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Progress of iteration theory since 1981

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Introduction

This survey tries to highlight a number of recent developments in iteration theory, and to point out a number of unsolved problems, thus also trying to predict the direction the evolution may take.

At least two things in this approach are arbitrary. "Recent" was chosen to mean "since about 1981", when [Targonski 81] was published, to the best of my knowledge the first book on iteration theory in general.

The choice of the topics is also, by necessity, somewhat arbitrary. Obviously I am talking more about the fields I know more about. In some cases, there are obvious objective reasons for the choice. For instance, numerical methods are in large part based on iteration; this immense field no longer can be counted as part of iteration theory proper. Also, one-dimensional discrete dynamics is now a field in its own right; while I devoted a part of [Targonski 81] to it, now I will only give references. I can however promise the following. The "iterated list of references", that is, the union of the lists of references in the books and papers listed in this paper united with the list of references itself, does contain a large part of what has been done in iteration theory in recent years.

The following fields within iteration theory will be treated.

(1) *Orbit theoretical iteration* theory, that is, study of the structure imprinted upon a set by a given self-mapping.
(2) *Algebraic iteration theory.*

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(3) *Iteration of formal power series.*
(4) *The Liedl transformation* (Pilgerschritt transformation) as a method of embedding a function in a one-parameter group of functions where the parameter takes on all real values (continuous iteration group).
(5) "*Aczél–Jabotinsky dynamics*". The three Aczél–Jabotinsky equations can be derived from the translation equation but not, in general, vice versa. Thus "weak dynamical systems" arise which have some but not all properties of a true dynamical system satisfying the translation equation.
(6) *The functional equations of Abel and of Schröder. Commuting functions. Real iterates.*
(7) *Cellular automata.* These automata, discovered by von Neumann and Ulam and then half forgotten now have a renaissance, with many applications. Their iteration (cellular automata are discrete, autonomous semi-dynamical systems!) poses many problems; there are interesting results.
(8) *Functional analysis.* Iteration of a function can be discussed by (crudely speaking) considering the linear operator of right composition with the function. This leads to new insights and results.
(9) *Phantom iterates.* Since functions in general have no iterative roots (fractional iterates) of every order (and thus no continuous time iterate) "generalized embeddings" have been sought for a long time. Since 1984, the idea of phantom iterates offers one such approach.
(10) An *Appendix* briefly discusses various topics which are outside the main fields outlined above but should be mentioned in our survey.

1. Orbit theory

In orbit theory we look solely at the structure imprinted on a set S by a self-mapping f. There is no other (algebraic, topological, ...) structure on S, which now is the union of (Kuratowski–Whyburn) orbits of f; for the simple properties of orbits see e.g. [Targonski 81], [Targonski 84], chapters 1 and 2.

We can give a few examples of purely "orbit theoretical" results. We need the notion of ultrastability ([Sklar 1969]). f is called ultrastable if $f|f(s)$ is bijective. Ultrastability is "almost as good as bijectivity", as the following result shows ([Weitkämper 85]): a mapping can be embedded in a $\mathbb{Z}$-group if and only if it is ultrastable.

Curiously, the perhaps best known unsolved problem in iteration theory appears in a purely orbit theoretical context. It is the $3x + 1$-problem, also called the Collatz problem, or the Ulam problem, or the Syracuse problem, or the Kakutani problem; Hasse's algorithm is also a usual term. We state the conjecture in the following simple form, following [Wagon 85]. Given the self-mapping f of $\mathbb{N}$ onto itself

$$f(n) = \begin{cases} \frac{n}{2} & (n \text{ even}) \\ 3n+1 & (n \text{ odd}) \end{cases}$$

one sees at once that $(1, 4, 2)$ is a 3-cycle of f.

Conjecture: every splinter (iteration sequence) of f terminates in the cycle (1, 4, 2). Much numerical work has gone into the problem, stochastical methods were used, the question of decidability raised. For a survey see [Lagarias 85]. Interestingly, the problem can be reformulated so as to involve chaos and fractals ([Agnes, Rasetti 88]). The problem was unsolved in 1991 ([Gale 91]).

2. Algebraic iteration theory

It seems clear, that many problems in "orbit theory" (cf. Section 1) could be treated by algebraic methods. A good example is the case of iterative square roots of self-mappings of arbitrary sets; the solution was given by [Isaacs 50]. Since the set of all self-mappings of a set is a semigroup, finding square roots of elements in a semigroup is a generalization of the Isaacs problem. This problem was solved in [Snowden, Howie 82] for the case of finite sets. It would be interesting to see whether the general solution of the iterative root problem in [Riggert 75], see also [Targonski 81], Section 2.1, (roots of arbitrary order on an arbitrary set) could be treated in the Snowden–Howie style.

An approach to "algebraization" of iteration could possibly be through unary algebras with research results already in the nineteen-sixties. As starting point one could take [Skornjakov 77] (with 41 references!) as well as [Chvalina, Matoušková 84] and [Blažková, Chvalina 84].

Since orbits are equivalence classes of an equivalence relation (existence of a common successor), recent work [Schleiermacher 93] in the direction of the Krasner theorem on invariant relations ([Krasner 38]) could become a tool in orbit theory.

Following the pioneering work of Carlo Bourlet ([Bourlet 97_1, 97_2]) right composition operators $T\varphi := \varphi \circ f$ (defined on suitable function spaces) have become part of iteration theory. It is of interest also to consider (nonlinear) left composition operators ($A\varphi := \alpha \circ \varphi$) and so on. An attempt to systematize all this is in [Targonski 90]. Results on algebraic right composition operators are in [Böttcher, Heidler 92].

Last, but not least, linear mappings $\mathbf{r} \mapsto A\mathbf{r}$ ($\mathbf{r} \in \mathbb{R}^n$), where A is an $n \times n$ matrix, pose nontrivial iteration problems for $n \geq 1$. There is of course a large amount of work on this, some by people who did not know they were doing iteration theory (seen from our point of view). A recent contribution (on square roots of uppertriangular matrices) is [Miller 91].

The orbit structure of any self-mapping of a set is invariant under conjugacy. For work in the important field of conjugacy see e.g. [Schweizer, Sklar 88].

3. Iteration of formal power series

Research by L. Reich and his colleagues and collaborators in Graz continued vigorously during the decade we are surveying.

[Reich, Schwaiger 80] introduced a linearization method for the solution of certain functional equations, which later turned out to be important also in another context, interpreted as a phantom iterate (see Section 9). In [Reich 85] the third Aczél–Jabotinsky equation for formal power series in one variable is solved. The relations between the three Aczél–Jabotinsky equations and their relations with the translation equation (5.1) was clarified around that time (see Section 5) in [Reich 88, 89, 91], [Aczél, Gronau 88, 88_1], [Gronau 91, 91_1].

The relationship between families of commuting functions and iteration groups have been investigated also in the context of formal power series ([Reich 88, 89]) see Section 6.

As these examples show, formal power series occupy an important position in iteration theory—many general problems are also treated in this context.

Results of the theory up to the end of 1980 have been surveyed in [Targonski 81], Section 6.2. A description of early results can be found in [Peschl, Reich 71]. An outline of the iteration theoretical aspects of formal power series (more specifically of formal biholomorphic mappings) and some open problems are given below. We closely follow L. Reich (personal communication).

Let $\mathbb{C}[[x]]$ be the ring of formal power series in the indeterminate $x = {}^t(x_1, \ldots, x_n)$ over $\mathbb{C}$, Γ the group of automorphisms F of $\mathbb{C}[[x]]$ continuous in the order topology, moreover $F|_{\mathbb{C}} = id$. The following are the central problems leading to further development.

I. Does there exist, for a given $F \in \Gamma$, a family $(F_t)_{t \in \mathbb{C}}$ in Γ ("iteration of F") such that $F_1 = F$ and $F_t \circ F_s = F_{t+s}$ for every $t, s \in \mathbb{C}$?

II. Does there exist, for a given $F \in \Gamma$ and for a given $r \in \mathbb{N}$ a $G \in \Gamma$ ("r-th iterative root of F") such that $g^r = F$ where g^r denotes the r-th iterate of g?

(1) Building on results of S. Sternberg, N. Lewis and E. Peschl, L. Reich looked for criteria ensuring the existence of analytic iterations for given $F \in \Gamma$ and given choice Λ of logarithms of eigenvalues of the linear part of F. $(F_t)_{t \in \mathbb{C}}$ is called analytic if the coefficients g_r in $r_t(x) = \sum_{r \in \mathbb{N}_0^n} g_r(t)x^r$ are entire functions. The problem was solved by L. Reich and J. Schwaiger using the normal forms under conjugation in Γ in particular the smooth normal forms for 1. The proofs also furnished the construction of the iteration groups and pointed to connections with autonomous differential systems and differential equations with complex linearization, following P. Erdös and E. Jabotinsky.

(2) Analogous results (using the normal forms mentioned above) were found for the existence of iterative roots.

(3) From these criteria grew results on connection between existence of analytic iterations and of "sufficiently many" iterative roots of an $F \in \Gamma$ (results of L. Reich and A. R. Kräuter; cf. (4) below).

(4) If one demands only continuity of the coefficients G_r ("continuous iteration") then the method leads to the normal forms. F is analytically iterable if and only if it is continuously iterable. The continuous iterations of an $F \in \Gamma$ are precisely the real-analytic iterations. (Results of L. Reich and W. Bucher.)

(5) The next question is this. What about iteration with no conditions whatsoever on the coefficients ("iterable F"). G. Mehring and C. Praagman showed, independently, that F is analytically iterable if and only if it is iterable. Using methods of algebraic geometry, C. Praagman even showed the following. If $F \in \Gamma$ has iterative roots of all orders then F is analytically iterable.

(6) In the case of one variable ($n = 1$) the theory of iterations, as sketched above, and results on the set of solutions of the equation

$$(G \circ \Phi)(x) = \frac{d\Phi}{dx} G(x)$$

(the Julia equation, a special case of the third Aczél–Jabotinsky equation) play a decisive part in the explicit description of the families of commuting automorphisms. For $n \geq 2$ this question is open.

(7) Another interesting problem seems to be the distribution of iterable (and of not iterable) automorphisms in the neighbourhood of a given $F \in \Gamma$. "Neighbourhood" may be defined by the order topology or by the coefficient-wise topology. First results in this direction (for $F = id$) were due to S. Sternberg.

So far the concise description of history and unsolved problems suggested by L. Reich. Let me add that [Reich 92, 93_1] are contributions concerning problem (7), about the distribution of iterable functions.

Closing this section let me express my personal feeling that the linearization method [Reich 71], [Reich, Schwaiger 80] will play a continued important part in Phantom Dynamics (cf. Section 9).

4. Liedl's Pilgerschritt transformation

Embedding of functions in "time continuous" iteration (semi-) groups is one of the central themes of iteration theory; it appears in several places also in the present survey.

There is an attempt to solve this problem which has been in the folklore of the theory for many years. It can be explained in the language of the (Bourlet) substitution operators. Assume f, the function to be embedded is a continuous self-map of I, the closed unit interval. Consider the following bounded linear operator A on $C(I)$: $A\varphi = \varphi \circ f$. One may say "Now determine the linear operator $\log A$, then you have the embedding of A: $A^t = e^{t \log A}$. Applying this semigroup of operators to $x \in C(I)$, we find $A^t x = f^t(x)$ and the embedding has been achieved".

This sounds too good to be true and in fact it is not true. The scheme does not work except in special cases. The first problem of course is with the existence of the logarithm of a substitution operator. Trying to sweep the problem under the carpet, we encounter divergence problems. Thus a method not using logarithms would be of advantage. Even so, any embedding would have to have a built-in failure mechanism for the case that the embedding does not exist and in general it does not exist.

R. Liedl's Pilgerschritt transformation is a "logarithm-free" method. Introduced in the nineteen-seventies, it is much more than a method of embedding. It has become a field of research in its own right, branching out in various directions. Work published before the end of 1980 was included in [Targonski 81], with the appropriate references. Since 1980 much new work was done, new concepts emerged and new results were found. We attempt an overview of some of these.

The Liedl transformation (Pilgerschritt transformation) is a method for finding one-parameter subgroups in topological groups. Given an element of the group in the same connected component as the unit element, an arbitrary path between the two is taken, and repeatedly subjected to the Liedl transformation. The resulting sequence of paths may converge to—or even reach in finitely many steps—a "homomorphic path", that is, a one-parameter subgroup.

One way of getting acquainted with Liedl's idea and early work by him and his co-workers, in English, is reading Chapter 4 in [Targonski 81]. The papers [Liedl, Netzer, Reitberger 81, 82] are introductions (in German) to the theory and its results at that stage.

The Pilgerschritt transform of a path $A(t)$ $(0 \leq t \leq 1)$ is given among others in $GL\ (n, \mathbb{R})$, for the case of a smooth path by

$$\tilde{A}(t) = M(1), \tag{4.1}$$

where M is the unique solution of the initial value problem

$$M'(\tau) = tA'(\tau)A^{-1}(\tau)M(\tau) \qquad M(0) = I. \tag{4.2}$$

In full generality the definition of the transformation is

$$\tilde{A}(t) + \lim_{|\pi| \to 0} \Pi(A; \pi, t) \tag{4.3}$$

where the Liedl product (Pilgerschritt product) Π is defined as

$$\Pi(A; \pi, t) := [A(a_{m-1}^*)A(a_{m-1})^{-1} \cdots A(a_0^*)A(a_0)^{-1}]. \tag{4.4}$$

Here π is a partition $0 = a_0 < a_1 < \cdots < a_m = 1$ of the unit interval, $|\pi| = \max a_{k+1} - a_k$ and $a_k^* := a_k + t(a_{k+1} - a_k)$ and $t \in [0, 1]$ is fixed. While [Netzer 82] deals with the convergence of the sequence of iterated Pilgerschritt transforms, [Netzer, Reitberger 82] investigates this problem for nilpotent Lie groups. In [Förg-Rob 85] it is shown that the transform can be computed within the Lie-algebra of the Lie group in question. Another part of the paper deals with the Pilgerschritt transform in the group of vectors of formal power series.

In [Förg-Rob, Netzer 85] the following problem is treated. Paths which are already homomorphic are invariant under the Pilgerschritt transformation, that is, fixed points. The decisive (and highly nontrivial) question is whether a given fixed point is attractive or not. This problem is treated in subgroups of the group of invertible matrices, using the method of product integration.

[Liedl 86] treats group valued power series.

[Förg-Rob 89] adds new results for the complex affine group.

[Liedl, Netzer 89] actually contains the material of two lectures at ECIT 87. Liedl's "short ruler" method uses an idea from differential geometry to solve the translation equation (with time-one condition), that is, achieve embedding. Netzer uses product Taylor expansion (PTE), that is, "Taylor products" to achieve the same goal. (For PTE and related topics see also [Cap 89]).

[Cap 91], somewhat different in style and approach compared to the rest of this Section, considers the Abel equation, an equation the author refers to as the (third) (Aczél-) Jabotinsky equation (actually it is a special case, the Julia equation) and the "inverse problem of ordinary differential equations". This last problem is the following. Given a time-one map of a time-continuous flow (it maps every point to the point where it will be in one time unit.) The task is to reconstruct the flow (or only its vector field). Thus we again have the embedding problem. This paper is an interesting meeting point of several topics discussed in the present survey.

[Netzer, Liedl 91] introduces an "improved version" of Liedl's transform: the fast Pilgerschritt transformation (FTP). The starting point is replacing the initial value problem (4.2) by a "better one". [Liedl, Netzer 91] is a more detailed paper of 54 pages, also discussing FTP. See also [Netzer 92], where a one-step Liedl transformation is described: under certain conditions the homomorphic path is reached in one step.

To conclude this section we note that the Liedl transformations may be used in the construction of phantom iterates (see Section 9).

5. The Aczél–Jabotinsky equations. "Weak dynamics"

The translation equation

$$F[F(x, s), t] = F(x, s + t) \tag{5.1}$$

describes an autonomous semi-dynamical system if x is interpreted as a point in the state space X ("phase space") of some "system", and the second variable of F ranges over an additive semigroup of $\mathbb{R}$. Customarily but by no means necessarily

$$F(x, 0) = x \tag{5.2}$$

is stipulated.

(For the general theory of (5.1) and a far-reaching generalization, the transformation equation, see the survey paper by Z. Moszner in this issue.)

The mapping

$$F(x, 1) =: f(x) \tag{5.3}$$

is the "time-one map" of the semidynamical system.

If t ranges over $\mathbb{R}$, then F is bijective for every fixed t and we have a dynamical system. (Often a semidynamical system is called, somewhat sloppily, a dynamical system.)

For example an iteration group $f^t(x)$ satisfies (5.1) if we put $F(x, t) = f^t(x)$.

The translation equation (5.1) requires no structure on X; a fairly general but still very useful condition is that X be a Banach space. In our present context we take x to be either a real number or a complex number. We present the idea formulated for this case.

As shown in [Aczél 49 and elsewhere] (see also [Jabotinsky 55, 63]), three equations can be derived from the Translation Equation under suitable differentiability conditions. Introducing

$$g(x) := \frac{\partial F(x, t)}{\partial t}\bigg|_{t=0} \tag{5.4}$$

we find the three Aczél–Jabotinsky equations

$$\frac{\partial F}{\partial t} = g \frac{\partial F}{\partial x} \tag{5.5a}$$

$$\frac{\partial F}{\partial t} = g \circ F \tag{5.5b}$$

and consequently

$$g \frac{\partial F}{\partial x} = g \circ F. \tag{5.5c}$$

An avalanche of research was started by D. Gronau asking the following question (see [Targonski 84], Gronau's problem (3.3.11)): Can the translation equation be deduced from the first or third Aczél–Jabotinsky equation? These questions and related ones were answered in the negative in [Aczél, Gronau 88_1, 88_2]; see also [Gronau 88, 91, 91_2] and general solutions were given of (5.5a,b,c) individually, by pairs and collectively. An attempt of a dynamical interpretation was made in [Targonski 91] (see also [Aczél 91]). Results in this direction are given in [Gronau 91]. It may be of interest to find applications of what could be called "Aczél–Jabotinsky dynamics": time evolutions satisfying the first, (5.5a) or the third, (5.5c) Aczél–Jabotinsky equations or both, with or without one or both conditions (5.2), (5.3).

The second Aczél–Jabotinsky equation (5.5b) is equivalent to the translation equation (5.1) if the initial value problem "(5.1b) with (5.2)" is well posed.

Time evolutions of this kind could emerge for example in biology, or in economics, but perhaps even in physics.

For formal power series the problem was solved in [Reich 91]; see also [Reich 85, 88, 89].

6. Iteration sequences. Groups and semigroups of iterates

Throughout this survey we emphasize new directions of research as well as generalizations. This section, however, is about "hard core, classical" iteration. Typically real functions of one variable are treated, but functions in $\mathbb{R}^n$, even in topological spaces make their appearance. Of course, new ideas and techniques emerge, but the flavour of research is the traditional one.

The first thing to read, of course, is chapter 1, Iteration, of the important book [Kuczma, Choczewski, Ger 90]. Also relevant to our topic in this section are the results about the Schröder and Abel equation, to be found in various sections of the book.

We note that there are two families of functions easy to embed in a group.

For $f(x) = \lambda x$ $(0 < \lambda < 1)$ we have $f^t(x) = \lambda^t x$, and for $g(x) = x + c$ $(c \neq 0)$ we have $g^t(x) = x + ct$. For functions conjugate to a linear transformation $f(x) = \varphi^{-1}[\lambda\varphi(x)]$ or to a translation $g(x) = \psi^{-1}[\psi(x) + c]$ we find the embedding $f^t(x) = \varphi^{-1}[\lambda^t\varphi(x)]$ ("Schröder form") or $g^t(x) = \psi^{-1}[\psi(x) + ct]$ ("Abel form"). Then existence of bijective solutions

$$\varphi[f(x)] = \lambda\varphi(x) \qquad \text{(Schröder equation)} \tag{6.1}$$

or

$$\psi[g(x)] = \psi(x) + c \qquad \text{(Abel equation)} \tag{6.2}$$

in a suitable domain implies embeddability. Thus the Schröder equation and the Abel equation (as well as the seldom occurring Böttcher equation $\varrho[h(x)] = \varrho(x)^a$) are closely linked to the problem of embedding ("continuous iteration" in an older and ambiguous terminology).

We also note that different elements of the same iteration group commute: $f^s \circ f^t = f^{s+t} = f^{t+s} = f^t \circ f^s$. The following question arises. Under what conditions are two commuting bijections "iterates of each other", that is, elements of the same iteration group? The question is even more profitably posed for maximal sets of commuting functions. The study of such families is interesting for its own sake.

All this has been known for a long time. In this section we also quote some new results in this general direction. We cite a few recent papers.

In [Zdun 85], homeomorphisms of the circle are embedded in a time-continuous flow, that is, in a real-parameter group of homeomorphisms.

In [Zdun 89] older results ([Zdun 79], [Smajdor 85]) are generalized: if f^t is an iteration semigroup on a compact metric space and $t \mapsto f^t(x)$ is measurable then it is continuous. This is the main result.

In [Zdun 90], quasi-continuous iteration groups and semigroups of real functions are characterized. These are semigroups $f^t(x)$ such that $f^t(x)$ is a continuous function of t, while no such condition is imposed with respect to x.

[Zdun 91] gives the structure of iteration groups of continuous functions such that the group elements with the exception of the identity mapping have no fixed points. In the representation of the group the "Schröder form" and a generalization of the "Abel form" appear.

In [Zdun 85_1] the representation for regular iteration semigroups $f^t(x) = \lim_{n \to \infty} f^{-n}[\lambda^t f^n(x)]$ appears. It may be interesting to look at this formula under conditions as general as possible. There is a strange formal similarity to the scattering operator in quantum theory.

$\mathbb{C}^r$ iteration groups are discussed in [Zdun 89_1].

In [Zdun 91_1], continuous iteration groups of fixed point free mappings in $\mathbb{R}^n$ are treated. The principal result is that these groups can be represented in the Abel form.

[Zdun 88] treats a case where, under appropriate conditions, two commuting functions are—as defined earlier in this Section—"iterates of each other". [Zdun 89_2] deals with simultaneous Abel equations, again leading to commuting functions and representation in the "Abel form" of an iterative group appearing in this context.

For systems of Abel equations and related topics see also [Neuman 82, 89].

[Zdun 92] discusses continuous, strictly increasing, commuting self-mappings of an open set. Relations between the iteration sequences of the two functions are established.

[Smajdor 89, 92] are recent additions to work of the author on the iteration of set-valued functions.

For the characterization of Zdun flows see [Sklar 87], and for the non-embedability of the baker's transformation [Schweizer, Sklar 90].

7. Cellular Automata

Cellular automata were introduced by J. von Neumann and S. Ulam; see the volume [von Neumann 66]. The matter lay dormant for a long time (the posthumous volume cited contains ideas of von Neumann from various times). During the past twenty-odd years, furious research activity sprang up, and now cellular automata is a large and rapidly growing field. For an overview of the field at one stage see [Farmer et al. 85].

A cellular automaton is a countable set (the "grid"; think for instance of a rectangular grid in the plane); each grid point is occupied by one of finitely many symbols (think for instance of 0, 1). Every grid point has a finite neighbourhood, consisting of grid points not necessarily neighbours in the ordinary geometrical sense; the neighbourhood may include the point itself. At every time impulse, every grid point gets a new symbol (possibly the old one) as a function of the "local configuration", that is, the way the neighbourhood is occupied by symbols; this is the local transition function.

A useful property is that every grid point should have the "same" neighbourhood and the same local transition function. These properties are expressed as "shift invariance". If we try to survey all possible or all useful "grids", questions of algebraic topology arise.

The local transition function is not everything, of course. Since all grid points get a new symbol at the "time impulse", the global configuration (a map of the countable set of grid points to the finite set of symbols) also changes. The map from the set of all configurations (the configuration space) to itself is the global transition function.

The problem arises of characterizing those global transition functions which are induced by a local transition function. In [Ferber 91] the following is shown. Introducing an appropriate metric on the configuration space, a self-mapping of the configuration space is the global transition function of a cellular automaton if and only if it is continuous.

It is a striking feature of cellular automata that a very simple local transition rule may give rise to very complicated and interesting global behaviour. The perhaps most famous example is John Horton Conway's "game of life" automaton (see [Gardner 70]).

Why is all this relevant to iteration theory?

From the definition it is clear that a cellular automaton is an autonomous discrete-time semidynamical system. The behaviour of such a system is given by the iteration of a continuous self-mapping of a configuration space. All results of iteration theory—starting with the orbit theoretical results—apply to the time evolution of cellular automata. We give a few examples: fractional iterates of cellular automata can be discussed; see [Ferber et al. 91]. The results on limit sets of continuous mappings [Graw 82, 84] can be applied to the time evolution of cellular automata (see [Langenberg 92]). Orbit theoretical results may be also applied ([Ferber 88]). A suitably modified version of the notion of orbit entropy ([Burkart 82]) can be used to study the behaviour of cellular automata (Langenberg 92]).

The number and scope of applications in computing, gas dynamics (to name only two fields) and so on, is breathtaking. Iteration theory has insights, results and methods which are (unfortunately) still largely unknown to outsiders. The challenge to iteration theory is clear.

8. Functional analysis

In [Bourlet 97, 97_1] the linear operator $F\varphi := \varphi \circ f$ was introduced. Bourlet's idea turned out to be a most fruitful one. The ramifications of this approach have been discussed in [Targonski 67, 81] and are outside the scope of this survey. However, work on the generators and co-generators of substitution semigroups continued until 1987; see [Targonski, Zdun 85, 87].

It seems that at the present time—and for some time to come—the most useful application of functional analysis to iteration theory will be the method of generalized embeddings (in particular, generalized iterative roots) we call phantom iterates (phantom dynamical systems). This will be discussed in Section 9.

9. Phantom iterates

As is well known, the functional equation $g^r = f$ (f, g are self-mappings of a set) has in general no solution;—in other words, the iterative root (fractional iterate) $f^{1/r}$ does not exist. To give a simple example: if f has exactly one 2-cycle ($x \neq y$, $f(x) = y$, $f(y) = x$), then f has no iterative square root. Then there exists no embedding f^t for f: $f^s \circ f^t = f^{s+t}$, $f^1 = f$, $t \geq 0$, since the choice $t = \frac{1}{2}$ would yield a (non-existing) square root. Thus the idea of generalized embedding arose quite naturally. Saying it quite simply: the trouble is that there are too many or too few orbits of a given type. (In our above example, extending the domain by two points forming a 2-cycle would help, unless there are additional obstacles.) In a similar sense, removing the offending 2-cycle would also help; but, in the presence of a topology, there now would be two holes in the domain. So, extension of domain is the better solution, and this idea was in fact carried out; see [Peschl, Reich 71], [Reich, Schwaiger 80], [Mira, Müllenbach 83].

Our approach is different. Formulated for the "maximal" problem, the problem of embedding, it can be crudely formulated as follows. Even if no embedding f^t exists, the Bourlet substitution operator $A\varphi := \varphi \circ f$ may be embeddable in a one-parameter semigroup of operators A^t; then A^t serves as a (weaker) version of the non-existing f^t. This is the phantom iterate, described intuitively and imprecisely. The idea was hinted at but not followed through in [Targonski 81], and introduced (under the temporary name of "weak iterate") in [Targonski 84]. Here the setting is quite general, but only the problem of iterative roots is addressed. The semigroup of all self-mappings of a set is isomorphically immersed in a larger semigroup, so that the equation $g^r = f$ has a solution in the larger semigroup. This approach was demonstrated on the case of self-mappings of a finite set, represented by "mapping matrices" with 0, 1 entries. If the r-th root of such a mapping is not itself a mapping matrix, it is still (in later terminology) a phantom root. This approach was followed up and elaborated in [Bartels 91].

Phantom iterates were formally introduced, rigorously defined and investigated in [Targonski 84_1]. The notion is introduced in a general form and then applied to certain continuous self-maps of the closed unit interval. A class of such functions was given which have non-trivial phantom square roots. This result was generalized in [Krause 88], [Bartels 91] and [Targonski 93].

Phantom iterates have been also established for formal power series. As already mentioned, a construction in [Reich, Schwaiger 80] can be used as phantom iterate, once one has the notion (Reich–Schwaiger phantom). For a thorough discussion, see [Schwaiger 89]; see also [Schwaiger 91].

Phantom roots of cellular automata were treated in [Ferber et al. 91], sections 2.3, 2.4, 4.1, 4.2, 4.3.

A direction in which so far nothing has been done is phantom roots of mappings from $\mathbb{N}$ to itself. As we saw, the finite case has been extensively dealt with, and there are also results on continuous self-mappings of [0, 1]; but $\mathbb{N} \to \mathbb{N}$ is terra incognita.

We conclude this section by describing a fairly general case of phantom iterates in the language of dynamics.

Consider the discrete autonomous semidynamical system (X, f). Here X is a compact topological space, the state space of some system, while f, the "next state function", is a continuous self-mapping of X. We are looking for a phantom embedding of f. Consider now Φ, the family of all real-valued continuous functions on X. (We could take complex valued functions on X, in the general case the functions could take values in any Banach algebra. Multiplication is defined pointwise). We interpret every φ as some special kind of measurement on X, so that $\varphi(x)$ is some real number partly characterizing the particular state $x \in X$ of the system. Introduce now the linear operator $A\varphi := \varphi \circ f$ on Φ, considered as a Banach space, and assume that an embedding of A exists in a semigroup A^t, where $t \geq 0$ is interpreted as time. Consider now the case that, for a particular t (for instance $t = \frac{1}{2}$), f^t does not exist, thus the state $f^t(x)$ does not exist. On the other hand, $(A^t\varphi)(x)$ is a numerical measurement of the non-existing state $f^t(x)$. For all $\varphi \in \Phi$ such a measurement can be carrried out and thus the non-existing state gains a kind of "phantom existence", hence the name.

Staying with the case $t = \frac{1}{2}$, it has been shown in [Targonski 84_1] that for certain f and some restriction on Φ, the fantom is the sum of two substitutions: $(A^{1/2}\varphi)(x) = \varphi[\alpha(x)] + \varphi[\beta(x)]$.

In this case (and in its generalizations) the system behaves as if it were in several states at the same time, and the individual measurement values have to be added to obtain the measurement values on the phantom states. An analogy with quantum mechanics is apparent; physicists have been quick to acknowledge this.

As already hinted at in Section 3, Liedl's transformation can be used to find a phantom embedding for invertible f ([Targonski 94]).

Appendix

In this Appendix, we briefly refer to fields which belong to iteration theory, but for reasons historical and/or practical, could not be included in this survey of research.

I. *Numerical methods.* A large part of this field is based on iteration. As an important example we cite [Deslauriers, Dubuc 91]. This field is immense and is now closely linked to computer science. An attempt to include iterative numerical methods in iteration theory would be like including the elephant in the Small Mammals House of a zoo.

Still, we mention one problem. It was noticed some time ago that by discretizing a continuous-variable problem (for instance, initial value problem) for computation purposes, chaos may appear in the solution which is not present in the rigorous solution of the original problem. Thus approximation may qualitatively falsify the solution. This phenomenon has been called "ghost dynamics" ([Ushiki 86]).

II. *Dynamics in one or two dimensions.* This important and rapidly growing field also belongs to the "hard core" of iteration theory and should be included in a survey such as this. In [Targonski 81] it was still possible to give a reasonably complete account of the field up to the time the manuscript went to the publisher. Since then, the field has grown so rapidly that I did not keep up entirely and can no longer attempt in good faith a survey.

Anyone interested in this field is well advised to look first at the book [Alsedà, Llibre, Misiurewicz 92]. In the list of references one finds at least eight more books on the subject, starting with [Collet, Eckmann 80], and a large number of papers by R. L. Adler, Ll. Alsedà, P. Blanchard, L. Block, R. Bowen, U. Burkart, A. Chenciner, P. Collet, J.-P. Eckmann, J. Franks, J. Guckenheimer, I. Gumowski, M. Hénon, M. R. Herman, P. Holmes, L. Jonker, P. E. Kloeden, A. G. Konheim, O. E. Lanford III, A. Lasota, T.-Y. Li, J. Llibre, E. N. Lorenz, A. M. McAndrew, J. Milnor, C. Mira, M. Misiurewicz, P. Mumbrú, Z. Nitecki, A. N. Sharkovsky,

C. Simó, S. Smale, J. Smítal, P. Štefan, P. D. Straffin, F. Takens, W. Thurston and many others.

All this is, of course, mainly in one dimension, but the key "key words" already appear: periodicity, chaos, topological entropy, strange attractors. For two dimensions, an introduction is provided by [Whitley 83].

For recent work applying symbolic dynamics, see [Lampreia, Sousa Ramos 91].

III. *Complex iteration.* This field would have to have a place of honour in any history of iteration theory because of the great discoveries of the XIX and early XX century. Here we hint mostly at work emerging during the past one or two decades. Complex iteration provides a particularly elegant way of dealing with certain dynamical systems in the plane. It is truly amazing what a variety of iterative behaviour can be seen even for the function $z^2 + \alpha$ as the parameter α is varied. Notions as Fatou set, Julia set, Mandelbrot set (not, of course, confined to complex iteration) arise here. For an introduction to this field see e.g. [Blanchard 84], [Douady, Hubbard 84/85]. J. Ecalle's theory of resurgent functions is an important contribution [Ecalle 81, 85].

IV. *Fractal sets.* These sets occur not only in iteration theory, but arise naturally as limit sets of splinters, boundaries of certain sets and so on. Such sets have been named "fractals" and extensively discussed by B. Mandelbrot (see [Mandelbrot 82]).

Fractals turn up also in connection with functional equations ([Dubuc 85]).

The notion of fractal seems to be related to C. Mira's concept of "frontière floue" ("vague boundary") see [Mira 79].

Also for application of fractals to computer graphics see [Barnsley 88].

For a survey of work of the Toulouse group up to 1987, see [Thibault 89].

Fractals, in particular the sets of limit points of iterative sequences in the plane can be very beautiful. This was discovered by I. Gumowski and C. Mira, who named this phenomenon "chaos esthétique" and showed an exhibition of attractive pictures in Toulouse, at the 1973 iteration conference. Others soon discovered commercial chances and now it is possible to buy posters and sets of slides showing fractal sets.

The beauty of the images can be enhanced by using different colours for the various constituents of the fractals. For a volume with such pictures see [Peitgen, Richter 86].

V. *Experimental mathematics.* This plays a part in iteration theory as in other parts of mathematics. It became feasible when high speed computing and also computer graphics became available. The "experimental results" can be used to find conjectures which then may be proved using the conventional methods of mathematics. Or, graphics may be used as "experimental proof" of statements. This is the extreme case. In general, the situation is that both "conventional" and "experimental" methods are used in a certain approach to dynamics which has an engineering flavour and which is close to applications. For an introduction see [Gumowski, Mira 80, 90_2]; see also numerous papers by C. Mira and his collaborators and students.

Concluding remark

Iteration can be considered as a field of research bordering on functional equations as well as on dynamics. This survey was prepared for publication in Aequationes Mathematicae. In order to keep the paper reasonably short, in this particular situation I leaned towards functional equations. Dynamics was emphasized less than it would have been in a more balanced treatment.

REFERENCES

[Aczél 49] ACZÉL, J., *Einige aus Funktionalgleichungen zweier Veränderlichen ableitbare Differentialgeichungen.* Acta Sci. Math. (Szeged) *13* (1949), 179–189.

[Aczél 91] ACZÉL, J., *Remarks on a problem of Gy. Targonski. Report of the 27th ISFE, Poland 1989.* Aequationes Math. *39* (1990), 314–315.

[Aczél, Gronau 88] ACZÉL, J. and GRONAU, D., *Some differential equations related to iteration theory.* Canad. J. Math. *40* (1988), 695–717.

[Aczél, Gronau 88_1] ACZÉL, J. and GRONAU, D., *Iteration, translation, commuting and differential equations.* In: Gronau, D. and L. Reich (eds), *Selected topics in functional equations.* [Grazer Math. Bericht Nr. 295], Math.-Stat. Sekt. Forschungsges. Joanneum, Graz, 1988.

[Agnes, Rasetti 88] AGNES, C. and RASETTI, M., *Complexity, undecidability and chaos: a class of dynamical systems with fractal orbits.* In: Livi et al. (eds), *Workshop on chaos and complexity, Torino, Oct. 5–11, 1987.* World Scientific, Singapore, 1988, pp. 3–25.

[Alsedá, Llibre, Misiurewicz 92] ALSEDÀ, LL., LLIBRE, J. and MISIUREWICZ, M., *Combinatorial dynamics and entropy in dimension one.* World Scientific, Singapore, 1992.

[Alsina et al. 89] ALSINA, C., LLIBRE, J., MIRA, C., SIMÓ, C., TARGONSKI, GY. and THIBAULT, R., (eds), *ECIT 87, Proc. Europ. Conf. on Iteration Th., Caldes de Malavella, Sept. 20–26, 1987.* World Scientific, Singapore, 1989.

[Barnsley 88] BARNSLEY, M., *Fractals everywhere.* Academic Press, New York, 1988.

[Bartels 91] BARTELS, A., *Über iterative Phantomwurzeln von Abbildungen (On iterative phantom roots of mappings).* Diploma (M.Sc.) Thesis, Universität Marburg, 1991.

[Blanchard 84] BLANCHARD, P., *Complex analytic dynamics of the Riemann sphere.* Bull. Amer. Math. Soc. *11* (1984), 95–141.

[Blažková, Chvalina 84] BLAŽKOVÁ, R. and CHVALINA, J., *Regularity and transivity of local-automorphism semigroups of locally finite forests.* Arch. Math. (Brno) 4, Scripta Fac. Sci. Nat. USEP Brunensis *20* (1984), 183–194.

[Böttcher, Heidler 92] BÖTTCHER, A. and HEIDLER, H., *Algebraic composition operators.* Integral Equations Operator Theory *15* (1992), 390–411.

[Bourlet 97] BOURLET, C., *Sur certaines équations analogues aux équations différentielles (On certain equations analogous to differential equations).* C.R. Acad. Sci. Paris *124* (1978), 1431–1433.

[Bourlet 97_1] BOURLET, C., *Sur les transmutations (On transmutations).* Bull. Soc. Math. France *25* (1897), 132–140.

[Burkart 82] BURKART, U., *Zur Charakterisierung diskreter dynamischer Systeme (On characterization of discrete dynamical systems).* Ph.D. Thesis, Universität Marburg, 1982.

[Cap 89] CAP, C. H., *Two approaches to the iteration problem of diffeomorphisms.* In [Alsina et al. 89], pp. 139–144.

[Cap 91] CAP, C. H., *Solving Abel, Jabotinsky and inverse ODE problems.* In [Mira et al. 91], pp. 19–28.

[Chvalina, Matoušková 84] CHVALINA, J. and MATOUSKOVÁ, K., *Coregularity of endomorphissm monoids of unars.* Arch. Math. (Brno) 1, Scripta Fac. Sci. Nat. USEP Brunensis *20* (1984), 43–48.

[Collet, Eckmann 80] COLLET, P. and ECKMANN, J. P., *Iterated maps on the interval as dynamical systems.* Birkhäuser, Boston, 1980.

[Deslauriers, Dubuc 91] DESLAURIERS, G. and DUBUC, S., *Continuous iterative iteration processes.* In [Mira et al. 91], pp. 71–78.

[Douady, Hubbard 84/85] DOUADY, A. and HUBBARD, J., *Etude dynamique de polynômes complexes (Dynamical study of complex polynomials) I, II.* [Publ. Math. Orsay 84-02 and 85-04], Univ. d'Orsay, Orsay, 1984–85.

[Dubuc 85] DUBUC, S., *Functional equations connected with peculiar curves.* In [Liedl et al. 85], pp. 33–40.

[Ecalle 81] ECALLE, J., *Les fonctions résurgents (On resurgent functions),* vol. 1, 2, [Publ. Math. Orsay], Univ. d'Orsay, Orsay, 1981.

[Ecalle 85] ECALLE, J., *Iteration and analytic classification of local diffeomorphisms of* $\mathbb{C}^v$. In [Liedl et al. 85], pp. 41–48.

[Farmer et al. 85] FARMER, D. et al. (eds), *Cellular automata.* North Holland, Amsterdam, 1985.

[Ferber 85] FERBER, R., *Zelluläre Automaten als dynamische Systeme (Cellular automata as dynamical systems).* Diploma (M.Sc.) Thesis, Universität Marburg, 1985.

[Ferber 88] FERBER, R., *Räumliche und zeitliche Regelmäßigkeiten zellularer Automaten (Spatial and temporal regularities of cellular automata).* Ph.D. Thesis, Universität Marburg, 1988.

[Ferber 91] FERBER, R., *Cellular automata are the continuous self-mappings of configuration spaces.* In [Mira et al. 91], pp. 79–85.

[Ferber et al. 91] FERBER, R., TARGONSKI, GY. and WEITKÄMPER, J., *Fractional-time states of cellular automata.* In [Mira et al. 91], pp. 86–106.

[Förg-Rob 85] FÖRG-ROB, W., *The Pilgerschritt transform in Lie algebras.* In [Liedl et al. 85], pp. 59–71.

[Förg-Rob 89] FÖRG-ROB, W., *Some results on the Pilgerschritt transform.* In [Alsina et al. 89], pp. 198–204.

[Förg-Rob, Netzer 85] FÖRG-ROB, W. and NETZER, N., *Product-integration and one-parameter subgroups of linear Lie groups.* In [Liedl et al. 85], pp. 71–82.

[Förg-Rob et al. 94] FÖRG-ROB, W., GRONAU, D., MIRA, C., NETZER, N. and TARGONSKI, GY., (eds), *Proceedings of ECIT 92,* Batschuns (Austria), September 1992. World Scientific, Singapore, 1994.

[Gale 91] GALE, D., *Conjectures.* Math. Intelligencer *13* (1991), 53–55.

[Gardner 70] GARDNER, M., *Mathematical games.* Scientific American, Oct. 1970 and Feb. 1971.

[Graw 82] GRAW, R., *Über die Orbitstruktur stetiger Abbildungen (On the orbit structure of continuous mappings.).* Ph.D. Thesis, Universität Marburg, 1982.

[Graw 84] GRAW, R., *Compact orbits and periodicity.* Nonlinear Anal. *8* (1984), 1473–1479.

[Gronau 91] GRONAU, D., *The Jabotinsky equations and the embedding problem.* In [Mira et al. 91], pp. 138–148.

[Gronau 91_1] GRONAU, D., *On the structure of the solution of the Jabotinsky equations in Banach spaces.* Zeitschr. Anal. Anw. *10* (1991), 335–343.

[Gumowski, Mira 80] GUMOWSKI, I. and MIRA, C., *Dynamique chaotique (Chaotic dynamics).* Cepadues Editions, Toulouse, 1980.

[Gumowski, Mira 80_1] GUMOWSKI, I. and MIRA, C., *Recurrences and discrete dynamic systems.* [Springer Lecture Notes in Mathematics, Nr. 809], Springer, Berlin, 1980.

[Isaacs 50] ISAACS, R., *Iterates of fractional order.* Canad. J. Math. *2* (1950), 409–416.

[Jabotinsky 55] JABOTINSKY, E., *Iteration.* Ph.D. Thesis, The Hebrew University, Jerusalem, 1955.

[Jabotinsky 63] JABOTINSKY, E., *Analytic iteration.* Trans. Amer. Math. Soc. *108* (1963), 457–477.

[Krasner 38] KRASNER, M., *Une géneralisation de la notion de corps (A generalization of the notion of field).* J. Math. Pures Appl. *17* (1938), 367–385.

[Krause 88] KRAUSE, G., Manuscript, 1988.

[Kuczma, Choczewski, Ger 90] KUCZMA, M., CHOCZEWSKI, B. and GER, R., *Iterative functional equations.* [Encyclopedia of Mathematics and its Applications, vol. 32], Cambridge University Press, Cambridge, 1990.

[Lagarias 85] LAGARIAS, J. C., *The $3x + 1$ problem and its generalizations.* American Math. Monthly *92* (1985), 3–23.

[Lampreia, Sousa Ramos 91] LAMPREIA, J. P. and SOUSA RAMOS, J., *Symbolic dynamics of trimodal maps.* In [Mira et al. 91], pp. 184–193.

[Lampreia et al. 93] LAMPREIA, J. P., LLIBRE, J., MIRA, C., SOUSA RAMOS, J. and TARGONSKI, GY. (eds), *ECIT 91, Proc. Europ. Conf. on Iteration Theory, Lisbon Sep. 15–21, 1991.* World Scientific, Singapore, 1993.

[Langenberg 92] LANGENBERG, H., *Zellulare Automaten und Iterationstheorie (Cellular automata and iteration theory).* Diploma (M.Sc.) Thesis, Universität Marburg, 1992.

[Liedl 86] LIEDL, R., *Gruppenwertige Potenzreihen (Group valued power series).* Anz. Österreich. Akad. Wiss. Math. Nat. Kl. 1986, No. 5, 57–58.

[Liedl, Netzer 89] LIEDL, R. and NETZER, N., *Group theoretic and differential geometric methods for solving the translation equation.* In [Alsina et al. 89], pp. 240–252.

[Liedl, Netzer 91] LIEDL, R. and NETZER, N., *Die Lösung der Translationsgleichung mittels schneller Pilgerschrittransformation (Solution of the translation equation by means of the fast Pilgerschritt transformation).* [Grazer Ber. Nr. 314], Forschungsinst., Graz, 1991.

[Liedl, Netzer, Reitberger 81] LIEDL, R., NETZER, N. and REITBERGER, H., *Eine Methods zur Berechnung von einparametrigen Untergruppen ohne Verwendung des*

Logarithmus (A method for calculation of one-parameter subgroups without using logarithm). Österreich. Akad. Wiss., Math.-Natur. Kl. Sitzungsberg. *II* (1981), 273–284.

[Liedl, Netzer, Reitberger 82] LIEDL, R., NETZER, N. and REITBERGER, H., *Über eine Methode zur Auffindung stetiger Iterationen in Lie-Gruppen (On a method of finding continuous iteration in Lie groups).* Aequationes Math. *24* (1982), 19–32.

[Liedl et al. 85] LIEDL, R., REICH, L. and TARGONSKI, GY. (eds), *Iteration Theory and its functional equations* (Proceedings, Schloss Hofen 1984). Springer Lecture Notes in Mathematics Nr. 1163.

[Mandelbrot 82] MANDELBROT, B. B., *The fractal geometry of nature.* W.H. Freeman, 1982.

[Miller 92] MILLER, J. B., *Square root of uppertriangular matrices.* [Analysis Paper No. 75], Dept. of Math., Monash University, Clayton, Vic., Australia, 1991.

[Mira 79] MIRA, C., *Frontière floue séparant des domaines d'attraction de deux attracteurs (Vague boundaries separating the domains of attraction of two attractors).* C. R. Acad. Sci. Paris *299* (1979), A591–A594.

[Mira, Müllenbach 83] MIRA, C. and MÜLLENBACH, S., *Sur l'itération fractionnaire d'un endomorphisme quasratique (On fractional iteration of a quadratic endomorphism).* C. R. Acad. Sci. Paris Sér. I. Math. *297* (1983), 369–372.

[Mira et al. 91] MIRA, C., NETZER, N., SIMÓ, C. and TARGONSKI, GY. (eds), *Proceedings of ECIT 89, European Conference on Iteration Theory, Batschuns, Austria, 10–16 Sept. 1989.* World Scientific, Singapore, 1991.

[Netzer 82] NETZER, N., *On the convergence of iterated pilgerschritt transforms.* Zeszyty Nauk. Uniw. Jagiellon. Prace Mat. *23* (1982), 91–98.

[Netzer 92] NETZER, N., *The convergence of fast Pilgerschritt transformation.* In [Förg-Rob et al. 92].

[Netzer, Liedl 91] NETZER, N. and LIEDL, R., *Fast Pilgerschritt transformation.* In [Mira et al. 91], pp. 279–293.

[Netzer, Reitberger 82] NETZER, N. and REITBERGER, H., *On the convergence of Pilgerschritt transformations in nilpotent Lie groups.* Publ. Math. Debrecen *29* (1982), 309–314.

[Neuman 82] NEUMAN, F., *Simultaneous solutions of a system of Abel equations and differential equations with several derivations.* Czechoslovak Math. J. *32* (1982) 488–494.

[Neuman 89] NEUMAN, F., *On iteration groups of certain functions.* Arch. Math. (Brno) *25* (1989), 185–194.

[Von Neumann 66] VON NEUMANN, J., *Theory of self-reproducing automata.* University of Illinois, Urbana-London, 1966.

[Peitgen, Richter 86] PEITGEN, H. O. and RICHTER, P. H., *The beauty of fractals.* Springer, Berlin, 1986.

[Peschl, Reich 71] PESCHL, E. and REICH, L., *Eine Linearisierung kontrahierender biholomorpher Abbildungen und damit zusammenhängender analytischer Differentialgleichungssysteme (A linearization method for contractive biholomorphic maps and for related systems of analytic differential equations).* Monatsh. Math. *75* (1971), 153–162.

[Reich 71] REICH, L., *Über analytische Iteration linearer und kontrahierender biholomorpher Abbildungen (On analytic iteration of linear and contractive biholomorphic mappings).* [Bericht Nr. 42] Ges. Math. Datenverarb., Bonn, 1971.

[Reich 79] REICH, L. (with the participation of J. Schwaiger), *Analytische und fraktionelle Iteration formal-biholomorpher Abbildungen (Analytic and fractional iteration of formal biholomorphic maps).* In *Jahrbuch Überblicke Mathematik 1979*, Bibliographisches Institut, Mannheim, 1979, pp. 123–144.

[Reich 85] REICH, L., *On a differential equation arising in iteration theory in rings of formal power series in one variable.* In [Liedl et al. 85], pp. 135–148.

[Reich 88] REICH, L., *On families of commuting formal power series.* In *Selected topics in functional equations* [Grazer Math. Bericht Nr. 294] Math-Stat. Sekkt. Forsch Ges. Joanneum, Graz, 1988, pp. 1–18.

[Reich 89] REICH, L., *Die Differentialgleichungen von Aczél–Jabotinsky, von Briot–Bouquet und maximale Familien vertauschbarer Potenzreihen (The differential equations of Aczél–Jabotinsky, of Briot–Bouquet and maximal families of commuting power series).* In *Complex methods on partial differential equations.* [Math. Res. Vol. 53]. Akademie Verlag, Berlin, 1989, pp. 137–150.

[Reich 91] REICH, L., *On the embedding problem for formal power series with respect to the Aczél–Jabotinsky equations.* In [Mira et al. 91], pp. 294–304.

[Reich 92] REICH, L., *On the local distribution of iterable power series transformation in one indeterminate.* In: *Functional analysis III, Proc. Postgrad School and Conf., Dubrovnik, Oct. 29–Nov. 2* (D Brutkovic et al. eds). [Aarhus Univ. Various Publications Series Nr. 40], Univ., Aarhus, 1992.

[Reich 93] REICH, L., *On power series transformations in one indeterminate having iterative roots of a given order and with given multiplier.* In [Lampreia et al. 93], pp. 210–216.

[Reich, Schwaiger 80] REICH, L. and SCHWAIGER J., *Linearisierung formal-biholomorpher Abbildungen und Iterationsprobleme.* Aequationes Math. *20* (1980), 224–243.

[Riggert 75] RIGGERT, G., *n-te iterative Wurzeln von beliebigen Abbildungen (n-th iterative roots of arbitrary sets).* In *Report of the 1975 International Symposium on Functional Equations.* Aequationes Math. *15* (1977), 288.

[Robert 86] ROBERT, F., *Discrete iterations.* Springer, Berlin, 1986.

[Schleiermacher 93] SCHLEIERMACHER, A., *On a theorem of Marc Krasner about invariant relations.* In [Lampreia et al. 93], pp. 230–240.

[Schwaiger 89] SCHWAIGER, J., *Phantom roots and phantom iterates of formal power series in one variable.* In [Alsina et al. 89], pp. 313–323.

[Schwaiger 91] SCHWAIGER, J., *On polynomials having different types of roots.* In [Mira et al. 91], pp. 315–319.

[Schweizer, Sklar 88] SCHWEIZER, B. and SKLAR, A., *Invariants and equivalence classes of polynomials under linear conjugacy.* In *Contributions to general algebra, No. 6.* Hölder–Pinchler–Tempsky, Vienna and Teubner, Stuttgart, 1988.

[Schweizer, Sklar 90] SCHWEIZER, B. and SKLAR, A., *The baker's transformation is not embeddable.* Found. of Phys. *20* (1990), 873–897.

[Simon 93] SIMON, K., *Hausdorff dimensions for certain near-hyperbolic maps.* In [Lampreia et al. 93], pp. 253–261.

[Sklar 69] SKLAR, A., *Canonical decompositions, stable functions, and fractional iterates.* Aequationes Math. *3* (1969), 118–129.

[Sklar 87] SKLAR, A., *The structure of one-dimensional flows with continuous trajectories.* Rad. Mat. *3* (1987), 111–142.

[Skornjakov 77] SKORNJAKOV, L. A., *Unars.* In *Universal algebra.* [Coll. Mat. Soc. János Bolyai 29], J. Bolyai Math. Soc. Budapest, 1977, pp. 735–743.

[Smajdor 85] SMAJDOR, A., *Iteration of multi-valued functions.* [Prace Nauk. Uniw. Śląsk. Katowic. No. 759], Silesian Univ. Katowice, 1985.

[Smajdor 89] SMAJDOR, A., *One-parameter families of set-valued contractions*. In [Alsina et al. 89].

[Smajdor 93] SMAJDOR, A., *Almost-everywhere set-valued semigroups*. In [Lampreia et al. 93], pp. 262–272.

[Smítal 88] SMÍTAL, J., *On functions and functional equations*. Adam Hilger, Bristol–Philadelphia, 1988.

[Snowden, Howie 82] SNOWDEN, M. and HOWIE, J. M., *Square roots in finite full transformation semigroups*. Glasgow Math. J. *23* (1982), 137–149.

[Targonski 67] TARGONSKI, GY., *Seminar on functional operators and equations*. [Springer Lecture Notes in Mathematics, No. 33], Springer, Berlin, 1967.

[Targonski 81] TARGONSKI, GY., *Topics in iteration theory*. Vandenhoeck & Ruprecht, Göttingen–Zürich, 1981.

[Targonski 84] TARGONSKI, GY., *New directions and open problems in iteration theory*. [Grazer Math. Bericht No. 229], Math. Stat. Sekt. Forschungszentrum, Graz, 1984.

[Targonski 84_1] TARGONSKI, GY., *Phantom iterates of continuous functions*. In [Liedl et al. 85], pp. 196–202.

[Targonski 89] TARGONSKI, GY., *Iteration theory and functional analysis*. In [Alsina et al. 89], pp. 74–93.

[Targonski 90] TARGONSKI, GY., *On composition operators*. Zeszyty Nauk. Politechn. Śląsk. Ser. Mat.-fiz. *64* (1990).

[Targonski 91] TARGONSKI, GY., *Problem 25*, In *Report of the 27th ISFE, Poland 1989*. Aequationes Math. *39* (1990), 313–314.

[Targonski 93] TARGONSKI, GY., *On a class of phantom fractional iterates*. In [Lampreia et al. 93], pp. 295–301.

[Targonski 94] TARGONSKI, GY., *Phantom iterates and Liedl's Pilgerschritt transformation*. In [Förg-Rob et al. 94].

[Targonski, Zdun 85] TARGONSKI, GY. and ZDUN, M. C., *Generators and co-generators of substitution semigroups*. Ann. Math. Sil. *1 (13)* (1985), 169–174.

[Targonski, Zdun 87] TARGONSKI, GY. and ZDUN, M. C., *Substitution operators on L^p-spaces and their semigroups*. [Grazer Math. Bericht No. 283], Math.-Stat. Sekt. Forsch. Ges. Joanneum, Graz, 1987.

[Thibault 89] THIBAULT, R., *Some results obtained in Toulouse on dynamical systems*. In [Alsina et al. 89], pp. 94–112.

[Ushiki 86] USHIKI, SH., *Chaotic Phenomena and Fractal Objects in Numerical Analysis*. In: Nishida, T. et al. (eds), *Patterns and waves—qualitative analysis of nonlinear differential equations*. [Stud. Math. Appl.], North Holland, Amsterdam, 1986, pp. 221–258.

[Wagon 85] WAGON, S., *The Collatz problem*. Math. Intelligencer *7* (1985), 72–76.

[Weitkämper 85] WEITKÄMPER, J., *Embeddings in iteration groups and semigroups with nontrivial units*. Stochastica *7* (1983), 175–195.

[Whitley 83] WHITLEY, D., *Discrete dynamical systems in dimensions one and two*. Bull. London Math. Soc. *15* (1983), 177–217.

[Zdun 79] ZDUN, M. C., *Continuous and differentiable iteration semigroups*. [Prace Nauk. Uniw. Śląsk. Katowic. No. 308], Silesian Univ., Katowice, 1979.

[Zdun 85] ZDUN, M. C., *On embedding of the circle in a continuous flow in* [Liedl et al. 85], pp. 218–231.

[Zdun 85_1] ZDUN, M. C., *Regular fractional iteration*. Aequationes Math. *28* (1985), 73–79.

[Zdun 88] ZDUN, M. C., *Note on commutable functions*. Aequationes Math. *36* (1988), 153–164.

[Zdun 89] ZDUN, M. C., *On continuity of iteration semigroups on metric spaces.* Annal. Soc. Math. Polon. *29* (1989), 113–116.

[Zdun 89_1] ZDUN, M. C., *On $\mathbb{C}^r$ iteration groups.* In [Alsina et al. 89], pp. 373–381.

[Zdun 89_2] ZDUN, M. C., *On simultaneous Abel equations.* Aequationes Math. *38* (1989), 163–177.

[Zdun 90] ZDUN, M. C., *On quasi-continuous iteration semigroups and groups of real functions.* Colloq. Math. *58* (1990), 281–289.

[Zdun 91] ZDUN, M. C., *The structure of iteration groups of continuous functions.* Aequationes Math. *46* (1993), 19–37.

[Zdun 91_1] ZDUN, M. C., *On continuous iteration groups of fixed point free mappings in $\mathbb{R}^n$ space.* In [Mira et al. 91], pp. 362–368.

[Zdun 93] ZDUN, M. C., *Some remarks on the iterates of commuting functions.* In [Lampreia et al. 93], pp. 336–342.

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