ANNALES
POLONICI MATHEMATICI
X (1961)

On monotonic solutions of a functional equation. II

by M. Kuczma (Kraków)

§ 1. The object of the present paper is the functional equation

$$\varphi[\varphi(x)] = g\big(x, \varphi(x)\big), \tag{1}$$

where $\varphi(x)$ denotes the unknown function and $g(x, y)$ is given. We shall suppose regarding the function $g(x, y)$ that:

(i) The function $g(x, y)$ is defined, continuous and strictly increasing with respect to each variable in the closure $\bar{\Omega}$ of the region (¹)

$$\Omega: \quad \begin{cases} a < x < b, \\ x < y < \beta(x). \end{cases}$$

(ii) $g(a, a) = a$, $g(b, b) = b$, $g(x, x) > x$ for $x \in (a, b)$, $g(x, y) > y$ in Ω.

(iii) $g\big(x, \beta(x)\big) = \beta(x)$ for $x \in \langle a, b\rangle$.

We admit also the case where one of the values a and b is infinite as well as where $\beta(x) \equiv \infty$.

It follows from hypotheses (i)-(iii) that if the function $\beta(x)$ is finite, then it is continuous and strictly increasing in $\langle a, b\rangle$, $\beta(a) = a$, $\beta(b) = b$.

In our preceding paper [1] we have proved that under suppositions (i)-(iii) for every $x_0 \in (a, b)$ there exist infinitely many solutions of equation (1) that are continuous and strictly increasing in the interval $\langle x_0, b\rangle$. In the present paper we consider the possibility of the continuation of these solutions to the left. The problem is not trivial and there arise some questions which we are not able to answer. These problems are formulated at the end of this paper.

At first we shall prove

Theorem I. *If hypotheses* (i)-(iii) *are fulfilled and if* $\varphi(x)$ *is a continuous solution of equation* (1) *which is defined in an interval* $I \subset (a, b)$ *and passes through an inner point of the region* Ω, *then* $\varphi(x)$ *remains in* Ω *for all* $x \in I$ *and is strictly increasing.*

(¹) Throughout this paper we shall use Greek capital letters for sets of the plane and Latin capital letters for points of the plane.

Proof. We shall denote by Γ and Ξ respectively the sets of the points of the curves

$$y=\beta(x), \quad x\in\langle a,b\rangle,$$
$$y=x, \quad x\in\langle a,b\rangle.$$

We shall prove that the graph of the function $\varphi(x)$ may intersect the curves Γ and Ξ only at the point $A(a,a)$ or $B(b,b)$. Hence it follows that for $x\in I$ the graph of $\varphi(x)$ remains in Ω.

Let $x_0\in I$ and $(x_0,\varphi(x_0))\in\Xi$. Consequently $\varphi(x_0)=x_0$. Putting $x=x_0$ in equation (1) we get $x_0=g(x_0,x_0)$, whence either $x_0=a$, or $x_0=b$. Similarly, let us suppose that $x_0\in I$ and $(x_0,\varphi(x_0))\in\Gamma$. Consequently $\varphi(x_0)=\beta(x_0)$. Putting $x=x_0$ in equation (1) we get $\varphi[\beta(x_0)]=\beta(x_0)$, whence it follows that either $\beta(x_0)=a$, or $\beta(x_0)=b$. But, since the function $\beta(x)$ is strictly increasing, it is possible only for $x_0=a$ or $x_0=b$.

Now we shall prove that the function $\varphi(x)$ is single-valued. In fact, let us suppose that there exist $x_1\in I$ and $x_2\in I$ such that $\varphi(x_1)=\varphi(x_2)=\varphi_0$. From equation (1) we have

$$g(x_1,\varphi_0)=g(x_2,\varphi_0),$$

whence, on account of the monotonity of the function g, follows $x_1=x_2$. Thus the function $\varphi(x)$, being continuous and single-valued, is monotonic. From the inequality $g(x,y)>y$ it follows that it must be increasing. This completes the proof.

In the paper [1] we have proved that for every point $P(x_0,y_0)\in\Omega$ there exist infinitely many functions $\varphi(x)$ that are continuous and strictly increasing in the interval $\langle x_0,b\rangle$, and satisfy equation (1) and the condition $\varphi(x_0)=y_0$. From the above theorem it follows that taking all possible continuous and strictly increasing solutions issuing from all points of the region Ω we obtain all continuous solutions of equation (1) passing through Ω.

§ 2. The question arises whether a given solution $\varphi(x)$ of equation (1) that is continuous and strictly increasing in an interval $\langle x_0,b\rangle\subset(a,b\rangle$ can be continued to the left even to the point A. As we shall see, the answer to this question is in general negative.

Let us write $\alpha(x)\overset{\text{df}}{=}g(x,x)$ and denote by Λ the set of the points of the curve

$$y=\alpha(x), \quad x\in\langle a,b\rangle.$$

The curve Λ divides the region Ω into two subregions:

$$\Omega_1:\quad\begin{cases}a<x<b,\\ x<y<\alpha(x),\end{cases}$$

and

$$\Omega_2:\quad\begin{cases}a<x<b,\\ \alpha(x)<y<\beta(x).\end{cases}$$

Definition. For an arbitrary point $P(x,y)\in\overline{\Omega}$ we shall denote by $R(P)$ and call the *right-hand correspondent of the point* P the point with the coordinates $(y,g(x,y))$. Similarly we shall denote by $L(P)$ and call the *left-hand correspondent of the point* P the point Q such that $R(Q)=P$ (if such exists).

Lemma. *The function $R(P)$ is defined for $P\in\overline{\Omega}$; moreover, $R(\Omega)=\Omega_1$, $R(\Xi)=\Lambda$, $R(\Gamma)=\Xi$. The function $L(P)$ (inverse to $R(P)$) is defined for $P\in\overline{\Omega}_1$; moreover, $L(\Lambda)=\Xi$, $L(\Xi)=\Gamma$. Both functions, $R(P)$ and $L(P)$, are continuous in their domain of definition.*

Proof. Let $P(x,y)\in\Omega$. Consequently

$$x<y<g(x,y)<g(y,y)=\alpha(y),$$

which proves that $R(P)\in\Omega_1$. On the other hand, let $Q(y,z)\in\Omega_1$. Let us put $x_1=\beta^{-1}(y)$ and $x_2=y$. The point (x,y) belongs to Ω for $x\in\langle x_1,x_2\rangle$ and according to (iii)

$$g(x_1,y)=g(x_1,\beta(x_1))=\beta(x_1)=y,$$
$$g(x_2,y)=g(y,y)=\alpha(y).$$

Since the function g is strictly increasing with respect to x, the equation

$$g(x,y)=z$$

has exactly one solution $x_0\in\langle x_1,x_2\rangle$, which means that $R((x_0,y))=Q$. Thus $R(\Omega)=\Omega_1$.

If $P(x,y)\in\Xi$, then $y=x$ and $g(x,y)=\alpha(y)$, whence $R(P)\in\Lambda$. If $P(x,y)\in\Gamma$, then $y=\beta(x)$ and $g(x,y)=y$; consequently $R(P)\in\Xi$. It is evident that for Q belonging to Λ resp. Ξ there always exists P belonging to Ξ resp. Γ such that $R(P)=Q$.

Thus we have proved the first part of the lemma. The second part (concerning the function $L(P)$) follows immediately from the first in view of the fact that for $P_1\neq P_2$, $R(P_1)\neq R(P_2)$. The continuity of the functions $R(P)$ and $L(P)$ follows from the continuity of the function $g(x,y)$. This completes the proof.

Now let $\varphi(x)$ be a solution of equation (1) defined in an interval $\langle x_0,b\rangle$ and let Φ denote the set of the points of the graph of

$$y=\varphi(x), \quad x\in\langle x_0,b\rangle. \tag{2}$$

If a point P lies on Φ, then also $R(P)$ lies on Φ. Consequently each point of the set Φ is a left-hand correspondent of a certain point of this set. Thus in order to continue solution (2) of equation (1) to the left from the point $P_0(x_0,\varphi(x_0))$ we form the set consisting of the left-hand correspondents of the points of that part of the set Φ which is contained between the points P_0 and $R(P_0)$ (this part will further be denoted by Φ_0).

If only a part of the set Φ_0 is contained in Ω_1, we can continue solution (2) to the left only to a certain point. Let Q be the common point of the sets Φ_0 and Λ. Then $L(Q)$ is the point with the same abscissa as Q, but lying in the set Ξ (cf. fig. 1). Hence it follows that Q as well as $L(Q)$ should belong to $\Phi \cup L(\Phi_0)$. Both these points have the same abscissae but different ordinates and thus $\Phi \cup L(\Phi_0)$ may not be a graph of any function. Thus as we see, if the set Φ_0 has a common part with the region Ω_2, then it is impossible to continue solution (2) to the left as far as to the point A. We can only continue it to a certain point Q_1, viz. to the point with the smallest abscissa among all points of the set $L(\Phi_0)$ (cf. fig. 1).

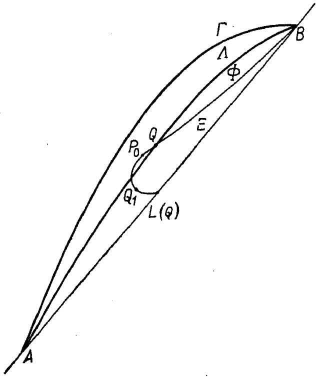

Fig. 1

§ 3. As we have seen, if $P_0 \in \Omega_1$ and $\Phi \subset \Omega_1$, then solution (2) may be continued to the left to the point $L(P_0)$. If such a continued solution is still contained in Ω_1, we can continue it further to the point $L^2(P_0)$ (²), etc. But if during this procedure one of the points $L^n(P_0)$ falls into the set Ω_2, then the possibility of the continuation of solution (2) becomes limited. Namely it can be continued a little beyond the point $L^n(P_0)$, but at any rate it cannot reach the point A. Thus the question arises whether among all solutions of equation (1) that pass inside Ω there exists a solution defined in the whole interval $\langle a, b\rangle$.

Let us fix an arbitrary $x_0 \in (a, b)$ and denote by Δ_0 the set of the points $P(x, y)$ such that $x = x_0$, $x_0 \leqslant y \leqslant a(x_0)$. Let us further put

(3) $$\Delta_{n+1} \overset{\mathrm{df}}{=} L(\Delta_n \cap \overline{\Omega}_1), \qquad n = 0, 1, 2, \ldots$$

On account of the continuity of the function L the sets Δ_n are arcs joining points on the curve Γ with points on the axis Ξ (cf. fig. 2). Moreover, all Δ_n are closed sets. Consequently also and $R^n(\Delta_n)$ are closed. Evidently
$$R^n(\Delta_n) \subset \Delta_0, \qquad n = 0, 1, 2, \ldots$$
We shall show that $R^n(\Delta_n)$ is a descending sequence of sets. In fact, we have by (3)
$$R(\Delta_{n+1}) = \Delta_n \cap \overline{\Omega}_1 \subset \Delta_n,$$

(²) $L^n(P)$, $R^n(P)$ denote the n-th iteration of the function $L(P)$ resp. $R(P)$.

whence
$$R^{n+1}(\Delta_{n+1}) = R^n[R(\Delta_{n+1})] \subset R^n(\Delta_n).$$
Consequently the set
$$\Delta(x_0) \overset{\mathrm{df}}{=} \bigcap_{n=0}^{\infty} R^n(\Delta_n)$$
is non-empty as a product of a descending sequence of closed sets.

Since x_0 has been arbitrary, the each point $x \in (a, b)$ there corresponds a non-empty set of points $\Delta(x)$. This is the set of points with the abscissa x such that all functions L^n are for these points defined. Evidently if $P(x, y) \in \Delta(x)$, then $R(P) \in \Delta(y)$. For each $x \in (a, b)$ we shall denote by V_x the set of the ordinates of the points of the set $\Delta(x)$ (i.e. the projection of the set $\Delta(x)$ on the y-axis).

In [1] we have described the construction of increasing solutions of equation (1) issuing (to the right) from a point $P_0(x_0, y_0)$. Namely, we may join the points P_0 and $R(P_0)$ by an arbitrary increasing curve with an equation $y = \varphi(x)$, and then relation (1) will uniquely determine the function $\varphi(x)$ satisfying equation (1) in the whole interval $\langle x_0, b\rangle$. As we see from the above considerations, if we define the function $\varphi(x)$ in the interval $\langle x_0, y_0\rangle$ in such a manner that

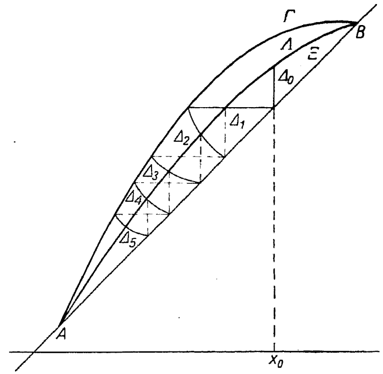

Fig. 2

(4) $$\varphi(x) \in V_x \quad \text{for} \quad x \in \langle x_0, y_0\rangle,$$

then the solution thus obtained will be continuable to the left even to the point A. Taking all possible increasing functions $\varphi(x)$ defined in the interval $\langle x_0, y_0\rangle$ and fulfilling condition (4) we obtain all solutions of equation (1) that are defined and increasing in the whole interval $\langle a, b\rangle$ and pass through the region Ω (³).

Thus we have the following

THEOREM II. *If the hypotheses* (i)-(iii) *are fulfilled, then equation* (1) *possesses a solution that is defined and increasing in the whole interval* $\langle a, b\rangle$ *and passes inside the region* Ω.

(³) Of course, all these solutions pass inside the set Ω_1.

We are not able, however, to prove the existence of a continuous solution defined in the whole interval $\langle a, b\rangle$. Also we cannot say anything about the number of solutions that are defined in the whole interval $\langle a, b\rangle$. It is our conjecture that under hypotheses (i)-(iii) equation (1) possesses exactly one solution defined in the whole interval $\langle a, b\rangle$, i.e. that each of the sets $\Delta(x)$ contains exactly one point. It can easily be shown that if the solution defined in the whole interval $\langle a, b\rangle$ is unique, then it is continuous and strictly increasing.

Reference

[1] M. Kuczma, *Monotonic solutions of a functional equation. I.* Ann. Polon. Math. 9 (1960), p. 295-297.

Reçu par la Rédaction le 24. 5. 1960

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POLONICI MATHEMATICI
X (1961)

On the continuous dependence of solutions of some functional equations on given functions. II

by J. Kordylewski and M. Kuczma (Kraków)

In the first part of this paper [2] we have proved (under suitable assumptions) continuous dependence on given functions for solutions of the functional equation

$$\varphi[f(x)]+\eta\varphi(x)=F(x), \tag{1}$$

where $\eta=\pm 1$. Presently we shall deal with a more general equation

$$\varphi(x)=H(x, \varphi[f(x)]), \tag{2}$$

where $\varphi(x)$ denotes the unknown function and $f(x)$ and $H(x, y)$ are given. Making use of the results obtained we shall prove a theorem about continuous dependence for solutions of equation (1) stronger than those proved in [2]. Although equation (1) is a particular case of equation (2), the hypotheses which we assume concerning equation (2) are not fulfilled in the case of equation (1). Thus the theorems proved in [2] do not follow from the results of the present paper.

II. Equation $\varphi(x)=H(x, \varphi[f(x)])$

§ 1. We assume the following hypotheses regarding the functions $f(x)$ and $H(x, y)$:

(i) The function $f(x)$ is defined, continuous and strictly increasing in an interval $\langle a, b\rangle$ and $f(x)>x$ for $x\in(a, b)$, $f(b)=b$.

(ii) The function $H(x, y)$ is continuous and has the continuous derivative $\partial H/\partial y\neq 0$ in a region Ω normal with respect to the x-axis.

(iii) $\Omega_x\neq 0$, $\Omega_x=\Gamma_{f(x)}$ for $x\in\langle a, b\rangle$, where Ω_x denotes the x-section of the region Ω:

$$\Omega_x=\{y:\ (x, y)\in\Omega\},$$

and Γ_x denotes the set of values of the function $H(x, y)$ for $y\in\Omega_x$ (1).

(1) In the case $f(a)\neq a$ it is enough to postulate only $\Omega_x\subset\Gamma_{f(x)}$, instead of the relation $\Omega_x=\Gamma_{f(x)}$.