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A ternary additive problem

8.1 A general conjecture

Suppose that $k_1, k_2, \ldots, k_s$ are s integers satisfying

$$2 \leqslant k_1 \leqslant k_2 \leqslant \ldots \leqslant k_s \quad \text{and} \quad \sum_{j=1}^{s} k_j^{-1} > 1. \tag{8.1}$$

Then the arguments discussed above, particularly in Chapters 2 and 4, suggest that the equation

$$\sum_{j=1}^{s} x_j^{k_j} = n \tag{8.2}$$

has a solution in natural numbers $x_1, \ldots, x_s$ whenever

(i) for each prime p and large k the equation (8.2) is soluble modulo p^k with $p \nmid x_j$ for some j;

(ii) n is sufficiently large.

There are some exceptions to this, see Exercise 5, but they all seem to have the general property that for some i there is a polynomial sequence of n for which $n - x^{k_i}$ has certain multiplicative properties arising from its polynomial factorisation which are at odds with the multiplicative properties of $\sum_{j=1, j \neq i}^{s} x_j^{k_j}$. A simplified form of this phenomenon occurs in Exercise 2. Even in these examples it should be true that (8.2) holds for almost all n.

There has been a great deal of work on questions of this kind, much of it rather inconclusive in nature because the treatment of the minor arcs in the present state of knowledge generally requires $\sum k_j^{-1}$ to be appreciably larger than unity.

The smallest value of s for which (8.1) is satisfied is $s = 3$. Then the only case which has been completely solved is that of $k_1 = k_2 = k_3 = 2$, the classical theorem of Legendre on sums of three squares. However, in all the remaining cases it has been shown that almost all numbers

This is readily done starting from the formula

$$N = p - \frac{1}{2} + \frac{1}{2}\left(\frac{-1}{p}\right)_L - \sum_{x=2}^{p-1} \frac{1}{4}\left(1 - \left(\frac{x}{p}\right)_L\right)\left(1 - \left(\frac{x-1}{p}\right)_L\right).$$

8.8 Exercises

1 Show that almost every natural number is of the form $p + x^k$.

2 (Babaev, 1958) Show that card $\{n : n \neq p + x^k, n \leqslant X\} \gg X^{1/k}$.

3 Let $R(n)$ denote the number of solutions of

$$x^2 + y^3 + z^6 = n$$

with $x > 0$, $y > 0$, $z > 0$. Show that

(i) $\sum_{n \leqslant X} R(n) = X\Gamma(\frac{3}{2})\Gamma(\frac{4}{3})\Gamma(\frac{7}{6}) + O(X^{5/6})$,

(ii) $\Gamma(\frac{3}{2})\Gamma(\frac{4}{3})\Gamma(\frac{7}{6}) = 0.73\ldots$,

(iii) $x^2 + y^3 + z^6 \equiv n \pmod q$ is always soluble with $(x, q) = 1$.

4 Obtain an asymptotic formula for the number of representations of a number as a sum of two squares, two cubes and two fifth powers.

5 (modified version of Jagy & Kaplansky, to appear) Suppose that $p \equiv 3 \pmod 4$, $p > 3$, $v = x^2 + y^2 = (18p)^3 - z^9$ and $u = 18p - z^3$. Show that $z \equiv 3 \pmod 4$, $(2p, u) = 1$, $(u, v/u) \mid 3^5$, $u \equiv 3 \pmod 4$ and 3 divides u to an even power. Prove that there is a prime number $q \equiv 3 \pmod 4$ and an odd natural number s such that $q^s \parallel u$ and $q \nmid v/u$. Deduce that $x^2 + y^2 + z^9 = (18p)^3$ is insoluble.

R. C. VAUGHAN
Professor of Pure Mathematics and EPSRC Senior Fellow
Imperial College, University of London

The Hardy–Littlewood method

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