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Representation of primes by the principal form of discriminant $-D$ when the classnumber $h(-D)$ is 3

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0. Notation and preliminary result. Throughout this paper p denotes a prime > 3. We shall be concerned with binary quadratic forms $ax^2+bxy+cy^2$, written (a, b, c), which are integral (that is, a, b, c are integers), positive-definite (that is, $a > 0$, $b^2-4ac < 0$) and primitive (that is, $\mathrm{GCD}(a, b, c) = 1$). The discriminant of the form (a, b, c) is the negative integer b^2-4ac. On the set of all such forms of fixed discriminant $-D$ ($D > 0$), we define an equivalence relation $\sim$ as follows: we write $(a, b, c) \sim (a', b', c')$ if there exist integers p, q, r, s with $ps-qr = +1$ such that

$$a(px+qy)^2+b(px+qy)(rx+sy)+c(rx+sy)^2 = a'x^2+b'xy+c'y^2.$$

It is well known that there are only finitely many such equivalence classes. The number of classes is called the classnumber of forms of discriminant $-D$ and is denoted by $h(-D)$. The principal form of discriminant $-D$ is the form p_{-D} given by

$$p_{-D} = \begin{cases} (1, 0, D/4), & \text{if } D \equiv 0 \pmod 4, \\ (1, 1, (D+1)/4), & \text{if } D \equiv 3 \pmod 4. \end{cases} \tag{0.1}$$

A positive integer m is said to be represented by the form (a, b, c) if there exist integers x and y such that $m = ax^2+bxy+cy^2$. If the prime p (not dividing $2D$) is represented by a form of discriminant $-D$, it is well known that the Legendre symbol $\left(\dfrac{-D}{p}\right) = +1$. In this paper we shall be concerned with the representability of a prime p (> 3) by the principal form p_{-D} of discriminant $-D$ when $h(-D) = 3$.

Recent deep work of Goldfeld, Gross, Mestre, Oesterlé and Zagier (see [6], [7], [12], [13], [14], [20]) has led to the complete determination of all the imaginary quadratic fields with classnumber 3 [12: Théorème 4], namely,

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$$Q(\sqrt{-n})\colon\ n = 23,\ 31,\ 59,\ 83,\ 107,\ 139,\ 211,\ 283,\ 307,$$
$$331,\ 379,\ 499,\ 547,\ 643,\ 883,\ 907.$$

The complete list of all the imaginary quadratic fields with classnumber 1 has been known for over twenty years [15], namely,

$$Q(\sqrt{-n})\colon\ n = 1,\ 2,\ 3,\ 7,\ 11,\ 19,\ 43,\ 67,\ 163.$$

From these results we can deduce

PROPOSITION. *$h(-D) = 3$ if and only if*

(0.2) $$D = 23, 31, 44, 59, 76, 83, 92, 107, 108, 124, 139, 172, 211, 243, 268, 283,$$
$$307,\ 331,\ 379,\ 499,\ 547,\ 643,\ 652,\ 883 \text{ or } 907.$$

Proof. Let d be the discriminant of the imaginary quadratic field given uniquely by

$$-D = f^2 d,$$

where f is a positive integer. Then, by a formula of Gauss, we have

$$h(-D) = h(f^2 d) = h(d)\psi_d(f)/u,$$

where

$$\psi_d(f) = f \prod_{q|f}\left(1 - \left(\frac{d}{q}\right)\frac{1}{q}\right)$$

and

$$u = \begin{cases} 3, & \text{if } d = -3, \\ 2, & \text{if } d = -4, \\ 1, & \text{if } d < -4. \end{cases}$$

Note that q runs through the distinct primes dividing f and $\left(\frac{d}{q}\right)$ is the Kronecker symbol. As $\psi_d(f)$ is a positive integer and $h(-3) = h(-4) = 1$, we see that

$$h(-D) = 3 \Leftrightarrow \text{(a) } d < -4,\ h(d) = 3,\ \psi_d(f) = 1 \quad \text{or}$$
$$\text{(b) } d < -4,\ h(d) = 1,\ \psi_d(f) = 3 \quad \text{or}$$
$$\text{(c) } \psi_{-4}(f) = 6 \text{ or}$$
$$\text{(d) } \psi_{-3}(f) = 9.$$

Now it is easy to check that

$$\psi_d(f) = 1 \Leftrightarrow f = 1 \text{ or } f = 2,\ d \equiv 1 \pmod 8;$$

$$\psi_d(f) = 3 \Leftrightarrow f = 2,\ d \equiv 5 \pmod 8 \text{ or}$$

$$f = 3,\ d \equiv 0 \pmod 3 \text{ or}$$

$$f = 6,\ d \equiv 1 \pmod 8,\ d \equiv 0 \pmod 3;$$

$$\psi_{-4}(f) = 6 \quad \text{cannot occur;}$$

$$\psi_{-3}(f) = 9 \Leftrightarrow f = 6 \text{ or } f = 9.$$

Thus, appealing to the lists of imaginary quadratic fields with classnumber 1 or 3, we see that:

(a) occurs if and only if $D = 23, 31, 59, 83, 107, 139, 211, 283, 307, 331, 379, 499, 547, 643, 883, 907, 23\cdot 2^2, 31\cdot 2^2$;

(b) occurs if and only if $D = 11\cdot 2^2, 19\cdot 2^2, 43\cdot 2^2, 67\cdot 2^2, 163\cdot 2^2$;

(c) cannot occur;

(d) occurs if and only if $D = 3\cdot 6^2, 3\cdot 9^2$.

This gives the twenty-five values of D listed in (0.2).

1. Introduction. Gauss [5] showed that 2 is congruent to a cube modulo a prime $p \equiv 1 \pmod 3$ if and only if there exist integers x and y such that $p = x^2 + 27y^2$, that is, if and only if p is represented by the principal form of discriminant -108. Moreover, when 2 is a cube (mod p), where $p \equiv 1 \pmod 3$, 2 has three distinct cube roots (mod p). If $p \equiv 2 \pmod 3$ then $\left(\frac{-108}{p}\right) = \left(\frac{-3}{p}\right) = -1$ and p is not represented by any form of discriminant -108, and 2 has a unique cube root (mod p). Since every positive-definite, primitive, integral binary quadratic form of discriminant -108 is equivalent to exactly one of the three forms $(1, 0, 27)$, $(4, -2, 7)$, $(4, 2, 7)$, Gauss' theorem can be expressed as follows:

THEOREM (Gauss). *The polynomial* $x^3 - 2$ *is*

(i) *the product of three distinct linear polynomials* (mod p) *if* $\left(\frac{-3}{p}\right) = +1$ *and* p *is represented by* $(1, 0, 27)$;

(ii) *the product of a linear polynomial and an irreducible quadratic polynomial* (mod p) *if* $\left(\frac{-3}{p}\right) = -1$;

(iii) *irreducible* (mod p) *if* $\left(\frac{-3}{p}\right) = +1$ *and* p *is represented by* $(4, \pm 2, 7)$.

Clearly Gauss' theorem can be reformulated as a criterion for p to be represented by the principal form of discriminant -108, namely,

THEOREM (Gauss). *The prime p is represented by $(1, 0, 27)$ if and only if* $\left(\frac{-3}{p}\right) = +1$ *and x^3-2 is congruent to the product of three distinct linear polynomials* (mod p).

Jacobi [10] showed that 3 is congruent to a cube modulo a prime $p \equiv 1 \pmod 3$ if and only if p can be written in the form $4p = A^2+243B^2$, where A and B are integers. If $4p = A^2+243B^2$ then we have $A \equiv B \pmod 2$ and $p = x^2+xy+61y^2$ with $x = \frac{1}{2}(A-B)$, $y = B$. Conversely, if $p = x^2+xy+61y^2$ then we have $4p = A^2+243B^2$ with $A = 2x+y$, $B = y$. Since every positive definite, primitive, integral binary quadratic form of discriminant -243 is equivalent to exactly one of the three forms $(1, 1, 61)$, $(7, -3, 9)$, $(7, 3, 9)$, Jacobi's theorem can be restated as follows:

THEOREM (Jacobi). *The prime p is represented by $(1, 1, 61)$ if and only if* $\left(\frac{-3}{p}\right) = +1$ *and x^3-3 is congruent to the product of three distinct linear polynomials* (mod p).

In this paper we generalize the results of Gauss and Jacobi to all D (>0) for which $h(-D) = 3$. These values of D are listed in (0.2). We prove

THEOREM 1. *Let D be a positive integer such that $h(-D) = 3$. Then the prime p ($p > 3$, $p \nmid D$) is represented by the principal form p_{-D} of discriminant $-D$ if and only if* $\left(\frac{-D}{p}\right) = +1$ *and $f_{-D}(x)$ is congruent to the product of three distinct linear polynomials* (mod p), *where $f_{-D}(x)$ is the monic cubic polynomial with integral coefficients listed in Table 1. Further we have*

$$\text{discriminant}(f_{-D}(x)) = \begin{cases} -D, & \text{if } D \equiv 3 \pmod 4 \text{ or } D \equiv 12 \pmod{32}, \\ -D/4, & \text{if } D \equiv 28 \pmod{32}. \end{cases}$$

Table 1

D	$f_{-D}(x)$	D	$f_{-D}(x)$
23	x^3-x+1	243	x^3-3
31	x^3+x+1	268	x^3+2x^2-2x+2
44	x^3+x^2-x+1	283	x^3+4x+1
59	x^3+2x+1	307	x^3-x^2+3x+2
76	x^3-2x+2	331	x^3-2x^2+4x+1
83	x^3+x^2+x+2	379	x^3+x^2+x+4
92	x^3-x+1	499	x^3+4x+3
107	x^3+x^2+3x+2	547	x^3+x^2-3x+4
108	x^3-2	643	x^3-2x+5
124	x^3+x+1	652	x^3+3x^2-5x+3
139	x^3-x^2+x+2	883	x^3+5x^2-5x+2
172	x^3-x^2-x+3	907	x^3+5x^2+x+2
211	x^3-2x+3		

The cases $D = 108$ and $D = 243$ of the theorem are the aforementioned results of Gauss and Jacobi respectively, so these two values of D will be excluded from further consideration. Furthermore, when $D = 92$ and $D = 124$, it is easy to check that p is represented by p_{-D} if and only if it is represented by $p_{-D/4}$, as $D/4 \equiv 7 \pmod 8$. Thus we can also exclude these two values of D from further consideration. We divide the remaining 21 values of D into two lists according as $D \equiv 3 \pmod 4$ or $D \equiv 0 \pmod 4$, namely,

(A) $D = 23, 31, 59, 83, 107, 139, 211, 283, 307, 331, 379, 499, 547, 643, 883, 907,$

(B) $D = 44, 76, 172, 268, 652.$

The proof of Theorem 1 for the 16 values of D listed in (A) is based on a theorem of Weinberger [18] and is given in Section 2. For the 5 values of D listed in (B), Weinberger's theorem does not apply and we give a proof (in § 3) using Artin's reciprocity law instead. We remark that the existence of such a polynomial $f_{-D}(x)$ is known by class field theory (see [3: Theorem 9.2 and Ex. 9.3]). Our Theorem 1 gives such a polynomial $f_{-D}(x)$ explicitly for all D with $h(-D) = 3$, and furthermore shows that $f_{-D}(x)$ may be chosen with discriminant $-D/4$ or $-D$ according as $D \equiv 28 \pmod{32}$ or not. In future work it is planned to determine $f_{-D}(x)$ explicitly when $h(-D) = 4, 5, 6, 7$ and 8, assuming that the known lists of such D are complete. For general D not much is known about $f_{-D}(x)$ or its discriminant.

The case $D = 124$ of Theorem 1 was treated by Kronecker [11], who showed that p is represented by $(1, 0, 31)$ if and only if the congruence

$$(x^3 - 10x)^2 + 31(x^2 - 1)^2 \equiv 0 \pmod p$$

is solvable. It is easy to check that this is equivalent to our result, namely, $p\,(\nmid 2 \cdot 3 \cdot 31)$ is represented by $(1, 0, 31)$ if and only if $\left(\frac{-31}{p}\right) = +1$ and the congruence $x^3 + x + 1 \equiv 0 \pmod p$ is solvable. Appealing to Theorem 1, a sextic polynomial analogous to that of Kronecker for $D = 124$ can be found for each D in (0.2).

In Section 4, we use Theorem 1 to construct explicitly some class fields. We prove

THEOREM 2. (i) *For those D in* (A), *the Hilbert class field over $Q(\sqrt{-D})$ is*

$$Q\left(\sqrt{-D}, \sqrt[3]{\varkappa_D} + \sqrt[3]{\varkappa'_D}\right),$$

where $\varkappa_D$ is given as follows:

D	$\varkappa_D$	D	$\varkappa_D$	D	$\varkappa_D$
23	$(-27+3\sqrt{69})/2$	139	$(-61+3\sqrt{417})/2$	379	$(-101+3\sqrt{1137})/2$
31	$(-27+3\sqrt{93})/2$	211	$(-81+3\sqrt{633})/2$	499	$(-81+3\sqrt{1497})/2$
59	$(-27+3\sqrt{177})/2$	283	$(-27+3\sqrt{849})/2$	547	$(-137+3\sqrt{1641})/2$
83	$(-47+3\sqrt{249})/2$	307	$(-79+3\sqrt{921})/2$	643	$(-135+3\sqrt{1929})/2$
107	$(-29+3\sqrt{321})/2$	331	$(-83+3\sqrt{993})/2$	883	$(-529+3\sqrt{2649})/2$
				907	$(-259+3\sqrt{2721})/2$

(ii) *For those* D *in* (B), *the ring class field of the order* $Z[\sqrt{-D/4}]$ *in* $Z[(-1+\sqrt{-D/4})/2]$ *is*

$$Q(\sqrt{-D/4},\ \sqrt[3]{\varkappa_D}+\sqrt[3]{\varkappa'_D}),$$

where

$$\begin{aligned}\varkappa_{44} &= -19+3\sqrt{33},\\ \varkappa_{76} &= -27+3\sqrt{57},\\ \varkappa_{172} &= -35+3\sqrt{129},\\ \varkappa_{268} &= -53+3\sqrt{201},\\ \varkappa_{652} &= -135+3\sqrt{489}.\end{aligned}$$

We remark that Hasse [9] has shown that the Hilbert class field over $Q(\sqrt{-23})$ is

$$Q(\sqrt{-23},\ \sqrt[3]{(25+3\sqrt{69})/2}+\sqrt[3]{(25-3\sqrt{69})/2})$$

and the Hilbert class field over $Q(\sqrt{-31})$ is

$$Q(\sqrt{-31},\ \sqrt[3]{(29+3\sqrt{93})/2}+\sqrt[3]{(29-3\sqrt{93})/2}).$$

Our results for $D=23$ and $D=31$ agree with those of Hasse since $\beta=(\alpha-9)/3$ for

$$\begin{cases}\alpha = \sqrt[3]{(-27+3\sqrt{69})/2}+\sqrt[3]{(-27-3\sqrt{69})/2} = -3.9741\ldots,\\ \beta = \sqrt[3]{(25+3\sqrt{69})/2}+\sqrt[3]{(25-3\sqrt{69})/2} = 3.2646\ldots;\end{cases}$$

and $\delta=(-\gamma-9)/\gamma$ for

$$\begin{cases}\gamma = \sqrt[3]{(-27+3\sqrt{93})/2}+\sqrt[3]{(-27-3\sqrt{93})/2} = -2.0469\ldots,\\ \delta = \sqrt[3]{(29+3\sqrt{93})/2}+\sqrt[3]{(29-3\sqrt{93})/2} = 3.3967\ldots\end{cases}$$

In Section 5, we use Theorem 1 and a theorem of Cauchy [2] to give a necessary and sufficient condition for the prime p to be represented by $p=x^2+ny^2$ (where n is in list (A) or list (B)) in terms of integer sequences defined by a second order linear recurrence relation which need only be considered modulo p. When $D=23$ our result agrees with that of Gurak [8]. We prove

THEOREM 3. *Let* D *denote one of the integers in list* (A) *or list* (B). *Let* p *be a prime* (>3) *such that* $\left(\frac{-D}{p}\right)=+1$. *Then*

$$p=\begin{cases}x^2+\dfrac{D}{4}y^2, & \text{if } D\equiv 0 \pmod 4,\\ x^2+xy+\left(\dfrac{1+D}{4}\right)y^2, & \text{if } D\equiv 3 \pmod 4,\end{cases}$$

is solvable in integers x and y if and only if

$$\begin{cases} u_{(p-1)/3} \equiv 2 \pmod{p}, & \text{if } p \equiv 1 \pmod 3, \\ u_{(p+1)/3} \equiv -2k \pmod{p}, & \text{if } p \equiv 2 \pmod 3, \end{cases}$$

where the sequence of integers $\{u_n\}_{n=0,1,2,\ldots}$ is given by

$$\begin{cases} u_0 = 2, \quad u_1 = l, \\ u_{n+2} = lu_{n+1} + k^3 u_n, \quad n = 0, 1, 2, \ldots, \end{cases}$$

and the integers k, l are given in Table 2:

Table 2

D	k	l	D	k	l
23	−1	+25	283	+12	+27
31	−1	+29	307	+8	−79
44	−4	−38	331	+8	+83
59	−4	−43	379	+2	+101
76	+8	−2	499	+12	+81
83	+2	−47	547	−10	+137
107	+8	+29	643	−6	+135
139	+2	−61	652	+20	+196
172	−4	+70	883	−40	+529
211	−6	−81	907	−22	+259
268	−10	+106			

The identities

$$u_{2m} = u_m^2 - 2(-1)^m k^{3m}, \qquad u_{3m} = u_m^3 - 3(-1)^m k^{3m} u_m,$$

are often useful in computing $u_{(p\pm 1)/3} \pmod p$. We illustrate Theorem 3 with a simple example.

EXAMPLE. Is the prime 1297 represented by the form (1, 0, 19)? Here we have $p = 1297$, $(p-1)/3 = 432$, $D = 76$, $k = 8$, $l = -2$. Making use of the above identities, we obtain successively modulo 1297

$$u_0 \equiv 2, \quad u_1 \equiv -2, \quad u_2 \equiv 1028, \quad u_4 \equiv 726, \quad u_8 \equiv 889,$$
$$u_{16} \equiv 904, \quad u_{48} \equiv 544, \quad u_{144} \equiv 1296, \quad u_{432} \equiv 2,$$

so that, by Theorem 3, 1297 is represented by (1, 0, 19). Indeed we have $1297 = 1 \cdot 9^2 + 19 \cdot 8^2$.

2. Proof of Theorem 1 for those D listed in (A). Throughout this section, D denotes one of the integers listed in (A). Note that D is a prime $\equiv 3 \pmod 4$.

Let p be a prime >3 with $p \nmid D$. If $\left(\frac{-D}{p}\right) = -1$ then p is not represented by $p_{-D} = (1, 1, \frac{1}{4}(D+1))$ and, as discrim$(f_{-D}(x)) = -D$, by a theorem of Stickelberger [16], $f_{-D}(x)$ is the product of a linear polynomial and an irreducible quadratic polynomial modulo p. Now suppose $\left(\frac{-D}{p}\right) = +1$. We must show that p is represented by $p_{-D} = (1, 1, \frac{1}{4}(D+1))$ if and only if $f_{-D}(x)$ is congruent to the product of three distinct linear polynomials (mod p).

We set

$$K_D = Q(\sqrt{3D}), \qquad K_D^* = Q(\sqrt{3D})\backslash\{0\}. \tag{2.1}$$

Let G_D be the group defined by

$$G_D = \{\alpha \in K_D^* : (\alpha) = A^3 \text{ for some ideal } A \text{ of } K_D\} \tag{2.2}$$

and let H_D be the subgroup of G_D given by

$$H_D = \{\alpha \in K_D^* : \alpha = \beta^3 \text{ for some } \beta \in K_D^*\}. \tag{2.3}$$

Then G_D/H_D is a group isomorphic with the direct sum of r_D+1 groups of order 3, where r_D is the rank of the 3-Sylow subgroup of the classgroup $H(K_D)$ of K_D. Now

$$H(K_D) \simeq \begin{cases} Z_3, & \text{for } D = 107, 331, 643, \\ Z_5, & \text{for } D = 547, \\ Z_1, & \text{otherwise}, \end{cases} \tag{2.4}$$

so

$$r_D = \begin{cases} 1, & \text{for } D = 107, 331, 643, \\ 0, & \text{otherwise}, \end{cases} \tag{2.5}$$

and thus

$$G_D/H_D \simeq \begin{cases} Z_3 \times Z_3, & \text{if } D = 107, 331, 643, \\ Z_3, & \text{otherwise}. \end{cases} \tag{2.6}$$

Let ε_{3D} denote the fundamental unit (>1) of K_D. When $D \neq 107, 331, 643$ a basis for the group G_D/H_D is $\{\varepsilon_{3D}H_D\}$. When $D = 107, 331$ or 643, $H(K_D)$ is generated by the class containing the ideal $A_D = (2, \frac{1}{2}(1+\sqrt{3D}))$. Since

$$A_D^3 = \begin{cases} (\tfrac{1}{2}(17+\sqrt{321})), & \text{if } D = 107, \\ (\tfrac{1}{2}(31-\sqrt{993})), & \text{if } D = 331, \\ (\tfrac{1}{2}(4963-113\sqrt{1929})) = (\tfrac{1}{2}(1258562169097-28655537523\sqrt{1929})), & \\ & \text{if } D = 643, \end{cases}$$

a basis for G_D/H_D is given by $\{\varepsilon_{3D}H_D, \mu_{3D}H_D\}$, where

$$\mu_{3D} = \begin{cases} (\tfrac{1}{2}(17+\sqrt{321})), & \text{if } D = 107, \\ (\tfrac{1}{2}(31-\sqrt{993})), & \text{if } D = 331, \\ (\tfrac{1}{2}(1258562169097-28655537523\sqrt{1929})), & \text{if } D = 643. \end{cases}$$

Hence, for every nonzero integer α of K_D, there is a unique integer γ_{3D} of K_D, a unique integer r $(= 0, 1, 2)$, and, if $D = 107$, 331 or 643, a unique integer s $(= 0, 1, 2)$, such that

$$(2.7) \qquad \begin{cases} \alpha\varepsilon_{3D}^r = \gamma_{3D}^3, & \text{if } D \neq 107, 331, 643, \\ \alpha\varepsilon_{3D}^r\mu_{3D}^s = \gamma_{3D}^3, & \text{if } D = 107, 331, 643. \end{cases}$$

The choice of generator μ_{3D} of A_D^3 with large coefficients in the case $D = 643$ is so that when α is taken to be α_D (see (2.12)) we have $r = 0$ and $s = 1$ (see Table 6 and (2.13)). The values of ε_{3D} for those D under consideration are taken from the table of Wada [17] and are listed in Table 3.

Table 3

D	ε_{3D}
23	$(25+3\sqrt{69})/2$
31	$(29+3\sqrt{93})/2$
59	$62423+4692\sqrt{177}$
83	$8553815+542076\sqrt{249}$
107	$215+12\sqrt{321}$
139	$85322647+4178268\sqrt{417}$
211	$440772247+17519124\sqrt{633}$
283	$1501654712948695+51536656330476\sqrt{849}$
307	$2522057712835735+83104627139412\sqrt{921}$
331	$2647+84\sqrt{993}$
379	$650468934487+19290626292\sqrt{1137}$
499	$22516718751127+581961430932\sqrt{1497}$
547	$4375+108\sqrt{1641}$
643	$126794455+2886916\sqrt{1929}$
883	$9973664921855379068224853 5+193782160811544821069727 6\sqrt{2649}$
907	$5231287949706796270736288215+100286934195999623391686388\sqrt{2721}$

Next we define $g_{-D}(x)$ to be the monic cubic polynomial

$$(2.8) \qquad g_{-D}(x) = x^3 + \frac{a_D}{3}x + \frac{b_D}{27},$$

where the integers a_D and b_D are listed in Table 4.

Table 4

D	a_D	b_D	D	a_D	b_D
23	−1	−25	307	+8	+79
31	−1	−29	331	+8	−83
59	−4	+43	379	+2	−101
83	+2	+47	499	+12	−81
107	+8	−29	547	−10	−137
139	+2	+61	643	−6	−135
211	−6	+81	883	−40	−529
283	+12	−27	907	−22	−259

The integers a_D and b_D were chosen so that the polynomials $f_{-D}(x)$ and $g_{-D}(x)$ have the same discriminant as well as the same number of roots (mod p). It is clear that

$$\text{discrim}(f_{-D}(x)) = \text{discrim}(g_{-D}(x))$$

as

$$\text{discrim}(f_{-D}(x)) = -D, \qquad \text{discrim}(g_{-D}(x)) = (-4a_D^3 - b_D^2)/27,$$

and

$$4a_D^3 + b_D^2 = 27D. \tag{2.9}$$

It is also clear that $f_{-D}(x)$ and $g_{-D}(x)$ have the same number of roots (mod p) as

$$f_{-D}(x) = (-1)^d x^e g_{-D}\left(\frac{tx+u}{vx+w}\right), \tag{2.10}$$

where the integers d $(=0, 1)$, e $(=0, 3)$, t, u, v, w are given in Table 5.

Table 5

D	d	e	t	u	v	w	D	d	e	t	u	v	w
23	1	3	1	−3	3	0	307	0	0	3	−1	0	3
31	1	3	−1	−3	3	0	331	1	0	−3	2	0	3
59	0	3	2	3	3	0	379	1	0	−3	−1	0	3
83	0	0	3	1	0	3	499	1	0	−1	0	0	1
107	1	0	−3	−1	0	3	547	1	0	−3	−1	0	3
139	0	0	3	−1	0	3	643	1	0	−1	0	0	1
211	0	0	1	0	0	1	883	1	0	−3	−5	0	3
283	1	0	−1	0	0	1	907	1	0	−3	−5	0	3

We can also see that $\text{discrim}(f_{-D}(x)) = \text{discrim}(g_{-D}(x))$ from (2.10)

Table 5, as in each case we have

$$\left(t^3+\frac{a_D}{3}tv^2+\frac{b_D}{27}v^3\right)^2 = \pm(tw-uv)^3. \tag{2.11}$$

Set

$$\alpha_D = \tfrac{1}{2}(b_D+3\sqrt{3D}), \tag{2.12}$$

so that by (2.9) α_D is of norm $(-a_D)^3$. For each D, we determine the values of r, s and $\gamma_{3D} = \frac{1}{2}(u_D+v_D\sqrt{3D})$ in (2.7) when $\alpha = \alpha_D$. These are listed in Table 6.

Table 6

D	r	s	u_D	v_D
23	1		−2	0
31	1		−2	0
59	1		+173	+13
83	1		+931	+59
107	1	0	+17	+1
139	1		+2185	+107
211	1		+4101	+163
283	1		+449331	+15421
307	1		+754117	+24849
331	1	0	+31	+1
379	1		+4687	+139
499	1		+92433	+2389
547	1		−41	−1
643	0	1	−55164	+1256
883	1		−3343018627	−64952791
907	1		−8124416167	−155749941

It is no coincidence that $r = 1$ for $D \neq 643$, this is a consequence of the choice of sign of b_D.

Summarizing we have

$$\begin{cases}\alpha_D\varepsilon_{3D} = \gamma_{3D}^3, & \text{for } D \neq 643,\\ \alpha_D\mu_{3D} = \gamma_{3D}^3, & \text{for } D = 643.\end{cases} \tag{2.13}$$

In view of (2.10), $f_{-D}(x)$ is the product of three distinct linear polynomials (mod p) if and only if $g_{-D}(x)$ is the product of three distinct linear polynomials (mod p). By a theorem of Dickson [4], as $\operatorname{discrim}(g_{-D}(x)) = -D$ and $\left(\frac{-D}{p}\right) = +1$, the polynomial $g_{-D}(x)$ is the product of three distinct linear polynomials (mod p) if and only if α_D is congruent to a cube (mod $\mathfrak{p}$), where $\mathfrak{p}$

is a prime ideal of the ring of integers of K_D which divides p. We note that $\alpha_D \not\equiv 0 \pmod{\mathfrak{p}}$, otherwise $p|a_D$, which is seen to be impossible from Table 4 remembering that $p > 3$ and $\left(\frac{-D}{p}\right) = +1$. In view of (2.13), α_D is a cube (mod $\mathfrak{p}$) if and only if ε_{3D} (if $D \neq 643$), μ_{3D} (if $D = 643$) is a cube (mod $\mathfrak{p}$).

Let $H(-9D)$ denote the group of classes of primitive, positive-definite, binary quadratic forms of discriminant $-9D$, so that, for those D under consideration, $H(-9D)$ is cyclic of order 12 (resp. 6) if $D \equiv 1 \pmod 3$ (resp. $D \equiv 2 \pmod 3$). As the 3-Sylow subgroup of $H(-9D)$ is of order 3, by a theorem of Weinberger [18], ε_{3D} (if $D \neq 643$), μ_{3D} (if $D = 643$) is a cube (mod $\mathfrak{p}$) if and only if $N(\mathfrak{p})$ is represented by one of the forms in the subgroup of sixth powers in $H(-9D)$, that is, by

$$\begin{cases} (1, 1, \tfrac{1}{4}(9D+1)) \text{ or } (9, 9, \tfrac{1}{4}(D+9)), & \text{if } D \equiv 1 \pmod 3, \\ (1, 1, \tfrac{1}{4}(9D+1)), & \text{if } D \equiv 2 \pmod 3. \end{cases} \tag{2.14}$$

In view of the identities

$$x^2 + xy + \frac{(9D+1)}{4}y^2 \equiv (x-y)^2 + (x-y)(3y) + \frac{(D+1)}{4}(3y)^2,$$

$$9x^2 + 9xy + \frac{(D+9)}{4}y^2 \equiv (3x+y)^2 + (3x+y)\,y + \frac{(D+1)}{4}y^2,$$

it is clear that if $N(\mathfrak{p})$ is represented by $(1, 1, \tfrac{1}{4}(9D+1))$ or $(9, 9, \tfrac{1}{4}(D+9))$ it is represented by $p_{-D} = (1, 1, \tfrac{1}{4}(D+1))$. In order to treat the converse, we first show that $N(\mathfrak{p}) \equiv 1 \pmod 3$. We have

$$N(\mathfrak{p}) = \begin{cases} p, & \text{if } \left(\frac{3D}{p}\right) = 1. \\ p^2, & \text{if } \left(\frac{3D}{p}\right) = -1. \end{cases}$$

Recalling that $\left(\frac{-D}{p}\right) = 1$, the condition $\left(\frac{3D}{p}\right) = 1$ (resp. -1) is equivalent to $p \equiv 1$ (resp. 2) (mod 3). Hence we have $N(\mathfrak{p}) \equiv 1 \pmod 3$. Thus, if $N(\mathfrak{p})$ is represented by $p_{-D} = (1, 1, \tfrac{1}{4}(D+1))$, then

$$N(\mathfrak{p}) = x^2 + xy + \tfrac{1}{4}(D+1)\,y^2,$$

with either (i) $y \equiv 0 \pmod 3$, or (ii) $x \equiv y \not\equiv 0 \pmod 3$, $D \equiv 1 \pmod 3$. If (i) holds then $N(\mathfrak{p})$ is represented by $(1, 1, \tfrac{1}{4}(9D+1))$ as

$$N(\mathfrak{p}) = \left(x + \frac{y}{3}\right)^2 + \left(x + \frac{y}{3}\right)\left(\frac{y}{3}\right) + \frac{(9D+1)}{4}\left(\frac{y}{3}\right)^2.$$

If (ii) holds then $N(\mathfrak{p})$ is represented by $(9, 9, \tfrac{1}{4}(D+9))$ as

$$N(\mathfrak{p}) = 9\left(\frac{x-y}{3}\right)^2 + 9\left(\frac{x-y}{3}\right)y + \frac{(D+9)}{4}y^2.$$

This completes the proof when $p \equiv 1 \pmod 3$ as in this case $N(\mathfrak{p}) = p$. When $p \equiv 2 \pmod 3$, we have $N(\mathfrak{p}) = p^2$, and since there are exactly three inequivalent forms of discriminant $-D$, p^2 is represented by p_{-D} if and only if p is represented by p_{-D}.

This completes the proof of Theorem 1 for those D listed in (A).

We conclude this section by noting that when $D = 44$, and p is a prime $\equiv 1 \pmod 3$ with $\left(\frac{-44}{p}\right) = 1$, Weinberger's theorem [18] gives a necessary and sufficient condition for p to be represented by the form (1, 1, 223), namely

p is represented by (1, 1, 223) if and only if $\varepsilon_{33} = 23 + 4\sqrt{33}$ is a cube (mod $\mathfrak{p}$), where $\mathfrak{p}$ is a prime ideal of $Q(\sqrt{33})$ with $N(\mathfrak{p}) = p$.

This result is not relevant to Theorem 1. Similar remarks apply to the other values of D in (B). Thus a different approach is needed to prove Theorem 1 for those D in (B), and this is done in the next section.

3. Proof of Theorem 1 for those D listed in (B). Throughout this section, D is one of the five integers listed in (B). Note that $D = 4D^*$, where D^* is a prime $\equiv 3 \pmod 8$. Let L_D denote the bicyclic biquadratic field $Q(\sqrt{-3}, \sqrt{-D^*})$. If $\theta \in L_D$ the conjugates of θ are θ, θ', $\bar{\theta}$, $\bar{\theta}'$, where

$$\begin{cases} \theta = a + b\sqrt{-3} + c\sqrt{-D^*} + d\sqrt{3D^*}, \\ \theta' = a - b\sqrt{-3} + c\sqrt{-D^*} - d\sqrt{3D^*}, \\ \bar{\theta} = a - b\sqrt{-3} - c\sqrt{-D^*} + d\sqrt{3D^*}, \\ \bar{\theta}' = a + b\sqrt{-3} - c\sqrt{-D^*} - d\sqrt{3D^*}, \end{cases} \tag{3.1}$$

where $a, b, c, d \in Q$. The ring of integers of L_D is denoted by R_D. It is known that R_D is a unique factorization domain [1].

Let p be a prime > 3 not dividing D. If $\left(\frac{-D}{p}\right) = -1$, p is not represented by $p_{-D} = (1, 0, D/4)$, and, as discrim $(f_{-D}(x)) = -D$, by a theorem of Stickelberger [16], $f_{-D}(x)$ is the product of a linear polynomial and an irreducible quadratic (mod p).

Suppose now that $\left(\frac{-D}{p}\right) = +1$. We must show that p is represented by $p_{-D} = (1, 0, D/4)$ if and only if $f_{-D}(x)$ is congruent to the product of three distinct linear polynomials (mod p). Define

$$g_{-D}(x) = x^3 + \frac{a_D}{3}x + \frac{b_D}{27}, \tag{3.2}$$

where the integers a_D and b_D are given in Table 7.

Table 7

D	a_D	b_D
44	−4	+38
76	+8	+2
172	−4	−70
268	−10	−106
652	+20	−196

We note that

$$\text{discrim}(g_{-D}(x)) = (-4a_D^3 - b_D^2)/27 = \begin{cases} -D, & \text{if } D \neq 652, \\ -4D, & \text{if } D = 652, \end{cases} \tag{3.3}$$

and that

$$f_{-D}(x) = \frac{1}{d}(vx+w)^e g_{-D}\left(\frac{tx+u}{vx+w}\right), \tag{3.4}$$

where the integers d, e $(= 0, 3)$, t, u, v, w are given in Table 8.

Table 8

D	d	e	t	u	v	w
44	+1	0	+3	+1	0	+3
76	+27	+3	+1	+2	+3	−3
172	−1	0	−3	+1	0	+3
268	−1	0	−3	−2	0	+3
652	−108	+3	−4	−2	−3	+3

From (3.4) we see that $f_{-D}(x)$ is congruent to the product of three distinct linear polynomials (mod p) if and only if $g_{-D}(x)$ is the product of three distinct linear polynomials (mod p). By (3.3) we have

$$\left(\frac{\text{discrim}(g_{-D})}{p}\right) = \left(\frac{-D}{p}\right) = +1,$$

so that by a theorem of Dickson [4], $g_{-D}(x)$ is the product of three distinct linear polynomials (mod p) if and only if

$$\left[\frac{\mu_D}{\lambda_D}\right]_3 = 1, \tag{3.5}$$

where

$$\mu_D = \begin{cases} 19+3\sqrt{33}, & \text{if } D = 44, \\ 1+3\sqrt{57}, & \text{if } D = 76, \\ -35+3\sqrt{129}, & \text{if } D = 172, \\ -53+3\sqrt{201}, & \text{if } D = 268, \\ -98+6\sqrt{489}, & \text{if } D = 652, \end{cases} \tag{3.6}$$

and λ_D is a prime divisor of p in R_D. (The symbol $\left[\frac{\mu}{\lambda}\right]_3$ in (3.5) is the cubic Legendre symbol.) The prime factorization of the prime 3 in R_D is given as follows:

$$3 = \begin{cases} -\pi_D^2 \bar{\pi}_D'^2, & \text{if } D = 44, \\ -\pi_D^2, & \text{if } D = 76, 172, 268, 652, \end{cases} \tag{3.7}$$

where

$$\pi_D = \begin{cases} \frac{1}{2}(1+2\sqrt{-3}+\sqrt{-11}), & \text{if } D = 44, \\ \sqrt{-3}, & \text{if } D = 76, 172, 268, 652. \end{cases} \tag{3.8}$$

By Artin's reciprocity law, we have

$$\left[\frac{\mu_D}{\lambda_D}\right]_3 = \begin{cases} \left(\frac{\mu_D, \lambda_D}{\pi_D}\right)_3 \left(\frac{\mu_D, \lambda_D}{\bar{\pi}_D'}\right)_3 \left[\frac{\lambda_D}{\mu_D}\right]_3, & \text{if } D = 44, \\ \left(\frac{\mu_D, \lambda_D}{\pi_D}\right)_3 \left[\frac{\lambda_D}{\mu_D}\right]_3, & \text{if } D \neq 44, \end{cases} \tag{3.9}$$

where $\left(\frac{\alpha, \beta}{\pi}\right)_3$ is the cubic Hilbert symbol. From (3.6) we see that

$$\mu_D \equiv 1 \pmod{(\sqrt{-3})^3}, \tag{3.10}$$

so that

$$\left(\frac{\mu_D, \lambda_D}{\pi_D}\right)_3 = \left(\frac{\mu_D, \lambda_D}{\bar{\pi}_D'}\right)_3 = 1. \tag{3.11}$$

Thus (3.9) reduces to

$$\left[\frac{\mu_D}{\lambda_D}\right]_3 = \left[\frac{\lambda_D}{\mu_D}\right]_3. \tag{3.12}$$

Next we observe that

$$\mu_D = \omega_D \theta_D \bar{\theta}_D'^2 \gamma_D^3, \tag{3.13}$$

where $\gamma_D \in R_D$, ω_D is a unit of R_D, and θ_D is the prime divisor of 2 in R_D given by

$$\theta_D = \begin{cases} \frac{1}{2}(\sqrt{-3}+\sqrt{-11}), & \text{if } D = 44, \\ \frac{1}{2}(3\sqrt{-3}+\sqrt{-19}), & \text{if } D = 76, \\ \frac{1}{2}(19\sqrt{-3}+5\sqrt{-43}), & \text{if } D = 172, \\ \frac{1}{2}(5\sqrt{-3}+\sqrt{-67}), & \text{if } D = 268, \\ \frac{1}{2}(715\sqrt{-3}+97\sqrt{-163}), & \text{if } D = 652. \end{cases} \tag{3.14}$$

We note that

$$\theta_D\overline{\theta}_D = \begin{cases} 2, & \text{if } D = 44, \\ -2, & \text{if } D \neq 44. \end{cases} \tag{3.15}$$

Appealing to (3.13) we see that

$$\left[\frac{\lambda_D}{\mu_D}\right]_3 = \left[\frac{\lambda_D}{\theta_D}\right]_3 \left[\frac{\lambda_D}{\overline{\theta}_D}\right]_3^2. \tag{3.16}$$

Thus we have shown:

$$p \text{ is represented by } p_{-D} \Leftrightarrow \left[\frac{\lambda_D}{\theta_D}\right]_3 = \left[\frac{\lambda_D}{\overline{\theta}_D}\right]_3. \tag{3.17}$$

From (3.14) and (3.15) we obtain

$$\pm\theta_D^3\overline{\theta}_D = 2\theta_D^2 = \begin{cases} -7-\sqrt{33}, & \text{if } D = 44, \\ -23-3\sqrt{57}, & \text{if } D = 76, \\ -1579-95\sqrt{129}, & \text{if } D = 172, \\ -71-5\sqrt{201}, & \text{if } D = 268, \\ -1533671-69355\sqrt{489}, & \text{if } D = 652, \end{cases}$$

from which we see that

$$\begin{cases} \sqrt{3D^*} \equiv r_D \pmod{\theta_D^3}, \\ \sqrt{3D^*} \equiv -r_D \pmod{\overline{\theta}_D^3}, \end{cases} \tag{3.18}$$

where

$$r_D = \begin{cases} 1, & \text{if } D = 44, \\ 3, & \text{if } D = 76, 172, 652, \\ 5, & \text{if } D = 268. \end{cases} \tag{3.19}$$

Multiplying (3.18) by $\sqrt{-3}$, we obtain

$$\begin{cases} \sqrt{-D^*} \equiv 3r_D\sqrt{-3} \pmod{\theta_D^3}, \\ \sqrt{-D^*} \equiv -3r_D\sqrt{-3} \pmod{\overline{\theta}_D^3}. \end{cases} \tag{3.20}$$

Next, as λ_D is a prime divisor of p in R_D, we have

$$p = \begin{cases} \lambda_D \bar{\lambda}_D \lambda'_D \bar{\lambda}'_D, & \text{if } p \equiv 1 \pmod 3, \\ \lambda_D \bar{\lambda}_D, & \text{if } p \equiv 2 \pmod 3. \end{cases} \tag{3.21}$$

As λ_D is an integer of $Q(\sqrt{-3}, \sqrt{-D^*})$, if $p \equiv 1 \pmod 3$, and of $Q(\sqrt{-D^*})$, if $p \equiv 2 \pmod 3$, there are integers x_0, x_1, x_2, x_3, if $p \equiv 1 \pmod 3$, and integers x_0, x_1, if $p \equiv 2 \pmod 3$, such that

$$\lambda_D = \begin{cases} \frac{1}{4}(x_0 + x_1\sqrt{-3} + x_2\sqrt{-D^*} + x_3\sqrt{3D^*}), & \text{if } p \equiv 1 \pmod 3, \\ \frac{1}{2}(x_0 + x_1\sqrt{-D^*}), & \text{if } p \equiv 2 \pmod 3, \end{cases} \tag{3.22}$$

with

$$\begin{cases} \begin{cases} x_0 \equiv x_1 \equiv x_2 \equiv x_3 \pmod 2 \\ x_0 - x_1 + x_2 + x_3 \equiv 0 \pmod 4 \end{cases}, & \text{if } p \equiv 1 \pmod 3, \\ x_0 \equiv x_1 \pmod 2, & \text{if } p \equiv 2 \pmod 3, \end{cases} \tag{3.23}$$

see [14]. (Note that $\sqrt{m_1 n_1}$ should be replaced by $\sqrt{m_1}\sqrt{n_1}$ in Theorem 1 of [19].) Set

$$\tfrac{1}{2}(u + v\sqrt{-D^*}) = \begin{cases} \lambda_D \lambda'_D, & \text{if } p \equiv 1 \pmod 3, \\ \lambda_D, & \text{if } p \equiv 2 \pmod 3, \end{cases} \tag{3.24}$$

so that u and v are integers such that

$$u = \begin{cases} (x_0^2 + 3x_1^2 - D^* x_2^2 - 3D^* x_3^2)/8, & \text{if } p \equiv 1 \pmod 3, \\ x_0, & \text{if } p \equiv 2 \pmod 3, \end{cases} \tag{3.25}$$

$$v = \begin{cases} (x_0 x_2 - 3x_1 x_3)/4, & \text{if } p \equiv 1 \pmod 3, \\ x_1, & \text{if } p \equiv 2 \pmod 3, \end{cases} \tag{3.26}$$

and

$$4p = u^2 + D^* v^2, \quad u \equiv v \pmod 2. \tag{3.27}$$

Clearly p is represented by p_{-D} if and only if $u \equiv v \equiv 0 \pmod 2$. Thus, in view of (3.17), we must show that

$$\left[\frac{\lambda_D}{\theta_D}\right]_3 = \left[\frac{\lambda_D}{\bar{\theta}_D}\right]_3 \Leftrightarrow \begin{cases} x_0 x_2 - 3x_1 x_3 \equiv 0 \pmod 8, & \text{if } p \equiv 1 \pmod 3, \\ x_1 \equiv 0 \pmod 2, & \text{if } p \equiv 2 \pmod 3. \end{cases} \tag{3.28}$$

Next, as θ_D is a prime divisor of 2 and λ_D is a prime divisor of the odd prime p, we have $\lambda_D \nmid \theta_D$ and

$$\lambda_D^3 \equiv \lambda_D^{N(\theta_D)-1} \equiv 1 \pmod{\theta_D}, \tag{3.29}$$

showing that

$$\lambda_D \equiv 1, \omega \text{ or } \omega^2 \pmod{\theta_D}, \tag{3.30}$$

where $\omega = (-1+\sqrt{-3})/2$. Appealing to (3.18) and (3.20), we obtain for $p \equiv 1 \pmod 3$

$$\lambda_D \equiv \begin{cases} 1 \pmod{\theta_D}, & \text{if } E \equiv 0 \pmod 4,\ F \equiv 4 \pmod 8, \\ \omega \pmod{\theta_D}, & \text{if } E \equiv 2 \pmod 4,\ F \equiv 4 \pmod 8, \\ \omega^2 \pmod{\theta_D}, & \text{if } E \equiv 2 \pmod 4,\ F \equiv 0 \pmod 8, \end{cases} \tag{3.31}$$

where

$$E = x_0 + rx_3, \qquad F = x_0 - x_1 - 3rx_2 + rx_3; \tag{3.32}$$

and for $p \equiv 2 \pmod 3$

$$\lambda_D \equiv \begin{cases} 1 \pmod{\theta_D}, & \text{if } x_0 \equiv x_1 \equiv 0 \pmod 2,\ x_0 + rx_1 \equiv 2 \pmod 4, \\ \omega \pmod{\theta_D}, & \text{if } x_0 \equiv x_1 \equiv 1 \pmod 2,\ x_0 + rx_1 \equiv 2 \pmod 4, \\ \omega^2 \pmod{\theta_D}, & \text{if } x_0 \equiv x_1 \equiv 1 \pmod 2,\ x_0 + rx_1 \equiv 0 \pmod 4. \end{cases} \tag{3.33}$$

We now treat the two cases $p \equiv 1 \pmod 3$ and $p \equiv 2 \pmod 3$ separately.

Case (i): $p \equiv 1 \pmod 3$. We have by (3.31)

$$\left[\frac{\lambda_D}{\theta_D}\right]_3 = \left[\frac{\lambda_D}{\bar{\theta}_D}\right]_3$$

$$\Leftrightarrow \left\{\begin{matrix}\lambda_D \equiv 1 \pmod{\theta_D} \\ \lambda_D \equiv 1 \pmod{\bar{\theta}_D}\end{matrix}\right\} \text{ or } \left\{\begin{matrix}\lambda_D \equiv \omega \pmod{\theta_D} \\ \lambda_D \equiv \omega \pmod{\bar{\theta}_D}\end{matrix}\right\} \text{ or } \left\{\begin{matrix}\lambda_D \equiv \omega^2 \pmod{\theta_D} \\ \lambda_D \equiv \omega^2 \pmod{\bar{\theta}_D}\end{matrix}\right\}$$

$$\Leftrightarrow \left\{\begin{matrix} x_0 \equiv -rx_3 \pmod 4 \\ x_0 - x_1 - 3rx_2 + rx_3 \equiv 4 \pmod 8 \\ x_0 \equiv rx_3 \pmod 4 \\ x_0 + x_1 - 3rx_2 - rx_3 \equiv 4 \pmod 8 \end{matrix}\right\} \text{ or}$$

$$\left\{\begin{matrix} x_0 + 2 \equiv -rx_3 \pmod 4 \\ x_0 - x_1 - 3rx_2 + rx_3 \equiv 4 \pmod 8 \\ x_0 + 2 \equiv rx_3 \pmod 4 \\ x_0 + x_1 - 3rx_2 - rx_3 \equiv 0 \pmod 8 \end{matrix}\right\} \text{ or}$$

$$\left\{\begin{matrix} x_0 + 2 \equiv -rx_3 \pmod 4 \\ x_0 - x_1 - 3rx_2 + rx_3 \equiv 0 \pmod 8 \\ x_0 + 2 \equiv rx_3 \pmod 4 \\ x_0 + x_1 - 3rx_2 - rx_3 \equiv 4 \pmod 8 \end{matrix}\right\}$$

$$\Leftrightarrow \begin{cases} x_0 \equiv x_1 \equiv x_2 \equiv x_3 \equiv 0 \pmod 2, \text{ say } x_i = 2y_i \ (i = 0, 1, 2, 3) \\ \text{and} \\ y_0 \equiv y_3 \pmod 2,\ y_0 - y_1 - 3ry_2 + ry_3 \equiv 2 \pmod 4,\ y_0 + y_1 - 3ry_2 - ry_3 \equiv 2 \pmod 4 \\ \text{or} \\ y_0 + 1 \equiv y_3 \pmod 2,\ y_0 - y_1 - 3ry_2 + ry_3 \equiv 2 \pmod 4,\ y_0 + y_1 - 3ry_2 - ry_3 \equiv 0 \pmod 4 \\ \text{or} \\ y_0 + 1 \equiv y_3 \pmod 2,\ y_0 - y_1 - 3ry_2 + ry_3 \equiv 0 \pmod 4,\ y_0 + y_1 - 3ry_2 - ry_3 \equiv 2 \pmod 4 \end{cases}$$

$$\Leftrightarrow \begin{cases} y_0 \equiv y_1 \equiv y_2 \equiv y_3 \pmod 2,\ y_0 - y_1 + ry_2 + ry_3 \equiv 2 \pmod 4 \\ \text{or} \\ y_0 \equiv y_3 + 1 \pmod 2,\ y_0 - y_1 - 3ry_2 + ry_3 \equiv y_0 + y_1 - 3ry_2 - ry_3 + 2 \pmod 4 \end{cases}$$

$$\Leftrightarrow \begin{cases} y_0 \equiv y_1 \equiv y_2 \equiv y_3 \pmod 2,\ y_0 - y_1 - y_2 - y_3 \equiv 2 \pmod 4 \\ \text{or} \\ y_0 \equiv y_1 \equiv y_2 + 1 \equiv y_3 + 1 \pmod 2 \end{cases}.$$

It should be noted that if $x_0 \equiv x_1 \equiv x_2 \equiv x_3 \equiv 0 \pmod 2$, with $x_i = 2y_i$ $(i = 0, 1, 2, 3)$, then by (3.23), we have

$$y_0 + y_1 + y_2 + y_3 \equiv 0 \pmod 2. \tag{3.34}$$

In view of (3.28) we must show that the assertion

$$x_0 x_2 - 3x_1 x_3 \equiv 0 \pmod 8 \tag{3.35}$$

is equivalent to

$$\begin{cases} x_i = 2y_i \ (i = 0, 1, 2, 3) \text{ and} \\ y_0 \equiv y_1 \equiv y_2 \equiv y_3 \pmod 2,\ y_0 - y_1 - y_2 - y_3 \equiv 2 \pmod 4, \text{ or} \\ y_0 \equiv y_1 \equiv y_2 + 1 \equiv y_3 + 1 \pmod 2, \end{cases} \tag{3.36}$$

under (3.23). It is clear that (3.36) implies (3.35) as

$$x_0 x_2 - 3x_1 x_3 = 4(y_0 y_2 - 3y_1 y_3) \equiv 4(y_0 y_2 - 3y_0 y_2) \equiv 0 \pmod 8.$$

Next we assume that (3.35) holds and begin by showing that the x_i are all even. We suppose that this is not the case, so that by (3.23) the x_i are all odd, say $x_i = 2z_i + 1$ $(i = 0, 1, 2, 3)$. Then, from (3.35), we have

$$2(z_0 z_2 + z_1 z_3) + (z_0 + z_1 + z_2 + z_3) \equiv 1 \pmod 4. \tag{3.37}$$

Further, as $u \equiv v \equiv 0 \pmod 2$, by (3.27) we see that $u + v \equiv 2 \pmod 4$, and so by (3.25) and (3.26), we have

$$(x_0^2 + 3x_1^2 - D^* x_2^2 - 3D^* x_3^2) + 2(x_0 x_2 - 3x_1 x_3) \equiv 16 \pmod{32},$$

and so (as $D^* \equiv 3 \pmod 8$) we obtain

$$(z_0^2 + 3z_1^2 - 3z_2^2 - z_3^2) + 2(z_0 z_2 + z_1 z_3) + 2(z_0 - z_2 + 2z_3) \equiv 7 \pmod 8. \tag{3.38}$$

From (3.37) we deduce

$$(2z_1+1)z_3 \equiv 1-z_0-z_1-z_2+2z_0z_2 \pmod 4. \tag{3.39}$$

Multiplying (3.39) by $(2z_1+1)$, we obtain

$$z_3 \equiv 1-(z_0+z_1+z_2)+2(z_0z_1+z_1z_2+z_2z_0) \pmod 4, \tag{3.40}$$

so that

$$\begin{cases} z_3 \equiv 1-A+2B \pmod 4, \\ z_3^2 \equiv 1+A^2-2A+4AB \pmod 8, \end{cases} \tag{3.41}$$

where

$$A = z_0+z_1+z_2, \qquad B = z_0z_1+z_1z_2+z_2z_0. \tag{3.42}$$

Using (3.41) in (3.38), we obtain

$$3+4(z_0+z_2)\big((z_0z_1+z_1z_2+z_2z_0)-z_1\big) \equiv 7 \pmod 8,$$

that is

$$(z_0+z_2)(z_0z_1+z_1z_2+z_2z_0-z_1) \equiv 1 \pmod 2,$$

showing that

$$z_0+z_2 \equiv z_0z_1+z_1z_2+z_2z_0-z_1 \equiv 1 \pmod 2,$$

which gives the contradiction

$$z_0+z_2 \equiv z_0z_2 \equiv 1 \pmod 2.$$

This completes the proof that (3.35) implies that all the x_i are even, say $x_i = 2y_i$ $(i = 0, 1, 2, 3)$. We complete the proof in the case $p \equiv 1 \pmod 3$ by showing that we must have either

$$y_0 \equiv y_1 \equiv y_2 \equiv y_3 \pmod 2, \qquad y_0-y_1-y_2-y_3 \equiv 2 \pmod 4$$

or

$$y_0 \equiv y_1 \equiv y_2+1 \equiv y_3+1 \pmod 2.$$

As $u \equiv 0 \pmod 2$, $v \equiv 0 \pmod 2$, $u+v \equiv 2 \pmod 4$ we have

$$y_0^2-y_1^2+y_2^2-y_3^2 \equiv 0 \pmod 4, \tag{3.43}$$

$$y_0y_2+y_1y_3 \equiv 0 \pmod 2, \tag{3.44}$$

$$y_0^2+3y_1^2-3y_2^2-y_3^2+2y_0y_2+2y_1y_3 \equiv 4 \pmod 8. \tag{3.45}$$

We begin by showing that $y_0 \equiv y_1 \pmod 2$. Suppose not, so that we have $y_0 \equiv y_1+1 \pmod 2$. Next (3.34) gives $y_2 \equiv y_3+1 \pmod 2$. Then, from either (3.43) or (3.44), we deduce that $y_1 \equiv y_3+1 \pmod 2$. Thus we have

$$y_0 \equiv y_1+1 \equiv y_2+1 \equiv y_3 \pmod 2. \tag{3.46}$$

If $y_0 \equiv 0 \pmod 2$ then (3.45) and (3.46) give

$$y_0^2 - y_3^2 + 2y_0 + 2y_3 \equiv 4 \pmod 8,$$

which gives the contradiction

$$0 \equiv (y_0+1)^2 - (y_3-1)^2 \equiv 4 \pmod 8.$$

If $y_0 \equiv 1 \pmod 2$ then (3.45) and (3.46) give

$$y_1^2 + y_2^2 + 2y_1 + 2y_2 \equiv 4 \pmod 8,$$

which gives the contradiction

$$2 \equiv (y_1+1)^2 + (y_2+1)^2 \equiv 6 \pmod 8.$$

Hence we must have

$$y_0 \equiv y_1 \pmod 2,$$

and so, by (3.34), we also have

$$y_2 \equiv y_3 \pmod 2.$$

If $y_1 \equiv y_2 + 1 \pmod 2$ we are finished. Otherwise $y_1 \equiv y_2 \pmod 2$ and we must show that $y_0 - y_1 - y_2 - y_3 \equiv 2 \pmod 4$. We have

$$y_0 \equiv y_1 \equiv y_2 \equiv y_3 \pmod 2.$$

If $y_0 \equiv y_1 \equiv y_2 \equiv y_3 \equiv 1 \pmod 2$ then (3.45) gives

$$y_0 y_2 + y_1 y_3 \equiv 2 \pmod 4,$$

and thus

$$\begin{aligned} y_0 - y_1 - y_2 - y_3 &\equiv 2y_0 - (y_0 + y_1 + y_2 + y_3) \pmod 4 \\ &\equiv 2 - (y_0+1)(y_2+1) - (y_1+1)(y_3+1) + (y_0 y_2 + y_1 y_3) \\ &\quad + 2 \pmod 4 \\ &\equiv 2 - 0 - 0 + 2 + 2 \pmod 4 \\ &\equiv 2 \pmod 4, \end{aligned}$$

as required. If $y_0 \equiv y_1 \equiv y_2 \equiv y_3 \equiv 0 \pmod 2$ then (3.45) gives (remembering that $n^2 \equiv 2n \pmod 8$ when n is even)

$$y_0 - y_1 + y_2 - y_3 \equiv 2 \pmod 4,$$

and thus

$$y_0 - y_1 - y_2 - y_3 \equiv (y_0 - y_1 + y_2 - y_3) - 2y_2 \equiv 2 \pmod 4,$$

as required. This completes the proof when $p \equiv 1 \pmod 3$.

Case (ii): $p \equiv 2 \pmod 3$. As $\lambda_D' = \lambda_D$ and $\theta_D' = -\theta_D$, we have $\left[\frac{\lambda_D}{\theta_D'}\right]_3 = \left[\frac{\lambda_D}{\theta_D}\right]_3^2$, and so $\left[\frac{\lambda_D}{\theta_D'}\right]_3 = \left[\frac{\lambda_D}{\theta_D}\right]_3$ holds if and only if $\left[\frac{\lambda_D}{\theta_D}\right]_3 = 1$, that is, if and only if $\lambda_D \equiv 1 \pmod{\theta_D}$. By (3.33) this condition is equivalent to $x_0 \equiv x_1 \equiv 0 \pmod 2$, $x_0 + r x_1 \equiv 2 \pmod 4$, which by (3.25), (3.26) and (3.27) is equivalent to $u \equiv v \equiv 0 \pmod 2$ as required.

The proof of Theorem 1 is now complete.

4. Proof of Theorem 2. Since $\sqrt[3]{\varkappa_D} + \sqrt[3]{\varkappa_D'}$ is the real root of $27 f_{-D}((x-r)/3)$, where r is the coefficient of x^2 in $f_{-D}(x)$, Theorem 2 follows immediately from Theorem 1 and [3: Theorem 9.2, Exercise 9.3].

5. Proof of Theorem 3. Theorem 3 follows from Theorem 1 and the following theorem (which is essentially due to Cauchy [2]) with $k = A_1 = a_D$, $l = -B = -b_D$ (see (2.8) and (3.2)).

THEOREM (Cauchy). *Let A and B be integers and let p be a prime such that*

$$p > 3, \quad p \nmid AB, \quad \left(\frac{-4A^3 - 27B^2}{p}\right) = +1.$$

Define an integer A_1 by $A \equiv 3A_1 \pmod p$. Let $\{u_n\}_{n=0,1,2,\ldots}$ be the sequence of integers defined by

$$u_{n+2} + B u_{n+1} - A_1^3 u_n = 0,$$

$$u_0 = 2, \quad u_1 = -B.$$

Then $x^3 + Ax + B$ is congruent to the product of three distinct linear polynomials $\pmod p$ if

$$\begin{cases} u_{(p-1)/3} \equiv 2 \pmod p, & p \equiv 1 \pmod 3, \\ u_{(p+1)/3} \equiv -2A_1 \pmod p, & p \equiv 2 \pmod 3, \end{cases}$$

and $x^3 + Ax + B$ is irreducible $\pmod p$ if

$$\begin{cases} u_{(p-1)/3} \equiv -1 \pmod p, & p \equiv 1 \pmod 3, \\ u_{(p+1)/3} \equiv A_1 \pmod p, & p \equiv 2 \pmod 3. \end{cases}$$

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References

[1] E. Brown and C. J. Parry, *The imaginary bicyclic biquadratic fields with classnumber 1*, J. Reine Angew. Math. 266 (1974), 118–120.

[2] A. Cauchy, Exercices de Mathématiques 4 (1829), 274–292. (*Oeuvres* (2), 9, 326–333.)

[3] David A. Cox, *Primes of the form x^2+ny^2* in *From Fermat to Class Field Theory and Complex Multiplication*, John Wiley and Sons, New York 1989.
[4] L. E. Dickson, *Criteria for the irreducibility of functions in a finite field*, Bull. Amer. Math. Soc. 13 (1906), 1–8.
[5] C. F. Gauss, *Theoria Residuorum Biquadraticorum, Commentatio Prima*, in *Werke*, II (1876), 65–92.
[6] D. Goldfeld, *Gauss' class number problem for imaginary quadratic fields*, Bull. Amer. Math. Soc. 13 (1985), 23–37.
[7] B. Gross and D. Zagier, *Points de Heegner et derivées de fonctions L*, C. R. Acad. Sci. Paris 297 (1983), 85–87.
[8] S. Gurak, *On the representation theory for full decomposable forms*, J. Number Theory 13 (1981), 421–442.
[9] H. Hasse, *Über den Klassenkörper zum quadratischen Zahlkörper mit der Diskriminante* -47, Acta Arith. 9 (1964), 419–434.
[10] C. G. J. Jacobi, *De residuis cubicis commentatio numerosa*, J. Reine Angew. Math. 2 (1827), 66–69.
[11] L. Kronecker, *Werke*, Vol. II, p. 93 and pp. 97–101, Vol. IV, pp. 123–129, Chelsea Publishing Co., New York, N.Y., 1968.
[12] J.-F. Mestre, *La méthode des graphes. Exemples et applications*, Proc. Int. Conf. on Class Numbers and Fund. Units, June 24–28, 1986, Katata, Japan, pp. 217–242.
[13] J. Oesterlé, *Nombres de classes des corps quadratiques imaginaires*, Séminaire Nicolas Bourbaki, 1983–1984, Exp. 631.
[14] – *Nombres des classes des corps quadratiques imaginaires*, Astérisque 121–122 (1985), 309–323.
[15] H. M. Stark, *A complete determination of the complex quadratic fields with classnumber one*, Michigan Math. J. 14 (1967), 1–27.
[16] L. Stickelberger, *Über eine neue Eigenschaft der Diskriminanten algebraischer Zahlkörper*, Verhand. I. Internat. Math. Kongress, Zürich (1897), 182–193.
[17] H. Wada, *A table of ideal class numbers and fundamental units of real quadratic number fields* $Q(\sqrt{m})$ $(2 \leqslant m < 8192)$, Sophia University, Tokyo, Japan.
[18] P. J. Weinberger, *The cubic character of quadratic units*, Proc. 1972 Number Theory Conference, University of Colorado, Aug. 14–18, 1972, pp. 241–242.
[19] K. S. Williams, *Integers of biquadratic fields*, Canad. Math. Bull. 13 (1970), 519–526.
[20] D. Zagier, *L-series of elliptic curves, the Birch–Swinnerton-Dyer conjecture, and the class number problem of Gauss*, Notices Amer. Math. Soc. 31 (1984), 739–743.

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