Math. Zeitschr. 86, 263—267 (1964)
Baker, I. N.

Entire functions with linearly distributed values

By
Irvine Noel Baker

1. Introduction and results

A complex number w will be called a linearly distributed value of the entire function $f(z)$ if there is a straight line l of the complex plane on which all the solutions of $f(z)=w$ lie. For functions of order less than one the occurrence of such values is completely described by

Theorem 1. *If $f(z)$ is an entire transcendental function of order less than one, then any two linearly distributed values are distributed on the same line; moreover, the set of such values forms a closed straight line segment (which may reduce to a single point or $\emptyset$) of the complex plane.*

That the theorem is no longer true for functions of order one is shown by e^z for which every value is linearly distributed. This is in fact a characteristic property of the exponential function:

Theorem 2. *If $f(z)$ is an entire function for which every value is linearly distributed, then $f(z)$ is either a polynomial of degree at most two or a function of the form $c+de^{az}$, c, d, a constant.*

2. Lemmas used in the proofs

Lemma 1. (Edrei [1].) *Given a meromorphic function $f(z)$ of the complex variable $z=re^{i\vartheta}$ and given the q radii defined by*

(1) $$r e^{i\omega_1},\ r e^{i\omega_2}, \dots, r e^{i\omega_q}, \qquad (r\geqq 0),$$

where

$$0\leqq\omega_1<\omega_2<\cdots<\omega_q<2\pi, \quad (q\geqq 1);$$

the roots of the equation $f(z)=a$ are said to be distributed on the radii (1) *if there exist at most a finite number of roots of the equation which do not lie on the radii* (1).

With this definition one has:

Let $f(z)$ be meromorphic and such that the roots of the three equations

(2) $$f(z)=0,$$

(3) $$f(z)=\infty,$$

(4) $$f^{(l)}(z)=1 \qquad (l\geqq 0,\ f^{(0)}\equiv f)$$

be distributed on the radii (1). *Denote by* $\delta(a, f^{(l)})$ *the deficiency of the value* a *of the function* $f^{(l)}$ *(in the sense of Nevanlinna), and assume*

$$\delta(0, f)+\delta(1, f^{(l)})+\delta(\infty, f)>0. \tag{5}$$

Then the order ρ *of* $f(z)$ *is necessarily finite and*

$$\rho \leqq \beta = \sup\left\{\frac{\pi}{\omega_2-\omega_1}, \frac{\pi}{\omega_3-\omega_2}, \ldots, \frac{\pi}{\omega_{q+1}-\omega_q}\right\}, \qquad [\omega_{q+1}=2\pi+\omega_1]. \tag{6}$$

Remark. In our applications $f(z)$ will be entire, so that $\delta(\infty, f)=1$ and (5) is satisfied. Moreover l will be 0. Linear transformations of z and of f will enable us to replace (2) and (4) by

$$f(z)=a, \tag{2'}$$

$$f(z)=b \ (\neq a) \tag{4'}$$

and to assume the rays (1) (which will be one or two complete straight lines corresponding to $q=2$ or 4) meet in some point other than $z=0$ without modifying the conclusion (6).

A consequence of Lemma 1 found by EDREI [*1*] is

Lemma 2. *Let* $f(z)$ *be an entire function which is real on the real axis and for which the equations* $f(z)=0$, $f(z)=1$ *have only real solutions. Then for* $0 \leqq h \leqq 1$ *all the roots of* $f(z)=h$ *are real.*

Lemma 3. *Let* $f(z)$ *be regular in the infinite angular sector* D *of aperture* π/α *bounded by two rays which meet in the origin. Suppose that* M, K, δ *are positive constants,* $\delta<\alpha$ *and that*

$$|f(z)|<M\exp(K r^{\delta}) \tag{7}$$

on the rays bounding D, *while*

$$|f(z)|=O(\exp r^{\beta}), \quad as \quad r=|z|\to\infty \tag{8}$$

holds uniformly in D *for some constant* $\beta<\alpha$. *Then for constant*

$$L=K\Big/\cos\left(\frac{\delta\pi}{2\alpha}\right)$$

one has

$$|f(z)|<M\exp(L r^{\delta}) \quad \text{in } D. \tag{9}$$

Proof. If (7) and (9) are replaced by the condition

$$|f(z)|<M \tag{7'}$$

the lemma reduces to the Phragmén-Lindelöf principle in the form given in [*3*, p. 177]. One may obviously assume without loss of generality that D is the angular sector

$$|\arg z|<\frac{\pi}{2\alpha}$$

and the slightly generalised form in Lemma 3 is obtained by applying the simple form of the principle to the function $f(z)\exp(-Lz^{\delta})$.

From Lemma 3 we obtain

Lemma 4. *If the order of the entire function $f(z)$ is $\leqq\beta$, $\beta>0$, and if as $z\to\infty$ outside a number of disjoint angular sectors of the form D:*

$$\vartheta_1<\arg z<\vartheta_2, \qquad \vartheta_2-\vartheta_1<\frac{\pi}{\beta}.$$

one has

$$|f(z)|=O(\exp(K r^{\beta'})), \qquad \beta'<\beta, \ K \text{ constant},$$

then the order of $f(z)$ is in fact $\leqq\beta'$.

Proof. For each D apply Lemma 3 to show f is $O(e^{Lr^{\beta'}})$ (for some L) in D.

Lemma 5. (BIEBERBACH [*1*].) *If there are two (non-infinite) values which are taken at most a finite number of times by the entire function $f(z)$ in the angular D of aperture π/α, then in every smaller sector contained in the interior of D*

$$f(z)=O(\exp(K|z|^{\alpha})) \quad \text{as} \quad z\to\infty$$

for suitable constant $K>0$.

3. Proof of theorem 1

Suppose that all the solutions of $f(z)=b$ lie on the line l: $z=\alpha+\beta t$, α, β constant, $-\infty<t<\infty$. Then

$$F(z)=f(\alpha+\beta z)-b \tag{10}$$

has only real zeros and can be written as a product

$$F(z)=A z^m \prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right), \tag{11}$$

where A is a constant, $m\geqq 0$ an integer and z_n real. The function $F(z)/A$ is real on the real axis, has only real zeros and has order less than one, so that by a theorem of Laguerre (c.f. [*3*, p. 266]) all the zeros of

$$F'(z)/A=\beta f'(\alpha+\beta z)$$

are real. Thus all the infinitely many zeros of $f'(z)$ lie on the straight line l, which is determined uniquely, independently of the value b.

From (10), (11) and the fact that $F(z)/A$ is real on the real axis it follows that Theorem 1 is completed if we show that the set S of values w for which $F(z)/A=w$ has only real roots form a closed segment of the real w-axis. Lemma 2 shows that S is connected (the values 0,1 can clearly be replaced by any other real numbers in this lemma). On the other hand it is well known that S cannot form the whole real line – indeed it is shown in [2] that S is bounded. If S contains more than two points it consists of an open or closed interval from a to b $(>a)$. If $f(z)=b$ has a non-real root $z=c$, then for arbitrarily small $\varepsilon>0$ the equation $f(z)=b-\varepsilon$ has a non-real root near $z=c$. Thus b (and similarly a) belong to $S=[a, b]$.

4. Proof of theorem 2

It is clear that a polynomial has all its values distributed linearly if and only if its degree is less than or equal to 2.

Suppose from now on that $f(z)$ is a transcendental function for which every value is linearly distributed. We prove firstly that the order of $f(z)$ is finite:

Either (i) there are two values a, b distributed on two lines l_a, l_b which intersect, and the result follows by Lemma 1, (Remark), with $q=4$, or (ii) all lines l_a are parallel. In case (ii) take the line l_a on which all solutions of $f(z)=a$ lie and find a z on l_a for which $f(z)=b\neq a$. Then all solutions of $f(z)=b$ will also lie on l_a and we can apply Lemma 1 (Remark) to a, b distributed on $q=2$ rays consisting of the two ends of l_a. Thus in either case f has finite order.

Next we show that the order of $f(z)$ is $\leqq 1$. There are two cases to consider:

Case (i): There are two values a, b $(\neq a)$ for which l_a, l_b are parallel. Then one can find two lines p, q which intersect in the origin at an arbitrarily small angle ε and such that two infinite angular sectors of aperture ε formed by p, q contain all but a bounded part of l_a and l_b. The values a, b are taken at most a finite number of times in the complementary sectors of aperture $\pi-\varepsilon$, so that by Lemma 5

$$|f(z)|=O \exp\{K|z|^{\pi/(\pi-\varepsilon)}\}$$

in the complementary sectors. If ε is small enough Lemma 4 shows that the order of $f(z)$ is at most $\pi/(\pi-\varepsilon)$ and hence is $\leqq 1$.

Case (ii): no two l_a are parallel. Then for arbitrarily small ϑ one can find a, b such that l_a, l_b intersect in an angle $<\vartheta$. The point of intersection may not be the origin, but one can find lines p, q which intersect in the origin, make an angle ϑ with one another and such that the two infinite angular sectors of aperture ϑ formed by p, q contain all but a bounded part of l_a and l_b. In the complementary sectors of aperture $\pi-\vartheta$ the values a and b are taken at most a finite number of times so that in these sectors

$$|f(z)|=O \exp\{K|z|^{\pi/(\pi-\vartheta)}\}$$

and by Lemmas 4, 5 it follows as in case (i) that the order of $f(z)$ is 1.

The final step is to shows that $f'(z)$ does not take the value 0. Suppose that there is a value $z=\alpha$ for which $f'(\alpha)=0$ and let $\beta=f(\alpha)$. Clearly the linear distribution of the value β implies that α is a simple zero of $f'(z)$. If the equation $f(z)=\beta$ has any root γ other than $z=\alpha$ we obtain a contradiction as follows: Choose $0<\delta<|\gamma-\alpha|/3$. If one chooses a suitable w sufficiently close to β the equation $f(z)=w$ will have a root z_1, in $|z-\gamma|<\delta$ and two roots z_2, z_3 in $|z-\alpha|<\delta$ which are so placed that the line joining z_2, z_3 does not meet $|z-\gamma|<\delta$ and hence does not contain z_1. It remains to dispose of the possibility that the equation $f(z)=\beta$ has no solution other than $z=\alpha$, i.e. that $f(z)$ is of the form $\beta+K(z-\alpha)^2 e^{az}$, where K and a are non-zero constants. This function has all its values linearly distributed if and only if the same is true of $g(z)=z^2 e^z$. For $0<c<e$ the equation $g(z)=c$ has precisely three real roots so that the values $0<c<e$ are not distributed linearly.

Thus we have found that $f'(z)$ has no zeros; like $f(z)$ it has order 1 and must take the form $f'(z)=Ke^{az}$, K, a constants; it follows that $f(z)$ has the form stated in Theorem 2.

References

[1] BIEBERBACH, L.: Über eine Vertiefung des Picardschen Satzes bei ganzen Funktionen endlicher Ordnung. Math. Z. **3**, 175–190 (1919).

[2] EDREI, A.: Meromorphic functions with three radially distributed values. Trans. Amer. Math. Soc. **78**, 276–293 (1955).

[3] TITCHMARSH, E.C.: The Theory of Functions, 2nd ed. Oxford 1939.

Dept. of Math., Imperial College, London S.W. 7 (United Kingdom)

(Received June 18, 1964)