) = \phi(e_3)$. Clearly, $\phi = f \cdot \sigma_\mathfrak{p}^n$, and $\theta(\phi) = \theta(f)\,\theta(\sigma_\mathfrak{p})^n \in \pi^n \cdot U_\mathfrak{p} \cdot F_\mathfrak{p}^{\times 2}$ since $L_\mathfrak{p}$ is half-regular. This completes the proof.

This proof is somewhat simpler than the one given in [SP_1]. An immediate consequence of this lemma is that the graph $R(L, \mathfrak{p})$ represents classes from at most two spinor genera, a result cited earlier. Furthermore, if $g(L, \mathfrak{p}) = 2$, then adjacent lattices belong to different spinor genera.

4.3. Suppose M is a neighbor of L in $R(L, \mathfrak{p})$. Then, $M = \Sigma(\mathfrak{p})L$ for a suitable choice of basis $\{e_1, e_2, e_3\}$ for $V_\mathfrak{p}$. But, $\theta(\Sigma(\mathfrak{p})) \equiv \boldsymbol{j}(\mathfrak{p}) \mod \theta(J_L)$. Hence, M is spinor-equivalent to L if and only if $\boldsymbol{j}(\mathfrak{p}) \in P_D J_F^G$. In other words, $|R(L, \mathfrak{p})|$ *contains lattices from only one spinor genus exactly when* $\boldsymbol{j}(\mathfrak{p}) \in P_D J_F^G$. This answers question (A). For an example, if L is the $\boldsymbol{Z}$-lattice $\langle 1 \rangle \perp \langle 17 \rangle \perp \langle 17^2 \rangle$ then $|Z(L, p)|$ contains two spinor genera for $p =$ 3, 5, 7, 11, 23, 29, 31, 37, 41, 67, $\cdots$etc. Here $\boldsymbol{G}_L$ has two spinor genera and eight classes, 4 classes in each spinor genus. A generalization of this example is the following:

EXAMPLE 4.4. Let $p_1, \cdots, p_r$ be primes each congruent to 1 (mod 8) and $(p_i/p_j) = 1$ for $i \neq j$. Consider the $\boldsymbol{Z}$-lattice $L = \langle 1 \rangle \perp \langle p_1 \cdots p_r \rangle \perp \langle (p_1 \cdots p_r)^2 \rangle$. Let p be a prime not dividing the half-discriminant of L, which is $-4(p_1 \cdots p_r)^3$. Then, $|Z(L, p)|$ contains lattices from only one spinor genus if and only if $(p_i/p) = 1$ for $i = 1, \cdots, r$. Note: in this example, $\boldsymbol{G}_L$ has 2^r spinor genera, but has no splitting integers.

4.5. Suppose now c is a splitting integer for $\boldsymbol{G}_L$ and $\mathfrak{p}$ a prime spot at which $L_\mathfrak{p}$ is half-regular. Let M be a neighbor of L in $R(L, \mathfrak{p})$. As before, $M = \Sigma(\mathfrak{p})L$ for a suitable basis of $V_\mathfrak{p}$, and $\theta(\Sigma(\mathfrak{p})) \equiv \boldsymbol{j}(\mathfrak{p}) \mod \theta(J_L)$. Thus, M and L belong to the same c-half-genus if and only if $\boldsymbol{j}(\mathfrak{p}) \in P_D \cdot N_c \cdot J_F^G = H_c$. By the proof of Lemma 3.1, this can happen if and only if $((E_c/F)/\boldsymbol{j}(\mathfrak{p})) = \boldsymbol{1}$, i.e., iff $-\delta_c \in F_\mathfrak{p}^{\times 2}$. In particular, the following is quite useful in applications:

PROPOSITION. *If c is a (primitive) spinor exceptional integer for $\boldsymbol{G}_L$ and $-\delta_c$ is a non-square at $\mathfrak{p}$, then the graph $R(L, \mathfrak{p})$ contains two spinor genera, one from the good c-half-genus and one from the bad. Whereas, if $g(L, \mathfrak{p}) = 2$ and $-\delta_c \in F_\mathfrak{p}^{\times 2}$ then either both spinor genera (primitively) represents c or both do not.*

Thus, the presence of splitting integers or (primitive) spinor exceptional integers sheds new informations on the representational properties of the graphs $R(L, \mathfrak{p})$ at various prime spots $\mathfrak{p}$. This answers, at least partly, our question (B).

§ 5. Appendix

We show how the results of § 4 can be efficiently used to construct the classes in the genus. In particular, if there is a complete system of spinor exceptional integers for $\boldsymbol{G}_L$ the construction of all the classes in $\boldsymbol{G}_L$ is quickly accomplished. We illustrate this point by considering the example given in 2.3, using the same notation for $\langle a, b, c, e, f, g\rangle$. We first make two preliminary, but useful, observations:

(1) If $X \in |Z(Y, p)|$ then $Z(X, p) = Z(Y, p)$.

(2) If $X \in \boldsymbol{S}_Y$ then $Z(X, p) \cong Z(Y, p)$ for all applicable p.

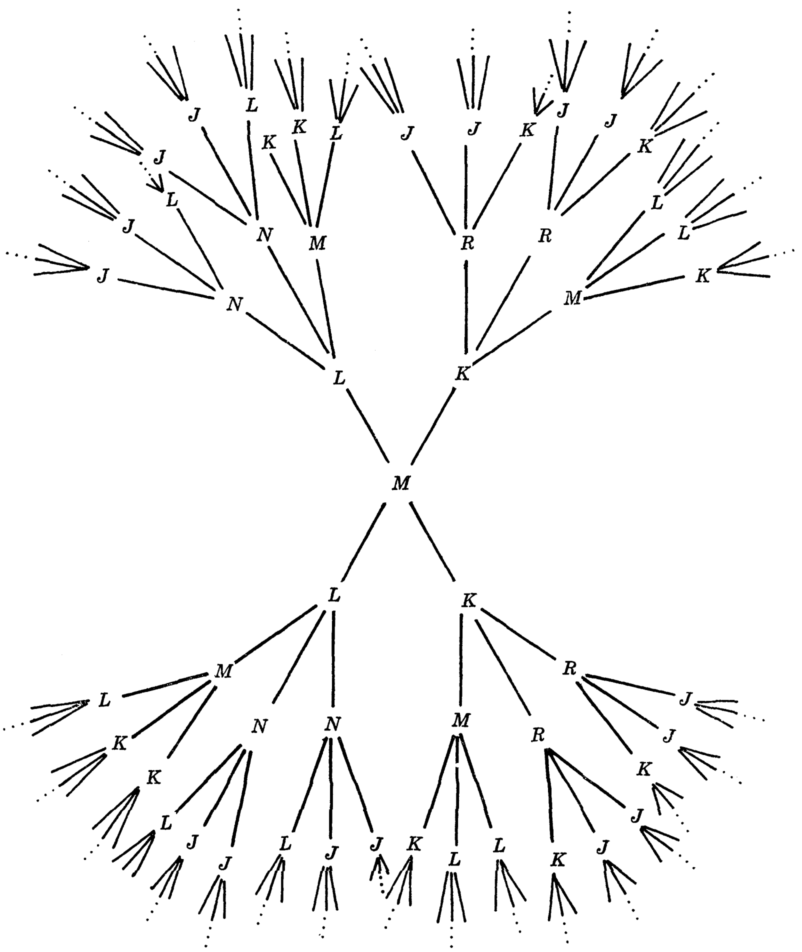
(1) is nearly obvious. To see (2) one just needs to remember that if a lattice M belongs to the graph $Z(N, p)$ then every class in $\boldsymbol{S}_M$ is represented in the graph as well.

We shall re-label the lattices as we construct them. We begin with $M = \langle 4, 5, 400\rangle$ in basis $\{e_1, e_2, e_3\}$. We construct the graph $Z(M, 3)$. Using the method developed in [SP$_1$] we see M has 4 neighbors

$$\begin{aligned} M_1 &= \boldsymbol{Z}(\tfrac{1}{3}(e_1 + e_2)) + \boldsymbol{Z}(3e_2) + \boldsymbol{Z}e_3 \\ M_2 &= \boldsymbol{Z}(\tfrac{1}{3}(e_1 - e_2)) + \boldsymbol{Z}(3e_2) + \boldsymbol{Z}e_3 \\ M_3 &= \boldsymbol{Z}(\tfrac{1}{3}(e_2 + e_3)) + \boldsymbol{Z}e_1 + \boldsymbol{Z}(3e_2) \\ M_4 &= \boldsymbol{Z}(\tfrac{1}{3}(e_2 - e_3)) + \boldsymbol{Z}e_1 + \boldsymbol{Z}(3e_2)\,. \end{aligned}$$

M_1 and M_2 both (Eisenstein) reduce to $\langle 1, 20, 400\rangle = L$, while M_3 and M_4 reduce to $\langle 4, 45, 45, -5, 0, 0\rangle = K$. Similarly, L has 4 neighbors which reduce to two classes: $N = \langle 1, 80, 100\rangle$ and, of course, M. From the neighbors of K we pick up a new class $R = \langle 16, 20, 29, 0, -8, 0\rangle$, and the class $J = \langle 9, 9, 100, 0, 0, -1\rangle$ from the neighbors of R. After this point no new class appears. Thus, each vertex is adjacent to 4 vertices representing two different classes, and the picture for the graph $Z(M, 3)$ is given below. Note that the prime 2 is not applicable since M is not half-regular there. Since $\{1, 5\}$ is a complete system of (primitive) spinor exceptional integers for $\boldsymbol{G}_M$ and since M manifestedly represents 5, we seek for a small prime p where $-\delta_5$ is a non-square in $\boldsymbol{Q}_p$ so as to have $Z(M, p)$ contain spinor genus from both the good and the bad half-genus with respect to $c = 5$, by the Proposition in 4.5. Such a prime is $p = 3$ since $\delta_5 = 1$. Therefore, the neighbors L, K of M belong to the bad half-genus w.r.t. 5. Since the genus $\boldsymbol{G}_M$ has 4 spinor genera, we need to go on. So far, we have caught the two spinor genera $\boldsymbol{S}^A = \mathrm{Cls}(M) \cup \mathrm{Cls}(N)$

The graph $Z(M, 3)$



$\cup \operatorname{Cls}(R)$, and $S^B = \operatorname{Cls}(L) \cup \operatorname{Cls}(K) \cup \operatorname{Cls}(J)$. S^A is regular because it represents both 1 and 5, while S^B represents 1 but not 5. We need a spinor genus which does not represent 1. Since $\delta_1 = 5$ and $(-5/7) = 1$ we cannot use $p = 5, 7$. However, $(-5/11) = -1$, so both $Z(M, 11)$ and $Z(L, 11)$ will yield a desired spinor genus.

Consider $Z(L, 11)$. L has 12 neighbors every 4 of which yield the same reduced forms, and the new classes obtained are: $F = \langle 5, 16, 100\rangle$, $G = \langle 4, 20, 101, 0, -2, 0\rangle$, and $H = \langle 4, 25, 80\rangle$. Now, the neighbors of F, G, H will all be in S_L by 4.2; their neighbors turn out to be again isometric to either F, G, or H. Hence, $Z(L, 11)$ adds only 3 new classes which form the third spinor genus S^C, and we see that S^C represents 5 but not 1. So, the spinor character theory of § 2 tells us that the fourth remaining spinor genus should represent neither 1 nor 5. By (2) above, the graphs $Z(K, 11)$ and $Z(J, 11)$ are all isomorphic to $Z(L, 11)$. On the other hand, since $(-\delta_1/11) = (-\delta_5/11) = -1$, the graph $Z(M, 11)$ should produce the final desired spinor genus according to 4.5. Indeed, this spinor genus $S^D = \mathrm{Cls}(W) \cup \mathrm{Cls}(X) \cup \mathrm{Cls}(Y)$, where $W = \langle 4, 20, 105, -10, 0, 0\rangle$, $X = \langle 4, 21, 100, 0, 0, -2\rangle$, and $Y = \langle 16, 20, 25\rangle$. This completes the genus G_M with 4 spinor genera and each having 3 classes.

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