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The existence of fixpoints of entire functions

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The existence and distribution of the fixpoints of entire functions are important in the study of the iteration of these functions; in [2] this is pointed out and reference is made to the literature. In the following if j is a positive integer $f_j(z)$ will denote the j-th iterate of the entire function $f(z)$. A fixpoint of exact order n of $f(z)$ is a solution of

$$f_j(z) - z = 0$$

for $j=n$ but not for any $j<n$. We prove the

THEOREM. *If $f(z)$ is an entire function other than a linear polynomial then there are fixpoints of exact order n of $f(z)$ except for at most one value of n.*

We must certainly exclude linear polynomials since, if

$$f(z) = \xi + a(z-\xi), \qquad a \neq 0 \text{ or a root of unity,}$$

then

$$f_n(z) = \xi + a^n(z-\xi)$$

and ξ is the only fixpoint (of order 1).

We use the following notation (c.f. [4]):

$n(f,r,a)$ = number of solutions of $f(z)=a$ in $|z| \leqq r$ counted according to multiplicity,

$$\bar{n}(f,r,a) = \text{number of } \textit{different} \text{ solutions of } f(z)=a \text{ in } |z| \leqq r,$$

$$N(f,r,a) = \int_0^r \frac{n(f,t,a) - n(f,o,a)}{t}\,dt + n(f,o,a)\log r,$$

$$\bar{N}(f,r,a) = \int_0^r \frac{\bar{n}(f,t,a) - \bar{n}(f,o,a)}{t}\,dt + \bar{n}(f,o,a)\log r,$$

$$T(f,r) = \text{Nevanlinna characteristic of } f(z),$$

$$M(f,r) = \max_{|z|=r} |f(z)|.$$

LEMMA 1 (PÓLYA [5]). *Let $e(z)$, $g(z)$ and $h(z)$ be entire functions satisfying*

(1) $$e(z) = g\{h(z)\}$$

(2) $$h(0) = 0.$$

There is a constant $c>0$ independent of e, g, h — with

(3) $$M(e,r) > M\left[g, c\,M\left(h, \frac{r}{2}\right)\right].$$

Condition (2) can be dropped provided (3) is to hold merely for all sufficiently great r.

LEMMA 2. *For $n>k$, n and k positive integers, we have*

$$\lim_{r\to\infty} T(f_k, r)/T(f_n, r) = 0. \tag{4}$$

PROOF OF LEMMA 2. From [4, p. 24] and lemma 1:

$$\begin{aligned} T(f_n, r) &\geqq \frac{1}{3}\log M\left(f_n, \frac{r}{2}\right) \\ &> \frac{1}{3}\log M\left[f_k, cM\left(f_{n-k}, \frac{r}{4}\right)\right] \\ &> \frac{1}{3}\log M(f_k, r^{N+1}) \end{aligned}$$

for any arbitrarily large but fixed N provided r is large enough. By [1, p. 124 Hilfssatz 1] the last expression is greater than

$$\frac{N}{3}\log M(f_k, r) > \frac{N}{3}\, T(f_k, r)$$

for all sufficiently large r. This proves the lemma.

PROOF OF THE THEOREM. I: *The case of a transcendental $f(z)$.*

We suppose that there is no fixpoint of exact order k and select a fixed integer $n>k$. The function

$$\varphi(z) = \frac{f_n(z) - z}{f_{n-k}(z) - z} \tag{5}$$

is meromorphic. For $T(\varphi, r)$ we have (c.f. [4, p. 14])

$$\left\{\begin{aligned} T(\varphi, r) &\leqq T\big(f_n(z) - z, r\big) + T\big(f_{n-k}(z) - z, r\big) + O(1) \\ &\leqq T(f_n, r) + T(f_{n-k}, r) + O(\log r) \\ &= \{1 + o(1)\}\, T(f_n, r) \quad \text{by lemma 2.} \end{aligned}\right. \tag{6}$$

By a similar argument it follows from

$$f_n(z) - z = \{f_{n-k}(z) - z\}\,\varphi(z)$$

that

$$T(f_n, r) \leqq T(f_{n-k}, r) + T(\varphi, r) + O(\log r)$$

so that

$$\{1 - o(1)\}\, T(f_n, r) \leqq T(\varphi, r)$$

which combined with (6) yields

$$T(\varphi, r) = \{1 + o(1)\}\, T(f_n, r). \tag{7}$$

In this calculation we have used the fact that the iterates of a transcendental function $f(z)$ are themselves transcendental so that their characteristics are not $O(\log r)$.

We now calculate the $\overline{N}$ functions of $\varphi(z)$ for the values 0, 1, ∞.

$$\overline{N}(\varphi, r, 0) \leqq \overline{N}\big(f_n(z) - z, r, 0\big) \tag{8}$$

$$\overline{N}(\varphi, r, \infty) \leqq \overline{N}\big(f_{n-k}(z) - z, r, o\big) < T(f_{n-k}, r) + O(\log r). \tag{9}$$

If $\varphi(z)=1$ then $f_n(z)=f_{n-k}(z)$ so that $\xi=f_{n-k}(z)$ is a solution of $f_k(\xi)=\xi$ and by the hypotheses also a solution of $f_j(\xi)=\xi$ for some integer j, $1<j<k-1$. Thus $f_{n-k+j}(z)=f_{n-k}(z)$ and

$$(10)\qquad \begin{cases} \bar{N}(\varphi, r, 1) \leqq \sum_{j=1}^{k-1} \bar{N}\big(f_{n-k+j}(z) - f_{n-k}(z), r, 0\big) \\ \qquad \leqq \sum_{j=1}^{k-1} T\big(f_{n-k+j}(z) - f_{n-k}(z), r\big) \\ \qquad \leqq \sum_{j=1}^{k-1} T(f_{n-k+j}, r) + (k-1)\, T(f_{n-k}, r) + O(1). \end{cases}$$

Using (8), (9), (10) and the second fundamental theorem [4, p. 70] in the form

$$T(\varphi, r) \leqq \bar{N}(\varphi, r, o) + \bar{N}(\varphi, r, 1) + \bar{N}(\varphi, r, \infty) + S(r)$$

where $S(r)$ is $O \log(r\, T(\varphi, r))$ except on a set of intervals of finite total length, we have

$$T(\varphi, r) < \bar{N}\big(f_n(z) - z, r, o\big) + k\, T(f_{n-k}, r) + \sum_{j=1}^{k-1} T(f_{n-k+j}, r) + S(r).$$

Dividing by $T(f_n, r)$ and taking the lower limit as $r \to \infty$ we have in view of (7) and lemma 2:

$$(11)\qquad 1 \leqq \varliminf_{r\to\infty} \frac{\bar{N}(f_n(z) - z, r, o)}{T(f_n, r)}.$$

Now if the number of different fixpoints of order $<n$ is measured by a counting function $N_1(r)$ we have

$$N_1(r) \leqq \sum_{j=1}^{n-1} \bar{N}\big(f_j(z) - z, r, o\big) \leqq \sum_{j=1}^{n-1} T(f_j, r) + O(\log r)$$

so that $\lim_{r\to\infty} \frac{N_1(r)}{T(f_n, r)} = 0$ by lemma 2. This together with (11) implies that there are fixpoints of exact order n. Thus the theorem is proved in the case when $f(z)$ is transcendental.

II. *The case when $f(z)$ is a polynomial*

Suppose $f(z)$ is a polynomial of degree $d \geqq 2$. Then $f_n(z)$ is a polynomial of degree d^n. We suppose that k and n are two positive integers with $n>k$ such that there are no fixpoints of order n or k. These numbers must satisfy

$$n > k \geqq 2$$

because the equation $f(z)-z=0$ always has d solutions. As in (5) we form

$$\varphi(z) = \frac{f_n(z) - z}{f_{n-k}(z) - z}$$

and perform any necessary cancellation to put φ in the form $\frac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are relatively prime polynomials of degrees $d^n - d^{n-k} + q$ and q respectively. $\varphi(z)$ has $d^n - d^{n-k}$ poles at $z=\infty$ and q poles at finite z values.

The number of poles (and hence of zeros) of $\varphi'(z)$ is therefore at most

$$d^n - d^{n-k} + 2q - 1. \tag{12}$$

We now count the number of different places where $\varphi(z)=0$. At any such place $f_n(z)-z=0$ and by hypothesis $f_j(z)-z=0$ for some $1\leqq j<n$. In fact j must divide n. Further if j divides k' and k' divides n every solution of $f_j(z)-z=0$ will be a solution of $f_{k'}(z)-z=0$ and will be counted among the solutions of this equation. Thus the *different* solutions of $\varphi(z)=0$ number at most

$$\sum{}' d^j \tag{13}$$

where the summation is taken over divisors j of n, $1<j<n$ excluding j for which there exists a k' with $j\,|\,k'$, $k'\,|\,n$, $j<k'<n$.

Similarly if $\varphi(z)=1$ we have $f_{n-k}(z)=f_n(z)=f_k(f_{n-k}(z))$ and $f_{n-k}(z)$ being a fixpoint of $f_k(z)$, is by hypothesis a fixpoint of $f_j(z)$ for some divisor j of k with $1<j<k$. Thus $f_j(f_{n-k}(z))=f_{n-k+j}(z)=f_{n-k}(z)$. The polynomial $f_{n-k+j}(z)-f_{n-k}(z)$ has degree d^{n-k+j} so that the number of different 1-points of $\varphi(z)$ is at most

$$\sum_{j|k} d^{n-k+j} \leqq \sum_{j=1}^{k-2} d^{n-k+j} \leqq d^{n-1} \quad \text{if} \quad k\geqq 3$$
$$\text{or} \quad = d^{n-1} \quad \text{if} \quad k=2.$$

Thus in any case the number is at most

$$d^{n-1}. \tag{14}$$

From (12), (13), (14) we conclude that the total number of solutions of the equations $\varphi(z)=0$ and $\varphi(z)=1$ (counting multiplicity) is at most

$$d^n - d^{n-k} + 2q - 1 + \sum{}' d^j + d^{n-1}$$

while from the form of $\varphi(z)$ it is exactly $2(d^n-d^{n-k}+q)$. This means that

$$2d^n - 2d^{n-k} + 2q \leqq d^n - d^{n-k} + 2q - 1 + \sum{}' d^j + d^{n-1}$$

or

$$d^n \leqq d^{n-k} + d^{n-1} - 1 + \sum{}' d^j$$
$$\leqq d^{n-2} + d^{n-1} - 1 + d^{n-2} \leqq d^n - 1$$

which is a contradiction. The last steps depend on the estimate that for $n=3, 4$ $\sum' d^j = d^{n-2}$ while for $n\geqq 5$ $\sum' d^j < d^1 + \cdots + d^{n-3} < d^{n-2}$.

Discussion of the result. A fixpoint ξ_1 of exact order n of $f(z)$ forms part of a *cycle* of order n. The members of the cycle are the values $\xi_1, \xi_2, \ldots, \xi_{n-1}, \xi_n$ which have the property

$$f(\xi_i) = \xi_{i+1}$$

or $f_j(\xi_i)=\xi_{i+j}$ where the $i+j$ is interpreted as a residue modulo n. No two of the n ξ_i are the same and all have the same value of $f_n'(\xi_i)$, namely

$$\prod_{j=1}^{n} f'(\xi_j)$$

which is called the multiplier of the cycle. Our result may be expressed in the form:

A non-linear entire function possesses cycles of all orders except for at most one exceptional order.

One exceptional value can indeed occur. The polynomial $f(z)=z^2-z$ has two fixpoints of order 1, namely 0 and 2. The fixpoints of $f_2(z)$ are 0 taken 3 times and 2 taken once. Thus there is no cycle of order 2. For transcendental functions there may be no fixpoints of order 1 as in the case of $f(z)=z+e^z$. One may ask if $n=1$ is the only possible exceptional order for transcendental functions.

For a certain class of functions including those with Picard exceptional values it has been shown in [3] that there are indeed fixpoints of exact order n for all n without exception.

If $f(z)$ is an entire transcendental function with no fixpoints of order k we can conclude from (11) not only that there are fixpoints of exact order n for $n>k$ but that their number provides an N-function of the same growth as $T(f_n, r)$. The question arises whether this remains true for *all* transcendental functions.

The results may also be expressed as necessary conditions that a given entire function be an iterate. As a sample of such statements we show: *If a transcendental entire function $F(z)$ is an iterate of nonprime order $n=pq$,* ($p>1$, $q>1$ integers) *then not all its fixpoints ξ (of first order) can have different multipliers $F'(\xi)$.*

Proof. Suppose $F(z)=f_n(z)$. Then $f(z)$ is transcendental and since $p \neq n$ there will be cycles of order p or of order n for $f(z)$. If there is a cycle of order n there are n numbers $\xi_1, \xi_2, \ldots, \xi_n$ with $f_n(\xi_j)=\xi_j$ and $F'(\xi_j)=f_n'(\xi_j)$ is the same for $j=1, 2, \ldots, n$ as noted at the beginning of this section. If there is no cycle of order n then there is one of order p and there are numbers $\eta_1, \ldots, \eta_p$ such that $f(\eta_j)=\eta_j$ for $j=1, \ldots, p$ and $f_p'(\eta_j)$ is independent of j. But then $f_n(\eta_j)=\eta_j$ since p divides n and $F'(\eta_j)=\{f_p'(\eta_j)\}^{n/p}$ is the same for each η_j of the cycle. Thus the statement is proved. If we knew that cycles of every order other than the first do occur we could drop the non-prime condition above and would have a generalisation of [2, theorem 3] where a similar theorem is proved with restrictions on the growth of the functions involved.

References

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