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The Iteration of Entire Transcendental Functions and the Solution of the Functional Equation $f\{f(z)\} = F(z)$.

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1. Introduction.

Let $F(z)$ be an entire analytic function. Define the n-th iterate of $F(z)$:

(1) $$F_0(z) = z,\ F_{n+1}(z) = F\{F_n(z)\} = F_n\{F(z)\},\quad n = 0, 1, 2, \ldots.$$

These iterates of positive integral order are entire functions satisfying the functional equations

(2) $$F_m\{F_n(z)\} = F_n\{F_m(z)\} = F_{m+n}(z)$$

where m and n are positive integers. One may also introduce the iterates of negative integral order by writing $F_{-m}(z)$ for the inverse function of $F_m(z)$. Thereby many-valued functions are introduced but the equations (2) remain valid for a suitable choice of branches.

SCHRÖDER [1], [2] originally studied the "natural" iterates (i. e. those of integral order) with a view to finding for $F_n(z)$ a closed analytic expression containing n as a parameter. The iterates of non-integral order then appeared simply by assigning non-integral values to this parameter. A modern work which treats iterates of "complex order" is that of TÖPFER [3]. JULIA [4] and FATOU [5], [6] have made a thorough investigation of the natural iteration of rational and entire functions, relying on MONTEL's theory of normal sequences of analytic functions.

In the following we shall be interested in (analytic) solutions $f(z)$ of the functional equation

(3) $$f\{f(z)\} = F(z),\quad F(z) \text{ entire.}$$

The problem which forms the basis of this paper is:

What restrictions must be placed on $F(z)$ if there are to be entire solutions $f(z)$?

The main theorem 1 bears directly upon this point, and in section 3 we discuss some cases not covered by the theorem. In the final section some applications of theorem 1 are made, in particular to discussing further KNESER's [7] treatment of the case $F(z) = e^z$.

2. The Case Where $F(z)$ is of Finite Order and Bounded on a Curve Leading to Infinity.

Theorem 1. If $F(z)$ is an entire function of finite order bounded on some continuous curve Γ which extends to infinity, then the functional equation

(3) $$f\{f(z)\} = F(z)$$

has no entire solution.

Proof: If $f(z)$ is an entire solution of (3) it must be of zero order, as follows from the following result of Fatou [6, p. 343]:

Let $E(z)$, $F(z)$ be two entire transcendental functions. On $|z| = r$ the maximum modulus of $E(z)$ is $M(r)$, that of $E\{F(z)\}$ is $M_1(r)$. Then however large we choose the positive constant q, we have for certain arbitrarily large values of r:

$$M_1(r) > M(r^q).$$

It follows that if $a > 0$ be the order of $f(z)$, the order of $F(z)$ is greater than $q(a - \varepsilon)$ however large q and however small ε; i. e. $F(z)$ is of infinite order, against assumption.

Now consider the behaviour of $f(z)$ on Γ. Either $f(z)$ is bounded on Γ or it maps Γ on to a continuous curve χ which extends to infinity. But $F(z) = f\{f(z)\}$ is bounded on Γ so that in the latter case $f(z)$ is bounded on χ, and in either event $f(z)$ is bounded on a continuous curve extending to infinity, which is impossible for an entire function whose order is less than $\frac{1}{2}$ — one recalls the result [8, p. 274] that for a function $g(z)$ with this property there is a sequence of values of r tending to infinity through which

$$\min_{|z| = r} |g(z)| \to \infty.$$

Corollary. If $F(z)$ is of finite order and there is an exceptional value A which $F(z)$ takes at most a finite number of times, then (3) has no entire solution. For such an exceptional value is an asymptotic value [9, p. 34] and on the curve extending to infinity along which $F(z)$ tends to A, $F(z)$ will be bounded.

3. Cases not Treated in the Last Section.

It is obvious that there are functions $F(z)$ such that $f(z)$ may be entire: $\exp(\exp z)$ is such a function, but its order is infinite. The question arises as to whether this is possible when $F(z)$ is of finite order, and this is answered by the provision of examples which demonstrate the possibility. In fact we can prove the

Theorem 2. There exist entire functions $f_1(z)$, $f_2(z)$ and $f_3(z)$ of zero order such that

(i) $f_1\{f_1(z)\}$ is of zero order

(ii) $f_2\{f_2(z)\}$ is of infinite order

(iii) $f_3\{f_3(z)\}$ is of finite non-zero order.

Remarks: Where it is necessary to check the order of the functions introduced below one uses the result that if ϱ is the order of the function $\sum_{n=0}^{\infty} a_n z^n$, one has [10, p. 40]

$$\frac{1}{\varrho} = \varliminf_{n\to\infty} \frac{-\log |a_n|}{n \log n}.$$

Use is made of the notations with respect to $f(z) = \Sigma a_n z^n$:

$\mu(r)$: the maximum term of the sequence $|a_n| r^n$.

$\nu(r)$: the central index, the index of the maximum term.

$M(r)$: the maximum for $|z| = r$ of $|f(z)|$.

$M'(r)$: the maximum for $|z| = r$ of $|f\{f(z)\}|$.

It is to be noted that these four functions are monotone increasing.

We reserve square brackets to have the meaning: $[A]$ is the greatest integer not exceeding A.

The discussion depends on the lemma:

$$(4) \qquad \mu(r) < M(r) < \mu(r)\left\{1 + 2\,\nu\left(r + \frac{r}{\nu(r)}\right)\right\},$$

proved in [9, p. 32] and the fact that we deal with functions for which the coefficients a_n are real and positive, so that

$$M(r) = f(r), \; M'(r) = f\{f(r)\} = M\{M(r)\}.$$

Proof of Theorem 2.

(i)
$$\text{Take } f_1(z) = \sum_0^\infty \frac{z^n}{\exp(n^2)}$$

$$\text{For } r > 1:\; \nu(r) = \left[\frac{1+\log r}{2}\right]$$

$$\mu(r) = \frac{r^{\left[\frac{1+\log r}{2}\right]}}{\exp\left\{\left[\frac{1+\log r}{2}\right]^2\right\}} \leqq \frac{r^{\frac{1+\log r}{2}}}{\exp\left\{\left(\frac{\log r - 1}{2}\right)^2\right\}}.$$

A simple consideration of (4) shows that for all large enough r

$$M(r) < \exp\{\tfrac{1}{2}(\log r)^2\}$$

and

$$M'(r) < \exp\{\tfrac{1}{8}(\log r)^4\},$$

so that the order of $f_1\{f_1(r)\}$ is 0.

(ii)
$$\text{Take } f_2(z) = \sum_1^\infty \frac{z^n}{\exp\{n(\log n)^2\}}.$$

Now for continuous n, $\exp\{n\log r - n(\log n)^2\}$ has its maximum where $n = n(r) = \exp\left(-1 + \sqrt{1+\log r}\right) > \exp\left(\tfrac{1}{2}\sqrt{\log r}\right)$ for $r > e^{\frac{16}{9}}$.
Certainly

$$\begin{aligned}\mu(r) &\geqq \exp\{[n(r)]\log r - [n(r)](\log[n(r)])^2\}\\ &\geqq \exp\{(n(r)-1)(\log r - \log n(r)\cdot\log n(r))\}\\ &> \exp\left\{\exp\left(\tfrac{1}{2}\sqrt{\log r}\right)\right\} \text{ for sufficiently large } r.\end{aligned}$$

From (4)

$$M'(r) > \mu\{\mu(r)\} > \exp\left\{\exp\tfrac{1}{2}\sqrt{\exp\sqrt{\tfrac{1}{4}\log r}}\right\}$$

which is not bounded by any expression of the form $\exp(r^A)$ and shows that $f_2\{f_2(z)\}$ is of infinite order.

(iii) We deal now with the comparatively intricate problem of constructing such an $f_3(z)$ that $f_3\{f_3(z)\}$ has finite non-zero order. The first step here is to produce a suitable real (nonanalytic) function $\Phi(r)$ satisfying $\Phi\{\Phi(r)\} = e^r$. By taking a series with positive coefficients, for which $\mu(r)$ is fairly close to $\Phi(r)$, a satisfactory $f_3(z)$ is obtained as is suggested by (4). Firstly then we prove the

Lemma: There is a continuously differentiable function $\Phi(r)$ such that $\Phi(\Phi(r)) = e^r$ and the four functions $\Phi(r)$, $\Phi'(r)$, $\frac{\Phi(r)}{r}$ and $\frac{r\Phi'(r)}{\Phi(r)}$ all tend monotonely (for $r > 1$) to infinity with r. We introduce the notation: I_0 is the interval $0 \leqq r \leqq 1$. In general I_n is defined inductively: I_{n+1} is the image of I_n under the mapping $x \to e^x$. $\sum_{n=0}^{\infty} I_n$ covers the positive real line. We assert that the conditions of the lemma are satisfied by the function defined in I_0 by

$$\left.\begin{aligned} \Phi(r) &= r + \tfrac{1}{2} \qquad & 0 \leqq r \leqq \tfrac{1}{2} \\ &= \exp(r - \tfrac{1}{2}) \qquad & \tfrac{1}{2} \leqq r \leqq 1 \end{aligned}\right\}$$

and defined in I_n generally by repeated application of the relation

$$\Phi(e^r) = \exp\{\Phi(r)\}. \tag{5}$$

Thus, for example, in I_1:

$$\left.\begin{aligned} \Phi(r) &= e^{1/2}\, r, \qquad & 1 \leqq r \leqq e^{1/2} \\ &= \exp(e^{-1/2}\, r), \quad & e^{1/2} \leqq r \leqq e \end{aligned}\right\}.$$

Obviously in I_0: $\Phi\{\Phi(r)\} = e^r$, and it then follows from (5) that this is so in every I_n.

Now the four functions mentioned in the lemma are all continuous and monotone increasing in I_1. Using the method of induction, suppose this is true in any I_n. Differentiating (5) gives

$$\frac{e^r\,\Phi'(e^r)}{\Phi(e^r)} = \Phi'(r),$$

so that $\frac{r\,\Phi'(r)}{\Phi(r)}$ is continuous and monotone increasing in I_{n+1}. A similar result follows for the other functions from the identities

$$\frac{\Phi(e^r)}{e^r} = \exp\{\Phi(r) - r\}$$

and $\Phi'(e^r) = \Phi'(r)\exp\{\Phi(r) - r\}$. Since the functions increase monotonely with r, they either tend to infinity or are bounded and tend to a finite limit. In each case the latter supposition conflicts with the fact that $\Phi\{\Phi(r)\} = e^r$. Thus for example if $\frac{r\,\Phi'(r)}{\Phi(r)}$ tends to l as r tends to infinity:

$$r = \frac{r\,\frac{d}{dr}(e^r)}{e^r} = r\,\frac{\Phi'\{\Phi(r)\}}{\Phi\{\Phi(r)\}}\,\Phi(r)\,\frac{\Phi'(r)}{\Phi(r)} \to l^2$$

as $r \to \infty$, since $\Phi(r) \to \infty$ monotonely with r; but this is obviously false.

If $\Phi(r)$, $\Phi'(r)$ or $\frac{\Phi(r)}{r}$ are bounded then certainly there is a K such that $\Phi(r) < K\,r$ and $e^r = \Phi\{\Phi(r)\} < K^2\,r$ which is impossible. This completes the proof of the lemma.

The function $f_3(z)$. The functions $\mu(r)$, $\nu(r)$ defined above are connected by the relation

$$\log \mu(r) = \log |a_0| + \int_0^r \frac{\nu(x)}{x}\,dx \tag{6}$$

where $a_0 \neq 0$ is the constant term in the expansion of the relevant entire function. $\nu(r)$ is a (monotonely increasing) step function with a denumerably infinite set of isolated discontinuities $r_n > 0$, $n = 1, 2, \ldots$, at each of which $\nu(r)$ increases by an integer. As VALIRON [10, p. 28—30] shows, to any such $\nu(r)$ given in advance one can construct a number of entire functions which have $\nu(r)$ as their central index function. For a given choice of $|a_0| \neq 0$, it is possible to find one of these whose coefficients are real and positive and which is in fact a majorant of all similar functions. Having chosen $\nu(r)$ and a_0 it is this function which, for definiteness, we shall use for $f_3(z)$.

We choose for $\nu(r)$:

$$\left.\begin{aligned} \nu(r) &= 1 + \left[\frac{r\Phi'(r)}{\Phi(r)}\right], & r &\geqq 1 \\ &= 0, & 0 \leqq r &< 0\cdot 1 \\ &= 2, & 0\cdot 1 \leqq r &< 1 \end{aligned}\right\}.$$

Then

$$\int_0^e \frac{\nu(x)}{x}\,dx > \int_0^e \frac{\Phi'(x)}{\Phi(x)}\,dx$$

and

$$\int_0^r \frac{\nu(x)}{x}\,dx > \int_0^r \frac{\Phi'(x)}{\Phi(x)}\,dx, \quad r > e.$$

When one makes $a_0 = \frac{1}{2}$, (6) becomes

$$\log \mu(r) = \log \tfrac{1}{2} + \int_0^r \frac{\nu(x)}{x}\,dx > \log \tfrac{1}{2} + \int_0^r \frac{\Phi'(x)}{\Phi(x)}\,dx > \log \Phi(r), \quad r > e.$$

Then $M'(r) > \mu\{\mu(r)\} > \Phi\{\Phi(r)\} = e^r$. But on the other hand

$$\log \mu(r) < \log \tfrac{1}{2} + \int_0^1 \frac{\nu(x)}{x}\,dx + \int_1^r \left\{\frac{\Phi'(x)}{\Phi(x)} + \frac{1}{x}\right\} dx,$$

so that for some positive K

$$\mu(r) < K\, r\, \Phi(r).$$

For all $r > 0\cdot 1$, $\nu(r) > 1$, so that from (4) and the line above:

$$M(r) < K\, r\, \Phi(r) \left\{1 + 2 + 2(r+1)\frac{\Phi'(r+1)}{\Phi(r+1)}\right\} < 3\, K\, r^2 \frac{\Phi'(r+1)}{\Phi(r+1)}\, \Phi(r)$$

for large enough r.

Now $\Phi\{\Phi(r)\} = e^r = \dfrac{d}{dr}(e^r) = \Phi'\{\Phi(r)\} \cdot \Phi'(r)$

from which one sees that $\Phi'(r) < \Phi(r)$ for all $r > 1$; so that:

$$M(r) < 3K\, r^2 \Phi(r). \tag{7}$$

However small $\delta > 0$ may be,

$$r \frac{d}{dr} \left\{ \log \frac{\Phi(r^{1+\delta})}{\Phi(r)} \right\} = (1+\delta)\, r^{1+\delta} \frac{\Phi'(1+\delta)}{\Phi(r^{1+\delta})} - \frac{r\,\Phi(r)}{\Phi(r)}$$

tends to infinity with r, and so remains greater than arbitrarily large N for $r > r(N)$. A simple integration shows that there exists a positive real C such that

$$\Phi(r^{1+\delta}) > C\, r^N \Phi(r), \quad r > r(N).$$

With this result (7) becomes

$$M(r) < \Phi(r^{1+\delta}) \tag{8}$$

Since $r \dfrac{\Phi'(r)}{\Phi(r)} \to \infty$ with r it follows that for large r,

$$r < \{\Phi(r)\}^{\alpha}$$

however small $\alpha > 0$ is chosen, and replacing r by $\Phi(r)$:

$$\Phi(r) < e^{\alpha r}.$$

From this result, (7) and (8) it follows that

$$\begin{aligned} M'(r) = M\{M(r)\} &< 3K\, \{\Phi(r^{1+\delta})\}^2\, \Phi\{\Phi(r^{1+\delta})\} \\ &< 3K \exp\{r^{1+\delta} + 2\alpha\, r^{1+\delta}\}. \end{aligned}$$

But α and δ may be chosen arbitrarily small, so that the order of $f_3\{f_3(z)\}$ is exactly 1. The order of $f_3(z)$ can only be zero.

4. Consequences of Theorem 1.

We now apply the main theorem to derive a property of analytic solutions of (3).

In the iteration of an entire function $F(z)$ there arises a set $\mathfrak{F}(F)$ which is of great importance: $\mathfrak{F}(F)$ is the set of points z where the iterates $F_n(z)$ of positive integral order do *not* form a normal family in the sense of MONTEL.

The term "Fatou exceptional point" is used to denote a point α such that the equation $F(z) = \alpha$ has no solution in z, except possibly $z = \alpha$. There can be at most one such point.

FATOU [6] has investigated the set $\mathfrak{F}(F)$ thoroughly and has proved the following result:

Let c be a small circle surrounding $z_0 \in \mathfrak{F}(F)$, and c_n its image under the transformation $z \to F_n(z)$. If Δ is any bounded region of the plane excluding a small circle d about the Fatou exceptional point if this exists, then for sufficiently large n, c_n will cover Δ.

From this follows

Theorem 3. If $f(z)$ is an analytic solution of (3), where $F(z)$ is bounded on some curve extending to infinity and $f(z)$ is analytic at any one point of $\mathfrak{F}(F)$ and also at the Fatou exceptional point if this exists, then $f(z)$ cannot be single-valued.

Proof: Suppose $f(z)$ single-valued and analytic at $z_0 \in \mathfrak{F}(F)$. Take for c a circle surrounding z_0 in which the expansion of $f(z)$ about z_0 converges. Then $f(z)$ may be continued into any c_n by repeated application of the relation

$$f\{F(z)\} = f[f\{f(z)\}] = F\{f(z)\}.$$

If a Fatou exceptional point α exists, take a circle d about α in which $f(z)$ is analytic. Then if D is any bounded region, of the plane our procedure of continuation shows that $f(z)$ is analytic in $D - \alpha$, and hence in D; that is $f(z)$ is entire, against theorem 1. q.e.d.

The instance of $F(z) = e^z$ is of particular interest and has been treated by Kneser [7], who has constructed a solution $f(z)$ of (3) which on the whole real axis is analytic and real. e^z has the exceptional value 0, and Töpfer [11] has shown that every point of the real axis belongs to $\mathfrak{F}(e^z)$. It follows that Kneser's solution is not single-valued.

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