SPINOR REGULAR POSITIVE TERNARY QUADRATIC FORMS

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ABSTRACT

Refining the notion of regularity introduced by Dickson, an integral quadratic form is said to be spinor regular if it represents all integers represented by its spinor genus. Examples of positive definite primitive integral ternary quadratic forms which have this property are presented, and it is proved that there exist only finitely many equivalence classes containing such forms.

Introduction

In terminology introduced by Dickson in 1927 [**5**], a positive definite integral ternary quadratic form is said to be *regular* if it represents all natural numbers not excluded by congruence considerations; that is, it represents all integers represented by its genus. An historical survey of the search for such forms can be found in [**7**], along with pertinent references to the original literature. In the present paper, a refinement of the notion of regularity, called spinor regularity, is introduced, it is proved that there exist only finitely many equivalence classes of positive definite primitive integral ternary quadratic forms which have this property, and all such forms are determined which lie in genera containing multiple spinor genera and have discriminant less than 2000. A form is said to be *spinor regular* if it represents all integers represented by its spinor genus; in particular, every regular form is spinor regular.

In [**13**, **7**], new regular forms were found through the utilization of spinor genus theory to analyse the representation properties of the forms in the table of Brandt and Intrau [**4**]. The work to be presented in this paper constitutes a completion of this application of the spinor genus theory. An exhaustive search was conducted to determine all forms in [**4**] which are spinor regular and lie in genera containing multiple spinor genera. This search resulted in the discovery of one previously unknown regular form (of discriminant 864), and eleven spinor regular forms which are not regular, of which one (also of discriminant 864) lies in a spinor genus containing more than one equivalence class. Details appear in Theorem 1.

A key step in the analysis carried out consists of the separation of classes within a genus into spinor genera. As there are no known invariants which distinguish between forms in different spinor genera, the method used here utilizing representation measures for this separation may be of some independent interest. This method, which is described and illustrated in §2, involves only simple machine computation and is efficient for genera of the size encountered in this project.

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This paper represents an extension of previous work of two of the authors. In **[1]**, the notion of spinor regularity was defined, the examples presented here in §3 were discovered, and a search was conducted for spinor regular positive definite primitive integral ternary quadratic forms in genera containing multiple spinor genera, at most four equivalence classes, and having discriminant less than 2000. All regular forms in genera satisfying the same restrictions but containing at most three classes were determined in **[7]**.

1. *Preliminaries*

The purpose of this section is to recount some basic facts from the representation theory of spinor genera. While many of the results to be mentioned here can be formulated in a more general context (for example, see **[6, 12, 2]**), we shall restrict to integral ternary forms in order to make the general theory as concrete as possible. For ease of reference to the original literature, the geometric language of quadratic lattices will be adopted throughout this section. For this discussion, terminology and notation will follow that of O'Meara's book **[11]**.

Let L be a $\mathbb{Z}$-lattice of discriminant d on a non-singular ternary quadratic space V with quadratic map Q and associated bilinear form B for which $B(x,x) = Q(x)$ for $x \in V$. We further assume that L is integral, in the sense that $B(x,y) \in \mathbb{Z}$ for all $x, y \in L$. Let Ω denote the set of all prime spots on $\mathbb{Q}$, and let S be the subset of Ω consisting of the finite prime spots on $\mathbb{Q}$. Throughout the following discussion, it will be assumed that c is a non-zero integer satisfying

$$-cd \notin \dot{\mathbb{Q}}^2. \tag{1.1}$$

For $p \in \Omega$, define

$$N_c(p) = \{\beta \in \dot{\mathbb{Q}}_p : (\beta, -cd)_p = 1\},$$

where $(\ ,\ )_p$ is the p-adic Hilbert symbol.

Let $J_{\mathbb{Q}}$ denote the idèle group of $\mathbb{Q}$, and let P_D and $J_{\mathbb{Q}}^L$ be the subgroups of $J_{\mathbb{Q}}$ as defined in **[11**, §101D**]**. Moreover, define a subgroup N_c of $J_{\mathbb{Q}}$ by

$$N_c = \{j \in J_{\mathbb{Q}} : j_p \in N_c(p), \text{ for all } p \in \Omega\}.$$

Equivalently, $N_c = N_{E/\mathbb{Q}}(J_E)$, where $E = \mathbb{Q}(\sqrt{(-cd)})$. The subgroup $H_c = N_c P_D J_{\mathbb{Q}}^L$ has index at most two in $J_{\mathbb{Q}}$ **[10, 6]**. An integer c is said to be a *splitting integer* for the genus $\operatorname{gen} L$ if c is represented by $\operatorname{gen} L$ and $[J_{\mathbb{Q}} : H_c] = 2$ **[2]**. In the case of primary interest in this paper, the splitting integers can be determined by the following result.

LEMMA 1. *Assume that L is positive definite and that the integer c is represented by* $\operatorname{gen} L$. *Then c is a splitting integer for* $\operatorname{gen} L$ *if and only if c satisfies* (1.1) *and*

$$\theta(O^+(L_p)) \subseteq N_c(p) \quad \text{for all } p \in S, \tag{1.2}$$

where θ denotes the spinor norm.

Proof. The necessity of (1.1) and (1.2) follows from the general theory **[9, 6]**. For the sufficiency, suppose that (1.1) and (1.2) hold, but $[J_{\mathbb{Q}} : H_c] = 1$. Then, in particular, the principal idèle (-1) lies in H_c. So there exist $\alpha \in \mathbb{Q}^+$, $j \in N_c$ and $k \in J_{\mathbb{Q}}^L$ such that $(-\alpha) = jk$. It then follows from (1.2) that $-\alpha \in N_c(p)$ for all $p \in S$; thus, $(-\alpha, -cd)_p = 1$ for all $p \in S$. However, since both c and d are positive by the assumption of positive definiteness, we have $(-\alpha, -cd)_\infty = -1$, giving a contradiction to the Hilbert reciprocity law.

The significance of splitting integers in the next section hinges upon the next result, the proof of which can be adapted from the arguments in **[10, 6]**. While experts in the field have surely been aware of this result for some time, it does not appear to be explicitly stated in the literature. For the sake of completeness, it is stated here without proof. For the statement, let $r(L,n)$ be the number of representations of the integer n by the positive definite lattice L, and let

$$\mathscr{M}(\operatorname{spn} L, n) = \sum r(L', n)/|O^+(L')|,$$

the sum running over a complete set of representatives L' of the classes in the spinor genus $\operatorname{spn} L$. We refer to $\mathscr{M}(\operatorname{spn} L, n)$ as the *representation measure of n by* $\operatorname{spn} L$. Finally, let J_V be the split rotation group of the underlying space V as defined in **[11**, §101D].

PROPOSITION 1. *Let L be positive definite and let c be represented by* gen L. *Then*

$$\mathscr{M}(\operatorname{spn} \Sigma L, c) = \mathscr{M}(\operatorname{spn} L, c)$$

for all $\Sigma \in J_V$ *such that* $\theta(\Sigma) \in H_c$.

COROLLARY 1. *If c is not a splitting integer for* gen L, *then*

$$\mathscr{M}(\operatorname{spn} K, c) = \mathscr{M}(\operatorname{spn} L, c)$$

for all $K \in \operatorname{gen} L$.

It follows from the proposition and the fact that $[J_{\mathbf{Q}} : H_c] \leqslant 2$ that if c is represented by L but not represented by every spinor genus in gen L, then it is represented by exactly half of these spinor genera. In this case, c is said to be a *spinor exceptional integer* for gen L. Of course, conditions (1.1) and (1.2) are necessary for c to be spinor exceptional. In fact, when L is positive definite, c is a spinor exceptional integer for gen L if and only if c is represented by gen L and (1.1), (1.2) and

$$\theta(L_p, c) = N_c(p) \quad \text{for all } p \in S \tag{1.3}$$

are satisfied. That result, the definition and calculations of the groups $\theta(L_p, c)$ appear in **[12]**.

In the remainder of the paper, it will be convenient to use the terminology of quadratic forms and quadratic lattices interchangeably. For an integer-valued ternary quadratic form $f = f(x_1, x_2, x_3)$, the discriminant $d = d(f)$ to be used is the determinant of the matrix $F = (\partial^2 f/\partial x_i \partial x_j)$. For such a form f, let cls$f$, spn$f$ and gen$f$ denote the equivalence class, spinor genus and genus of f, respectively. A lattice L corresponding to the form $2f$ (in the sense of **[11**, §41]) is a quadratic lattice for which there is a $\mathbb{Z}$-basis $\{e_1, e_2, e_3\}$ in which $(B(e_i, e_j)) = F$. This L is a lattice having integral scale, norm contained in $2\mathbb{Z}$, and discriminant d. When f and L are so related, we shall write $\mathscr{M}(\operatorname{spn} f, a)$ for $\mathscr{M}(\operatorname{spn} L, 2a)$.

2. *Separating a genus into spinor genera*

One method which can be used in some cases to separate the isometry classes in a ternary genus into spinor genera utilizes the partial character theory available when there are sufficiently many spinor exceptional integers for the genus [**2**]. A general computer-applicable method for this separation which is based on graph-theoretic techniques is described in [**3**]. In this section, a third method will be discussed which is based on Corollary 1. It is this method which was used to analyse the larger genera in the table of Brandt and Intrau [**4**] in order to produce the results to be described in §3. In that table, representatives are listed for all of the classes within each genus of positive definite primitive integral ternary quadratic forms of discriminant less than 2000. Let $f_1, \ldots, f_h$ be such a list. For each form f_i, a simple machine computation can be used to calculate the numbers $r(f_i, j)$ of representations of the integers j from 1 to n, for some convenient value of n, and to calculate the ratios $r_i(j) = r(f_i, j)/o^+(f_i)$, where $o^+(f_i)$ is the number of proper integral automorphs of f_i. An analysis of how such terms can be combined so that the representation measures within each spinor genus are equal, in accordance with Corollary 1, then leads to the desired separation into spinor genera.

In order to illustrate the method described above, consider one of the genera of largest class number encountered in [**4**] which contains two spinor genera, namely the genus $\mathscr{G}$ of discriminant 1998 and class number 10 containing the form

$$f = x^2 + y^2 + 333z^2 + xy.$$

For convenience, representatives $f = f_1, \ldots, f_{10}$ of the ten classes in this genus are listed in Table 1, along with the numbers of proper automorphs of each.

TABLE 1. *Representatives of classes in the genus* $\mathscr{G}$

i	f_i	$o^+(f_i)$
1	$x^2+y^2+333z^2+xy$	12
2	$x^2+3y^2+84z^2+3yz$	4
3	$x^2+9y^2+28z^2+xz$	4
4	$x^2+9y^2+30z^2+9yz$	4
5	$x^2+12y^2+21z^2+3yz$	2
6	$x^2+16y^2+19z^2-13yz+xz+xy$	2
7	$3x^2+3y^2+37z^2+3xy$	12
8	$3x^2+9y^2+10z^2+3xz$	4
9	$4x^2+4y^2+16z^2-yz+xz+xy$	2
10	$4x^2+7y^2+9z^2+xy$	2

For each of the forms $f_1, \ldots, f_{10}$, the numbers of representations of j were calculated for $j = 1, \ldots, 40$. In Table 2, the resulting values $r_i(j)$ are listed for only those integers j for which they are explicitly used in subsequent arguments.

To begin the analysis of the genus $\mathscr{G}$, consider the lattice L corresponding to the form $2f$. A straightforward computation of the group index $[J_Q : P_Q J_Q^L]$ shows that the genus of L (hence also $\mathscr{G}$) consists of two spinor genera, say $\mathscr{S}_1$ and $\mathscr{S}_2$. If $c = 2a$ is a splitting integer for gen L, then by condition (1.2), $\theta(O^+(L_{37})) \subseteq N_c(37)$. By [**9**, Satz

TABLE 2. *Ratios $r_i(j)$ for forms in $\mathscr{G}$*

$j \diagdown i$	1	2	3	4	5	6	7	8	9	10
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	0	0	0
3	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0
4	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	0	2	1
10	0	0	1	1	0	0	0	1	0	1
12	$\frac{1}{2}$	$\frac{3}{2}$	0	0	1	0	$\frac{1}{2}$	$\frac{3}{2}$	0	1
18	0	0	1	1	0	2	0	0	2	0
19	1	1	0	0	0	2	0	2	2	2
21	1	1	0	0	3	3	1	1	3	3
28	1	3	1	0	2	2	0	1	1	1

3], $\theta(O^+(L_{37})) = \dot{\mathbb{Q}}_{37}$. So $(\beta, -cd)_{37} = 1$ for all $\beta \in \dot{\mathbb{Q}}_{37}$. This forces $-cd \in \dot{\mathbb{Q}}_{37}^2$, or, equivalently, $a \in 37\dot{\mathbb{Q}}_{37}^2$. Thus, it follows from Corollary 1 that $\mathscr{M}(\mathscr{S}_1, 2j) = \mathscr{M}(\mathscr{S}_2, 2j)$ for all integers j listed in Table 2.

Proceeding now to the separation of classes into spinor genera, consider first the representation measure of 18. As $r_3(18) = r_4(18) = 1$, $r_6(18) = r_9(18) = 2$ and $r_i(18) = 0$ for all other i, it follows that f_6 and f_9 are in opposite spinor genera, as are f_3 and f_4. Similarly, by considering the values $r_i(19)$, one obtains that f_8 and f_{10} are in opposite spinor genera, as are f_1 and f_2. Using this information and the values $r_i(21)$, it further follows that f_5 and f_{10} are in opposite spinor genera, as are f_7 and f_8. So to this point it has been established that f_5 and f_8 lie in one spinor genus, while f_7 and f_{10} lie in the opposite spinor genus. Now, the contributions to $\mathscr{M}(\operatorname{spn} f_5, 12)$ by f_5 and f_8 sum to $\frac{5}{2}$, while the contributions to $\mathscr{M}(\operatorname{spn} f_7, 12)$ by f_7 and f_{10} sum to $\frac{3}{2}$. In order for $\mathscr{M}(\operatorname{spn} f_5, 12)$ to equal $\mathscr{M}(\operatorname{spn} f_7, 12)$, f_1 must lie in $\operatorname{spn} f_5$ and f_2 must lie in $\operatorname{spn} f_7$. Next, the contributions to $\mathscr{M}(\operatorname{spn} f_1, 4)$ by f_1, f_5 and f_8 sum to $\frac{3}{2}$, the contributions to $\mathscr{M}(\operatorname{spn} f_2, 4)$ by f_2, f_7 and f_{10} sum to $\frac{5}{2}$, and each measure receives a contribution of $\frac{1}{2}$ from f_3 and f_4. Consequently, $f_9 \in \operatorname{spn} f_1$ and $f_6 \in \operatorname{spn} f_2$. Finally, the sum of contributions to $\mathscr{M}(\operatorname{spn} f_1, 28)$ by the known forms in $\operatorname{spn} f_1$ is 5, while for $\mathscr{M}(\operatorname{spn} f_2, 28)$ the sum is 6. Thus, $f_3 \in \operatorname{spn} f_1$ and $f_4 \in \operatorname{spn} f_2$. This yields the desired separation

$$\operatorname{spn} f_1 = \{\operatorname{cls} f_1, \operatorname{cls} f_3, \operatorname{cls} f_5, \operatorname{cls} f_8, \operatorname{cls} f_9\},$$
$$\operatorname{spn} f_2 = \{\operatorname{cls} f_2, \operatorname{cls} f_4, \operatorname{cls} f_6, \operatorname{cls} f_7, \operatorname{cls} f_{10}\}.$$

3. *Spinor regular forms*

An integral quadratic form (or lattice) or a spinor genus of such forms (or lattices) is said to be *regular*, in the sense of [**5**], if it represents all integers represented by its genus. This definition of a regular spinor genus differs somewhat from that in [**2**]. A form (or lattice) is said to be *spinor regular* if it represents all integers represented by its spinor genus. In this terminology, a form (or lattice) is regular if it is spinor regular in a regular spinor genus. If a spinor regular form lies in a spinor genus containing more than one class, it will be referred to as *non-trivially* spinor regular.

REMARKS. For indefinite forms of rank exceeding two, the class and spinor genus coincide (for example, [**11**, 104: 5]); hence, all such forms are trivially spinor regular. For arbitrary forms of rank exceeding three, every spinor genus is regular (for example, [**6**]).

The main result of this section is the following.

THEOREM 1. *Let f be a positive definite primitive integral ternary quadratic form of discriminant* $d \leqslant 2000$. *Further assume that the genus of f contains more than one spinor genus. Then f is spinor regular if and only if f is equivalent to one of the regular forms listed in* [7, Theorem 3] *or to one of the following forms:*

$$(d=128),\quad 2x^2+2y^2+5z^2+2yz+2xz, \tag{3.1}$$
$$(d=216),\quad 3x^2+3y^2+4z^2+3xy, \tag{3.2}$$
$$(d=216),\quad 3x^2+4y^2+4z^2+4yz+3xy, \tag{3.3}$$
$$(d=256),\quad x^2+4y^2+9z^2+4yz, \tag{3.4}$$
$$(d=512),\quad 4x^2+4y^2+5z^2+4xz, \tag{3.5}$$
$$(d=512),\quad 2x^2+5y^2+8z^2+4yz+2xy, \tag{3.6}$$
$$(d=648),\quad x^2+7y^2+12z^2+xy, \tag{3.7}$$
$$(d=686),\quad 2x^2+7y^2+8z^2+7yz+xz, \tag{3.8}$$
$$(d=864),\quad x^2+3y^2+37z^2+3yz+xz, \tag{3.9}$$
$$(d=864),\quad 3x^2+7y^2+7z^2-2yz+3xz+3xy, \tag{3.10}$$
$$(d=864),\quad 3x^2+4y^2+9z^2, \tag{3.11}$$
$$(d=864),\quad 4x^2+4y^2+9z^2+4xy. \tag{3.12}$$

Of these, only (3.9) *is regular.*

As a first step in proving this theorem, the spinor regularity of two forms in the list will be established.

PROPOSITION 2. *The form* (3.9) *is regular, and the form* (3.10) *is non-trivially spinor regular, but not regular.*

Proof. These forms lie in the same genus; from [4], representatives of the four classes in this genus are

$$g_1 = x^2+y^2+144z^2-xy,$$
$$g_2 = x^2+3y^2+37z^2-3yz-xz,$$
$$g_3 = 3x^2+7y^2+7z^2+5yz+3xz+3xy,$$
$$g_4 = 3x^2+3y^2+16z^2-3xy.$$

Of these, g_1 and g_2 lie in one spinor genus $\mathscr{S}_1$, and g_3 and g_4 lie in another spinor genus $\mathscr{S}_2$. Spinor exceptional integers for the genus are of the form m^2; clearly, all such integers are represented by g_1. Thus, $\mathscr{S}_1$ is a regular spinor genus. However, 1 is not represented by $\mathscr{S}_2$, so $\mathscr{S}_2$ is not a regular spinor genus.

To show that g_2 is spinor regular, it suffices to show that g_2 represents every integer represented by g_1. This follows immediately from the equations

$$g_1(x,y,z) = g_2(-x+\tfrac{1}{2}y+z, \tfrac{1}{2}y+z, 2z) \tag{3.13}$$
$$= g_2(\tfrac{1}{2}x-y+z, \tfrac{1}{2}x+z, 2z) \tag{3.14}$$
$$= g_2(-\tfrac{1}{2}x-\tfrac{1}{2}y+z, \tfrac{1}{2}x-\tfrac{1}{2}y+z, 2z). \tag{3.15}$$

For, if a is represented by g_1, there exist $x, y, z \in \mathbb{Z}$ such that $a = g_1(x, y, z)$. If x is even, then (3.14) gives a representation of a by g_2; if y is even, then (3.13) gives a representation of a by g_2; if both x and y are odd, then (3.15) gives such a representation. As g_2 is spinor regular in a regular spinor genus, it follows that g_2, and thus (3.9), is regular.

The spinor regularity of g_3, and thus (3.10), follows from an analogous argument using the equations

$$\begin{aligned} g_4(x,y,z) &= g_3(x - \tfrac{1}{2}y + z,\ -\tfrac{1}{2}y - z, \tfrac{1}{2}y - z) \\ &= g_3(-\tfrac{1}{2}x + y + z,\ -\tfrac{1}{2}x - z, \tfrac{1}{2}x - z) \\ &= g_3(\tfrac{1}{2}x + \tfrac{1}{2}y + z,\ -\tfrac{1}{2}x + \tfrac{1}{2}y - z, \tfrac{1}{2}x - \tfrac{1}{2}y - z). \end{aligned}$$

Proof of Theorem 1. By the assumption that only those genera containing multiple spinor genera are considered, we are restricted to the discriminants in the list appearing in [7, p. 235], with the correction that 1280 should be deleted from that list and 1372 added. For the remainder of the proof, we shall consider only genera having one of these discriminants d. For each such genus containing multiple spinor genera, the number of spinor genera is two. Representatives of all classes within these genera were obtained from the table [4], and these classes were separated into spinor genera through a combination of the techniques described in the previous section.

In those genera having class number $h = 2$, forms in both classes are trivially spinor regular. Of these, the ones which are in fact regular appear in [7, Theorem 3], and those classes which are not regular have representatives (3.1), (3.2), (3.3), (3.5), (3.11) and (3.12). Genera with $h = 3$ occur for $d = 162$, 256, 486, 648, 686 and 1944. In these cases, one spinor genus necessarily contains a single class, which is again trivially spinor regular. Among these classes, those which are regular appear in [7, Theorem 3], while those which are not regular have representatives (3.4), (3.7) and (3.8). The fact that there are no spinor regular forms in the opposite spinor genera which contain two classes each is easily established by producing integers represented by one class but not the other. Genera with $h = 4$ occur for $d = 378$, 512, 640, 648, 864 (two genera), 1024, 1152 (two genera), 1512 and 1944. Of these genera, all separate into two spinor genera of two classes each, with the exception of one genus having $d = 512$. That genus has one spinor genus containing only one class, represented by (3.6). This form fails to represent 1, so it is spinor regular, but not regular. Of the classes in the remaining genera, all except the ones represented by (3.9) and (3.10) can be eliminated by producing integers represented by one class, but not the other, in the spinor genus. Details of the splittings into spinor genera of the genera of class number 4 can be found in [1].

No spinor regular forms in the scope of [4] occur in genera with $h \geqslant 5$ and two spinor genera. Of the genera which need to be investigated, $h = 5$ occurs for $d = 486$, 1372 and 1944, $h = 6$ occurs for $d = 702$, 1024, 1026, 1350 (two genera), 1512 and 1664, $h = 7$ occurs for $d = 1134$, $h = 8$ occurs for $d = 1674$ and 1944, and $h = 10$ occurs for $d = 1458$ and 1998. In all cases, the occurrence of spinor regular forms is ruled out by finding integers represented by the spinor genus, but not by the individual form. As all cases follow similarly, we give the computations only for the genus $\mathscr{G}$ of discriminant 1998 described in §2. As observed there, no integer less than 37 can be spinor exceptional for $\mathscr{G}$ (in fact, $\mathscr{G}$ has no spinor exceptional integers). The desired conclusion then follows from an investigation of the computer-generated representation numbers. It can be seen from Table 2 that the integers 1, 3 and 10 are represented by at least one form in $\mathscr{G}$, but f_7, f_8, f_9 and f_{10} fail to represent 1; f_3, f_4, f_5 and f_6 fail to represent 3; and f_1 and f_2 fail to represent 10. Thus, no form in $\mathscr{G}$ is spinor regular.

One additional example outside the range of **[4]** of a form which is non-trivially spinor regular is given in the next proposition.

PROPOSITION 3. *The form* $f = 9x^2+16y^2+48z^2$ *of discriminant* 55,296 *is non-trivially spinor regular, but not regular.*

Proof. The form lies in the genus of class number 4 containing the form $x^2+48y^2+144z^2$, which was proved to be regular by Jones and Pall **[8]**. This genus contains two spinor genera of two classes each. A representative for the other class in the spinor genus of f is

$$g = 16x^2+25y^2+25z^2+14yz+16xz+16xy.$$

The spinor regularity of f follows from the equations

$$\begin{aligned} g(x,y,z) &= f(y-z, x+\tfrac{1}{2}y+\tfrac{1}{2}z, -\tfrac{1}{2}y-\tfrac{1}{2}z) \\ &= f(y-z, \tfrac{1}{2}x+y+z, -\tfrac{1}{2}x) \\ &= f(y-z, \tfrac{1}{2}x+\tfrac{1}{2}y+\tfrac{1}{2}z, -\tfrac{1}{2}x+\tfrac{1}{2}y+\tfrac{1}{2}z). \end{aligned}$$

That f is not regular follows from the obvious fact that f fails to represent 1.

4. *A finiteness result*

Let $\mathscr{F}$ denote the collection of all positive definite primitive integral ternary quadratic forms. In this section, a finiteness theorem of Watson **[14]**, for the number of classes of forms in $\mathscr{F}$ which are regular, is extended to those forms which are spinor regular. For this purpose, it is convenient to introduce some additional notation. For a subset X of $\mathscr{F}$, let $R(X)$ be the set of all integers represented by at least one form in X, and for a positive real number ξ, let

$$R_\xi(X) = \{\alpha \in R(X) : 0 < \alpha < \xi\}.$$

For $f \in \mathscr{F}$, let

$$E(f) = R(\text{gen} f)\backslash R(f), \qquad E(\text{spn} f) = R(\text{gen} f)\backslash R(\text{spn} f),$$

and

$$SE(f) = R(\text{spn} f)\backslash R(f).$$

Define $E_\xi(f) = R_\xi(\text{gen} f)\backslash R_\xi(f)$, and define $E_\xi(\text{spn} f)$ and $SE_\xi(f)$ analogously. Note that

$$SE_\xi(f) = E_\xi(f)\backslash E_\xi(\text{spn} f). \tag{4.1}$$

For $f \in \mathscr{F}$, an asymptotic lower bound for $|SE(f)|$ in terms of the discriminant d of f will be established, from which the desired finiteness result will follow.

LEMMA 2. *For any positive real number* ξ *and any* $f \in \mathscr{F}$ *of discriminant* d,

$$|E_\xi(\text{spn} f)| \leqslant 2^{\omega(2d)}\xi^{\frac{1}{2}},$$

where $\omega(2d)$ *is the number of distinct prime divisors of* $2d$.

Proof. If $a \in E(\mathrm{spn} f)$, then $2a$ is a spinor exceptional integer for the genus of the lattice L corresponding to $2f$. For primes $p \nmid 2d$, $\theta(O^+(L_p)) = u_p \dot{\mathbb{Q}}_p^2$ [**11**, 92: 5], so it follows from condition (1.2) that $\mathrm{ord}_p(2a) \equiv 0 \pmod 2$. Thus, any prime q for which $\mathrm{ord}_q(a) \equiv 1 \pmod 2$ must be a divisor of $2d$. So the integers in $E(\mathrm{spn} f)$ lie in at most $2^{\omega(2d)}$ square classes, and the lemma follows.

LEMMA 3. *For any $\delta > 0$ and sufficiently small $\varepsilon > 0$*

$$|SE_{d^{1-\delta/3}}(f)| > d^{1-\delta/2-\varepsilon}$$

for all $f \in \mathcal{F}$ of sufficiently large discriminant d.

Proof. For any $\varepsilon > 0$, $2^{\omega(2d)} \leqslant \tau(2d) = O(d^{\varepsilon})$, where $\tau(2d)$ is the number of positive divisors of $2d$. So, by Lemma 2,

$$|E_{d^{1-\delta/3}}(\mathrm{spn} f)| = O(d^{\frac{1}{2}-\delta/6+\varepsilon}). \tag{4.2}$$

For sufficiently large d, by [**14**],

$$|E_{d^{1-\delta/3}}(f)| > d^{1-\delta/2-\varepsilon}. \tag{4.3}$$

The lemma now follows from (4.1), (4.2) and (4.3).

The desired asymptotic lower bound for $|SE(f)|$ now follows immediately from Lemma 3.

THEOREM 2. *For any $\delta > 0$, $|SE(f)| > d^{1-\delta}$ holds for all $f \in \mathcal{F}$ of sufficiently large discriminant d.*

As f is spinor regular if and only if $|SE(f)| = 0$, we obtain the following.

COROLLARY 2. *There exist only finitely many equivalence classes of spinor regular positive definite primitive integral ternary quadratic forms.*

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