PROBLEMS AND SOLUTIONS

Edited by:
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Proposed problems should be sent to the MONTHLY PROBLEMS address given on the inside front cover. Please include solutions, relevant references, etc. Three copies are requested.

Solutions of published problems should arrive before June 30, 1995 at the MONTHLY PROBLEMS address given on the inside front cover. Solutions should be typed with double spacing, including the problem number and the solver's name and mailing address. Two copies suffice. A self-addressed postcard or label should be included if an acknowledgement is desired.

An asterisk (*) after the number of a problem, or part of a problem, indicates that no solution is currently available. Partial solutions will be useful in such cases. Otherwise, the published solution is likely to be based on a solution which is complete and correct. Of course, an elegant partial solution or a method leading to a more general result is always useful and welcome. In addition, references to other appearances of MONTHLY problems or to solutions of these problems in the literature are also solicited.

PROBLEMS

10424. *Proposed by Ira Gessel, Brandeis University, Waltham, MA.*

Evaluate the sum

$$\sum_{0\le k\le n/3} 2^k \frac{n}{n-k}\binom{n-k}{2k}.$$

10425. *Proposed by Allen Barnes, Queensborough Community College, Bayside, NY.*

A circle of radius r is centered at the point $(c, 0)$. Whether or not the sine wave $y = A\sin(wx+b)$ *hits* the circle (i. e., touches or passes through it) depends on the values of r, c, A, w and b. Suppose that A is much larger than r and that b is chosen uniformly at random between 0 and 2π. Find the asymptotic behavior as $r \to 0$ of the probability of a hit.

10426. *Proposed by Noam Elkies, Harvard University, Cambridge, MA, and Irving Kaplansky, Mathematical Sciences Research Institute, Berkeley, CA.*

Show that any integer can be expressed as a sum of two squares and a cube. Note that the integer being represented and the cube are both allowed to be negative.

(**b**) Show that $S(k)$ is the union of a finite number of intervals with the sum of the lengths of those intervals equal to $(k^2 - 3k + 6)/2$.

Composite solution by O. P. Lossers, Technical University Eindhoven, Eindhoven, The Netherlands, and National Security Agency Problems Group, Fort Meade, MD. The largest integer in $S(k)$ is $k(k-2)$. Since the equation holds for x if and only if it holds for $\lfloor x \rfloor$, it suffices to determine the integers in $S(k)$.

The desired equation is $AD + \lfloor x/k \rfloor = BC + \lfloor x/k - 1 \rfloor$, where $A = \lfloor (x+k-2)/k \rfloor$, $B = \lfloor (x+k-1)/k \rfloor$, $C = \lfloor (x+k-2)/(k-1) \rfloor$, and $D = \lfloor (x+k-1)/(k-1) \rfloor$. When x is an integer, we have $A = B - \epsilon$ and $C = D - \epsilon'$, where ϵ is 1 if $x \equiv 1 \bmod k$ and 0 otherwise, and ϵ' is 1 if $x \equiv 0 \bmod (k-1)$ and 0 otherwise.

When $A = B$ and $C = D$, we require $\lfloor x/k \rfloor = \lfloor x/(k-1) \rfloor$. Letting $x = a(k-1) + b$, the equation holds when $0 \le a \le b < k-1$. We must exclude $b \in \{0, a+1\}$. There remain $\sum_{b=1}^{k-2} b = (k-1)(k-2)/2$ solutions, of which $x = (k-2)k$ is the largest.

When $A = B - 1$ and $C = D$, we require $\lfloor x/k \rfloor = \lfloor x/(k-1) \rfloor + D > \lfloor x/(k-1) \rfloor$. This inequality fails for $x \ge 0$ and $k \ge 3$, so there are no solutions in this case.

When $A = B$ and $C = D - 1$, we require $\lfloor x/k \rfloor + \lfloor (x+k-1)/k \rfloor = \lfloor x/(k-1) \rfloor$. Also x must be divisible by $k-1$ and not congruent to 1 modulo k. This holds only when x is 0 or $k-1$.

When $A = B - 1$ and $C = D - 1$, we require $\lfloor x/k \rfloor + \lfloor (x+k-1)/k \rfloor = \lfloor x/(k-1) \rfloor + \lfloor (x+k-1)/(k-1) \rfloor$. Also $x \equiv 0 \bmod (k-1)$ and $x \equiv 1 \bmod k$. Since x is a multiple of $k-1$, we have $\lfloor (x+k-1)/(k-1) \rfloor > \lfloor (x+k-1)/k \rfloor$, and always $\lfloor x/(k-1) \rfloor \ge \lfloor x/k \rfloor$. Thus equality cannot hold in this case.

Altogether, we obtain $(k-1)(k-2)/2 + 2 = (k^2 - 3k + 6)/2$ unit-length half-open intervals where the condition holds, with the largest integer being $k(k-2)$.

Solved also by J. C. Binz (Switzerland), R. J. Chapman (U. K.), J. Christopher, J. H. Lindsey II, D. K. Nester, A. A. Tarabay (Lebanon), D. B. Tyler, A. N. 't Woord (Netherlands), WMC Problems Group, and the proposer.

Sums of Two Squares and a Cube

10426 [1995, 70]. *Proposed by Noam Elkies, Harvard University, Cambridge, MA, and Irving Kaplansky, Mathematical Sciences Research Institute, Berkeley, CA.* Show that any integer can be expressed as a sum of two squares and a cube. Note that the integer being represented and the cube are both allowed to be negative.

Solution by Andrew Adler, University of British Columbia, Victoria, British Columbia, Canada.

$$\begin{aligned}
2x + 1 &= (x^3 - 3x^2 + x)^2 + (x^2 - x - 1)^2 - (x^2 - 2x)^3 \\
4x + 2 &= (2x^3 - 2x^2 - x)^2 + (2x^3 - 4x^2 - x + 1)^2 - (2x^2 - 2x - 1)^3 \\
8x + 4 &= (x^3 + x + 2)^2 + (x^2 - 2x - 1)^2 - (x^2 + 1)^3 \\
16x + 8 &= (2x^3 - 8x^2 + 4x + 2)^2 + (2x^3 - 4x^2 - 2)^2 - (2x^2 - 4x)^3 \\
16x &= (x^3 + 7x - 2)^2 + (x^2 + 2x + 11)^2 - (x^2 + 5)^3
\end{aligned}$$

Editorial comment. Other identities were supplied by readers, but all solutions used a similar division into cases. John P. Robertson notes that the representation for odd integers follows from Theorem 2 on page 113 of L. J. Mordell, *Diophantine Equations*, Academic Press, 1966. He and the proposers mention related open problems, including sums of a square and two cubes.

Solved also by J. P. Robertson, A. N. 't Woord (The Netherlands), and the proposers.