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# Permutable Entire Functions

By
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## 1. Introduction and results

Two functions $f(z)$ and $g(z)$ are called permutable if

$$f(g(z)) = g(f(z)) \tag{1}$$

holds for all values of $z$. We shall be concerned only with the case where $f(z)$ and $g(z)$ are entire functions of the complex variable $z$. JULIA [7], RITT [8, 9] and, more recently JACOBSTHAL have treated the related cases of permutable polynomials and permutable rational functions.

A problem of some interest is the determination of the class $P(f)$ of those entire functions $g(z)$ which satisfy (1) with a given entire function $f(z)$. We define the natural iterates $f_n(z)$, $n=0, 1, 2, \ldots$ of $f(z)$ by

$$f_0(z) = z; \qquad f_n(z) = f(f_{n-1}(z)), \qquad n = 1, 2, \ldots, \tag{2}$$

so that in particular $f_1(z)=f(z)$; for all $n$ we have $f_n(z) \in P(f)$. The natural iterates of a given entire function are all different except in the cases

$$f(z) \equiv \alpha, \quad \alpha \text{ constant}, \tag{3}$$

and

$$f(z) \equiv \beta + \gamma(z-\beta), \quad \beta \text{ constant and } \gamma \text{ a root of unity}. \tag{4}$$

Now $P(\alpha)$ is the (non-denumerably) infinite set of those entire functions $g(z)$ which have $\alpha$ as a fixpoint, i.e. for which

$$g(\alpha) = \alpha. \tag{5}$$

On the other hand $P(\beta+\gamma(z-\beta))$ contains the non-denumerably infinite subset of functions of the form $\beta+\delta(z-\beta)$, $\delta$ constant. Thus in every case $P(f)$ is an infinite set.

GANAPATHY IYER [5] and the present author [1] have shown independently:
*A. If $f(z)$ is a polynomial, then $P(f)$ contains entire transcendental functions if and only if $f(z)$ has one of the forms $f(z)=$const. or $f(z)=\gamma z+\delta$, $\delta$ constant and $\gamma$ a root of unity.*

The quite different case $f(z)=e^z$ was discussed in [1, p. 147] where it was proved that:

*B. $P(e^z)$ consists of the natural iterates of $e^z$ together with the constants $c$ (fixpoints of $e^z$) such that $e^c=c$. In particular $P(e^z)$ is denumerably infinite.*

We quote also the following result of JACOBSTHAL [6]:

*C.* *If $f(z)$ is a polynomial of degree greater than one, then the set of polynomials permutable with $f(z)$ is denumerably infinite.*

Together $A$ and $C$ give

*D.* *If $f(z)$ is a polynomial of degree greater than one, then $P(f)$ is denumerably infinite.*

The results $B$ and $D$ suggest the

*Conjecture. E.* *If $f(z)$ is an entire function, other than a polynomial of degree less than two, then $P(f)$ is denumerably infinite.*

While $E$ remains undecided, the partial result of theorem 1 offers some support in favour of the conjecture. The following terminology is used: If $m$ is a positive integer, the complex number $\xi$ is called *a fixpoint of order $m$ of $f(z)$* if

$$f_m(\xi) = \xi \tag{6}$$

while

$$f_j(\xi) \neq \xi, \qquad j = 1, 2, \ldots, m-1; \tag{7}$$

$f'_m(\xi)$ is called the multiplier of $\xi$. A fixpoint is repulsive if $|f'_m(\xi)| > 1$.

THEOREM 1. *If the entire function $f(z)$ is not a polynomial of degree less than two and if $f(z)$ has a fixpoint of some order which is either repulsive or of multiplier $+1$, then $P(f)$ is denumerably infinite.*

THEOREM 2. *Theorem 1 includes the results $B$ and $D$ as special cases.*

These theorems will be proved in sections 3 and 4. It may be noted in connexion with theorem 1 that $f(z)$ possesses fixpoints of every order $m = 1, 2, \ldots$ except for at most one order [2]. However, very little is known about the possible values of the multipliers of these fixpoints. For instance an unsolved problem of FATOU [4] asks whether every transcendental entire function has a repulsive fixpoint of some order. An affirmative answer to FATOU's problem would establish the truth of conjecture $E$.

## 2. Lemmas used in the proofs

LEMMA 1 (BAKER [1, p. 145]). *If $f(z)$ and $g(z)$ are permutable entire transcendental functions, then there exist a positive integer $n$ and a real positive constant $R$, such that*

$$M(g, r) < M(f_n, r) \tag{8}$$

*holds for all $r > R$.*

$M(g, r)$ and $M(f_n, r)$ are the maximum modulus functions.

LEMMA 2 (FATOU [4]). *If $f(z)$ is an entire transcendental function, then the set $\mathfrak{F}(f)$ of points about which the sequence $\{f_n(z)\}$ is not a normal family is a nonempty perfect set. $\mathfrak{F}(f)$ contains all repulsive fixpoints and fixpoints of multiplier $+1$ (of any order) of $f(z)$. Every point of $\mathfrak{F}(f)$ is a point of accumulation of fixpoints (of varying orders) of $f(z)$.*

Lemma 3 (Fatou [3, §§10, 11]). *If $\xi$ is a fixpoint of order 1 and multiplier $+1$ of the function $f(z)$, which therefore has an expansion*

$$f(z) = z + a_{m+1}(z-\xi)^{m+1} + \cdots \quad \text{all } z, \tag{9}$$

*and if $f_{-1}(z)$ is the inverse series* (10) *to* (9):

$$f_{-1}(z) = z - a_{m+1}(z-\xi)^{m+1} + \cdots \tag{10}$$

*convergent in say $|z-\xi|<\varrho$, $\varrho>0$, then any open neighbourhood (which we may assume to be interior to the circle $|z-\xi|<\varrho$) contains open sets $\mathfrak{G}$, $\mathfrak{H}$ with the properties*

(i) $\mathfrak{G}\cup\mathfrak{H}\cup(\xi)$ *is a neighbourhood of* $\xi$,

(ii) $f(\mathfrak{G})\subset\mathfrak{G}$,

(iii) $f_n(z)\to\xi$ *uniformly for* $z\in\overline{\mathfrak{G}}$ *as* $n\to\infty$,

(iv) $f_{-1}(\mathfrak{H})\subset\mathfrak{H}$,

(v) $(f_{-1})_n(z)=f_{-n}(z)\to\xi$ *uniformly for* $z\in\overline{\mathfrak{H}}$ *as* $n\to\infty$,

(vi) $f(z), f_{-1}(z)$ *are schlicht in* $\mathfrak{G}\cup\mathfrak{H}\cup(\xi)$.

A description of the form of $\mathfrak{G}$ and $\mathfrak{H}$ will be found in Fatou [3], but is irrelevant to the present discussion.

## 3. Proof of the theorem 1

Since theorem 1 is already known in the case $D$ where $f(z)$ is a polynomial, we shall discuss only the case where $f(z)$ is entire and transcendental. Moreover $P(f)\subset P(f_m)$ for every $m=1,2,\ldots$ so that we may and henceforth do assume the order of the fixpoint in the statement of theorem 1 to be one.

The constants in $P(f)$ are solutions of $f(z)=z$ and thus form a denumerable set. By $A$ any other polynomial members of $P(f)$ have the form $\gamma z+\delta$ where $\delta$ is a constant and $\gamma$ is a root of unity. Now if $\xi$ is a fixpoint of order one of $f(z)$ and if $g(z)\in P(f)$, then $f(\xi)=\xi$ and $g(\xi)=g(f(\xi))=f(g(\xi))$, so that $g(\xi)$ is a solution of $f(z)=z$. Thus if $\xi$ is a given fixpoint of order one of $f(z)$ and $\gamma z+\delta\in P(f)$, then $\gamma\xi+\delta$ is a solution of $f(z)=z$. For a fixed root of unity $\gamma$ then, at most a denumerable set of values $\delta$ yield $\gamma z+\delta\in P(f)$. The set of roots of unity is denumerable and so in consequence is the set of all polynomial members of $P(f)$.

We introduce the set $P(n,R,f)$, $n$ a positive integer, $R>0$, which is the set of those transcendental members $g(z)\in P(f)$ for which (8) holds for all $r>R$.

Lemma 4. *For any integer $n>0$, and for any $R>0$, the set $P(n\,R,f)$ is finite.*

Theorem 1 follows from Lemma 4 since by Lemma 1 the set of all transcendental members of $P(f)$ may be written as

$$\bigcup_{n=1}^{\infty}\bigcup_{m=1}^{\infty} P(n,m,f).$$

PROOF OF LEMMA 4. Let $\xi$ be a solution of $f(\xi)=\xi$ with $|f'(\xi)|>1$ or $f'(\xi)=+1$. Choose $Q$ so that

$$Q > \operatorname{Max}(|\xi|+2, R). \tag{11}$$

Then for any $g(z)\in P(n, R, f)$ and $|z|<Q$ one has

$$|g(z)| \leqq M(g, |z|) < M(g, Q) < M(f_n, Q). \tag{12}$$

In particular

$$|g(\xi)| < M(f_n, Q). \tag{13}$$

We note also that from (11) and (12) follows

$$|g'(z)| = \left| \frac{1}{2\pi i} \int_{|z-t|=1} \frac{g(t)}{(t-z)^2}\, dt \right| < M(f_n, Q) \quad \text{for} \quad |z| < |\xi|+1. \tag{14}$$

Now $g(\xi)$ is by (13) one of the finite set $(\eta_1, \eta_2, \ldots, \eta_k)$ of fixpoints which satisfy

$$f(\eta_i) = \eta_i, \quad |\eta_i| < M(f_n, Q), \quad i = 1, 2, \ldots, k. \tag{15}$$

We note moreover that if $g'(\xi)=g''(\xi)=\cdots=g^{m-1}(\xi)=0$ while $g^{(m)}(\xi)\neq 0$, then

$$f'(g(\xi)) = (f'(\xi))^m.$$

Thus, if $f'(\xi)=+1$ then $f'(g(\xi))=+1$, while if $|f'(\xi)|>1$, then $|f'(g(\xi))|>1$.

Consider $P(n, R, \eta_i, f) \subset P(n, R, f)$, namely the subset of those $g(z)\in P(n, R, f)$ for which $g(\xi)=\eta_i$.

CASE I. $|f'(\xi)|>1$. In this case we have

$$|f'(\eta_i)| > 1. \tag{16}$$

We take a circular disc $K$ centered in $\eta_i=g(\xi)$ and of radius so small that

(17)
- (i) the expansion $f_{-1}(z)=\eta_i+f'(\eta_i)^{-1}(z-\eta_i)+\cdots$ of the inverse function to $f(z)$ is convergent in $K$,
- (ii) $f_{-1}(K)\subset K$,
- (iii) $f_{-n}(z)=(f_{-1})_n(z)\to\eta_i$ uniformly in $K$ as $n\to\infty$,
- (iv) $f(z)$ is schlicht in $K$.

We then take a circular disc $C$ centred in $\xi$ and of radius so small that

(18)
- (i) the expansion of $f^*_{-1}(z)=\xi+f'(\xi)^{-1}(z-\xi)+\cdots$ of the inverse function to $f(z)$ converges in $C$,
- (ii) $f^*_{-1}(C)\subset C$,
- (iii) $f^*_{-n}(z)\to\xi$ uniformly in $C$ as $n\to\infty$,
- (iv) $f(z)$ is schlicht in $C$,
- (v) $g(C)\subset K$ for every $g(z)\in P(n, R, \eta_i, f)$.

Condition (18) (v) may be satisfied because the inequality (14) holds for every $g(z)\in P(n, R, f)$.

By lemma 2 the point $\xi$ belongs to $\mathfrak{F}(f)$ and the neighbourhood $C$ of $\xi$ must contain a fixpoint $\varphi \neq \xi$ of order (say) $p$ of $f(z)$:

$$f_p(\varphi) = \varphi.$$

Now for $g(z) \in P(n, R, \eta_i, f)$ the value $g(\varphi)$ must be a solution of $f_p(z) = z$, which by (18, v) lies in $K$. Suppose $g(z)$ and $h(z)$ are members of $P(n, R, \eta_i, f)$ for which

$$g(\xi) = h(\xi) = \eta_i$$

$$g(\varphi) = h(\varphi) = \alpha \quad \text{(say)}.$$

The value $f^*_{-1}(\varphi)$ is the only solution in $C$ of $f(z) = \varphi$, $g(f^*_{-1}(\varphi)) \in K$, and

$$f(g(f^*_{-1}(\varphi))) = g(f(f^*_{-1}(\varphi))) = g(\varphi) = \alpha;$$

so that $g(f^*_{-1}(\varphi))$ is the unique solution in $K$ of $f(z) = \alpha$. Replacing $g(z)$ by $h(z)$ in this argument one has also that $h(f^*_{-1}(\varphi)) = g(f^*_{-1}(\varphi)) = f_{-1}(\alpha)$. By induction we define the sequences $f^*_{-n}(\varphi)$, $n = 1, 2, 3, \ldots$ such that $f^*_{-n}(\varphi) = f^*_{-1}(f^*_{-n+1}(\varphi))$, and $f_{-n}(\alpha)$, $n = 1, 2, 3$ such that $f_{-n}(\alpha) = f_{-1}(f_{-n+1}(\alpha))$. The first of these sequences belongs to $C$ and tends to $\xi$ as $n \to \infty$, while the second belongs to $K$ and tends to $\eta_i$ as $n \to \infty$ (17 iii and 18 iii). Moreover $f^*_{-n}(\varphi)$ is the unique solution in $C$ of $f(z) = f_{-n+1}(z)$, while $f_{-n}(\alpha)$ is the unique solution in $K$ of $f(z) = f_{-n+1}(\alpha)$. Also $g(f^*_{-n}(\varphi)) \in K$, $h(f^*_{-n}(\varphi)) \in K$. From

$$f(g(f^*_{-n}(\varphi))) = g(f(f^*_{-n}(\varphi))) = g(f_{-n+1}(\varphi)) = f_{-n+1}(\alpha)$$

it follows that

$$g(f^*_{-n}(\varphi)) = f_{-n}(\alpha),$$

and similarly

$$h(f^*_{-n}(\varphi)) = f_{-n}(\alpha).$$

Since $g(z)$ and $h(z)$ agree on a convergent sequence of points they must be identically equal. Thus $g(z) \in P(n, R, \eta_i, f)$ is completely determined by the choice of $g(\varphi)$, which must take one of the finite set of values $\alpha$ which satisfy

$$f_p(\alpha) = \alpha$$

and lie in $K$. Since the set $P(n, R, \eta_i, f)$ is finite, it follows that $P(n, R, f)$ which by (15) is a union of finitely many such sets, is also finite.

CASE II. $f'(\xi) = +1$.

In this case we have

$$f'(\eta_i) = 1 \quad \text{where} \quad \eta_i = g(\xi). \tag{19}$$

Again we will show that $P(n, R, \eta_i, f)$ is finite. We take a pair of sets $\mathfrak{G}$, $\mathfrak{H}$ satisfying the requirements (i)—(vi) of Lemma 3 *with respect to the fixpoint* $\eta_i$ (not $\xi$) of $f(z)$. Then $\mathfrak{G} \cup \mathfrak{H} \cup (\eta_i)$ contains a circular disc $K$ of centre $\eta_i$. We choose a circular disc $C$, of centre $\xi$, such that for all $g(z) \in P(n, R, \eta_i, f)$ one has

$$g(C) \subset K.$$

This is possible by (14). We now choose sets $\mathfrak{G}_0$, $\mathfrak{H}_0$ which are contained in $C$ and satisfy the conditions of Lemma 3 with respect to the fixpoint $\xi$ of $f(z)$.

Now $\xi \in \mathfrak{F}(f)$, so that by Lemma 2 there is a fixpoint $\varphi \neq \xi$ of order (say) $p$ of $f(z)$ lying in $\mathfrak{G}_0 \cup \mathfrak{H}_0$. One has $f_p(\varphi) = \varphi$. For all $n = 1, 2, \ldots$ one has $f_{np}(\varphi) = \varphi \neq \xi$, so that $f_n(\varphi) \not\rightarrow \xi$ as $n \to \infty$ and by Lemma 3 we have $\varphi \notin \mathfrak{G}_0$, $\varphi \in \mathfrak{H}_0$. For any $g \in P(n, R, \eta_i, f)$ one has $g(\varphi) = \alpha$ where (i) $\alpha \in K \subset \mathfrak{G} \cup \mathfrak{H} \cup (\eta_i)$, (ii) $f_p(\alpha) = \alpha$. If also $h(\varphi) = \alpha$ for $h \in P(n, R, \eta_i, f)$ we again introduce (analogously to Part I) the uniquely defined sequences of points $\{f^*_{-n}(\varphi)\}$ and $\{f_{-n}(\alpha)\}$, $n = 1, 2, 3, \ldots$. Here $f^*_{-n}(\varphi)$ is the unique solution in $\mathfrak{G}_0 \cup \mathfrak{H}_0$ of $f_n(z) = \varphi$ and $f_{-n}(\alpha)$ is the unique solution in $\mathfrak{G} \cup \mathfrak{H} \cup (\eta_i)$ of $f_n(z) = \alpha$. Moreover $f^*_{-n}(\varphi)$ is the value taken at $z = \varphi$ by the $n$-th iterate of the expansion $f^*_{-1}(z) = z - a_{m+1}(z - \xi)^{m+1} \ldots$ of the inverse of $f(z)$, while $f_{-n}(\alpha)$ is the value taken at $z = \alpha$ by the $n$-th iterate of the expansion $f_{-1}(z) = z - a'_{m'+1}(z - \eta_i)^{m'+1} \ldots$ In fact it follows from $\varphi \in \mathfrak{H}_0$ that $f^*_{-n}(\varphi) \in \mathfrak{H}_0$ for all $n$, and that $f^*_{-n}(\varphi) \to \xi$ as $n \to \infty$. Moreover exactly the calculation used in case I shows that

$$g\left(f^*_{-n}(\varphi)\right) = h\left(f^*_{-n}(\varphi)\right) = f_{-n}(\alpha),$$

so that $g(z) \equiv h(z)$ and $g(z)$ is determined by the choice of $g(\varphi)$ subject to (i) and (ii). Thus again $P(n, R, \eta_i, f)$ and hence $P(n, R, f)$ are finite sets.

## 4. Proof of theorem 2

Theorem 2 includes result $B$ on $P(e^z)$, since $e^z$ possesses an infinity of repulsive fixpoints of order one. The pair nearest the origin are $0{,}3181 \ldots \pm i 1{,}3372 \ldots$.

To show that $D$ is included in the formulation of theorem 1, we must show that every polynomial of degree $d \geqq 2$ has a fixpoint of the required type. Let $f(z)$ be a polynomial of degree $d \geqq 2$, and let $\xi_1, \xi_2, \ldots, \xi_d$ be its fixpoints of order one. If any $\xi_j$ is a multiple root of $f(z) = z$, then $f'(\xi_j) = +1$. If no $\xi_j$ is a multiple root of $f(z) = z$, then $\xi_1, \ldots, \xi_d$ are all different with multipliers $f'(\xi_j) = S_j \neq 1$, and

$$\{f(z) - z\}^{-1} = \sum_{j=1}^{d} (S_j - 1)^{-1} (z - \xi_j)^{-1}. \tag{20}$$

Expanding (20) in powers of $1/z$ and comparing coefficients of $1/z$ gives

$$0 = \sum_{j=1}^{d} (S_j - 1)^{-1}.$$

Putting $t_j = (S_j - 1)^{-1}$ so that

$$0 = \sum_{j=1}^{d} t_j,$$

we note that the interior of $|S| < 1$ is mapped by $t = (S - 1)^{-1}$ onto $\operatorname{Re} t < \frac{-1}{2}$. But at least one $t_j$ must have $\operatorname{Re} t_j \geqq 0 > \frac{-1}{2}$ and for this value of $j$ one has $|S_j| = |f'(\xi_j)| > 1$. Thus in any case $f(z)$ has a fixpoint (or order 1) of the kind postulated in theorem 1.

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