Early days in complex dynamics: a history of complex dynamics in one variable during 1906–1942, by Daniel S. Alexander, Felice Iavernaro, and Alessandro Rosa, History of Mathematics, 38, American Mathematical Society, Providence, RI, London Mathematical Society, London, 2012, xviii+454 pp., ISBN 978-0-8218-4464-9, US $99.00

Some time ago, on my way to a special lecture at the Fields Institute, I overheard a couple of graduate students whispering in the corridor:

- Are you coming to Shishikura's talk?
- Who's that?
- The guy who proved that the boundary of the
  Mandelbrot set has Hausdorff dimension two.
- Sweet!

That was not the topic of the talk (in fact, $\mathrm{HD}(\partial M) = 2$ had been proven 15 years earlier), so it was telling that they even knew what the notions were in the statement. To me, this conversation is symbolic of how much the language of complex dynamics has permeated mathematical culture since the early 1980s, when it was preposterous to think of doing serious research on the properties of quadratic polynomials.[1]

Back then much of the subject's popularity had to do with the pretty pictures. A few lines of computer code were enough to start exploring an endless variety of fractal shapes, and ever since then, Julia sets have decorated calculus books, art galleries, and cereal boxes. But there *was* serious research to be done, and complex dynamics is today a vital branch of dynamical systems that has developed deep connections to differential equations, geometric group theory, harmonic analysis, algebraic number theory, and many other subjects. It is also a field with a rich tradition. As a student, I got to learn quite a number of historical trivia facts, such as the string of Paris cafés where Douady and Hubbard did most of their original work, or the unconfirmed existence of T. Cherry's notebooks, which may contain valuable insights on rotation domains.

One peculiarity of the subject's evolution is the 60-year hiatus between the groundbreaking results of Fatou and Julia, and the explosion of research started around 1980. To be sure, there were some sporadic results (notably Siegel's treatment of the small denominator problem [14]), but the floodgates were open only after computer graphics became available. Without this powerful aid to intuition, the work of the early pioneers appears all the more remarkable. What mental representation did they have to work on? How much did they suspect about the richness of the world they were uncovering?

These are open-ended questions, but they suggest that here is a historical topic worth pursuing. The present book does this job admirably well. The authors seem to have read every single paper written on the subject of complex iteration between

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[1] "I must say that in 1980, whenever I told my friends that I was just starting with J. H. Hubbard a study of polynomials of degree 2 in one complex variable (and more specifically those of the form $z \mapsto z^2 + c$), they would all stare at me and ask: Do you expect to find anything new?" A. Douady in [11].

1870 and 1942; the main results being digested for the benefit of readers interested primarily in the mathematical story. They also had access to dozens of academic and personal documents that enabled them to tell a more human tale; a tale in which personalities are important and national traits evident. To put this story in context, I will introduce a (very) rough division of the history of complex dynamics into five periods. As a disclaimer, do not expect a full treatment; I am omitting much in order to concentrate on highlights that might tempt readers from outside the field to explore it.

1. Local Theory. The first modern study of iteration was due to Ernst Schröder, a Gymnasium teacher in Germany who published two papers in *Mathematische Annalen* in 1870–71. Although his treatment is not very rigorous, he was the first to suggest the use of conjugation as a means to studying the dynamical behavior of an analytic function f near a fixed point z_0. The idea is to find a local change of coordinates $\zeta = \varphi(z)$ around z_0 that conjugates f into an easy-to-describe model. When possible, this facilitates the study of f because conjugation respects iteration, so the dynamical behavior of f and its "simpler" model are essentially the same. To illustrate this idea, consider the function

$$(1) \qquad\qquad f(z) = \sum_{n=1}^{\infty} a_n z^n,$$

which has 0 as a fixed point. The derivative $f'(0) = a_1$ (the *multiplier* of the fixed point) dictates the dynamical behavior of f near 0. If $a_1 \neq 0$, we have $f(z) \approx a_1 z$ near the origin, and we are led to ask, following Schröder, if f is conjugate to the linear map $L(\zeta) = a_1 \zeta$.

For the case $0 < |a_1| < 1$, Gabriel Kœnigs gave a positive answer in 1884. It is an instructive exercise to prove that

$$\varphi(z) = \lim_{n \to \infty} \frac{f^{\circ n}(z)}{a_1^n}$$

furnishes an explicit solution to the Schröder conjugating equation $\varphi(f(z)) = a_1 \varphi(z)$. Since L is a contraction, successive iterates $f^{\circ n}(z)$ converge to 0 whenever z is in the domain of the conjugating map φ, so we call 0 an *attracting* fixed point. The case $|a_1| > 1$ follows trivially from the attracting case by considering a local inverse f^{-1}. In this situation we call 0 a *repelling* fixed point.

Two students of Kœnigs, Auguste-Clémente Grévy and Leopold Leau, studied the remaining cases $a_1 = 0$ and $|a_1| = 1$. They formed, at the turn of the century, the core of a "French school of iteration" concerned with the problem of studying similar conjugating equations and their domains of definition.

Grévy's interest was to generalize Kœnigs's linearization result, so he missed the correct model in the *superattracting* case $a_1 = 0$ (although he did find some degenerate particular cases). The solution was found by Lucyan Böttcher, an obscure Polish mathematician, who showed that the function f is conjugate to the map $z \mapsto z^d$, where the degree d is given by the smallest index such that $a_d \neq 0$. Böttcher published almost exclusively in Polish and Russian, so his results went unnoticed for a long time. Böttcher coordinates will be mentioned later as an essential tool in polynomial dynamics.

Leau, on the other hand, faced the hard problem of finding a natural model for the dynamics of f around 0 when the multiplier a_1 is a root of unity, i.e., when the

fixed point is *parabolic*. As it turns out, the origin is repelling in certain directions and attracting in others, so f cannot be conjugate to its linear part. Leau's Flower Theorem decomposes a punctured neighborhood of 0 into a union of attracting and repelling *petals* that alternate around 0 and have 0 as their common boundary point. The dynamical behavior of f in each of these petals is conjugate to the translation $z \mapsto z + 1$.

The case of a multiplier $a_1 = \mathrm{e}^{2\pi \mathrm{i} \theta}$ with irrational θ had to wait for half a century. As we will see, it is a problem with connections to celestial mechanics and number theory, and is still not completely solved.

2. Global theory. Newton's method applied to the polynomial $P(z) = z^2 - 1$ yields the rational function $N(z) = \frac{z^2+1}{2z}$. It is an instructive exercise to prove that iteration of N quickly converges to either of the two roots ± 1 of P, provided the initial point is not on the imaginary axis. In dynamical terms, N has superattracting points at ± 1, and their *basins of attraction* are the half-planes with positive and negative real parts. This particular result appears in Schröder's papers, but is often accredited to Arthur Cayley who did a much better job of promoting it in England and France.

The authors are likely right in saying that Cayley's work was "probably tangential to the development of complex dynamics", although it has historical significance for three reasons. First, there is reason to believe that it carried more influence in French circles than did his German counterpart. Also, Newton's method for $z^3 - 1$ splits $\mathcal{C}$ in three regions all of which share the same boundary and are thus *fractal*. Cayley's surprise at the difficulty of this situation (which he did not understand) has become part of folk mathematical culture and was eventually popularized as an early forerunner of chaos theory. The third reason is that this decomposition of the plane into different dynamical behaviors (whether attributed to Schröder or Cayley) can be considered the first dynamical result of a global nature.

Some nonlocal ideas can also be found in Böttcher's work, and in Salvatore Pincherle's 1912 entry on functional equations in the French Mathematical Encyclopedia. The best indication however that global problems were ripe to be attacked was probably the short note of Pierre Fatou in 1906 where he showed that the rational functions $f_k(z) = \frac{z^k}{z^k + 2}$ have an attracting fixed point at 0 whose basin of attraction consists of the complex plane with a Cantor set removed.

Two events converged to make the transition possible. Paul Montel's development of the theory of normal families around 1907–11, and the announcement late in 1915 of a *Grand Prix des Sciences Mathématiques* to be awarded by the French Académie des Sciences at the beginning of 1918. The topic for the prize? Iteration in one or several variables from a global point of view...

The story of this prize is one of the most cherished bits of trivia in the field. The main protagonists were involved in a (civil) controversy over priority, and the fact that Julia was a wounded war hero played a part in the whole situation resolving to his advantage. Most versions of this story are inaccurate. For instance, a common misconception has Fatou as one of the competitors. In fact, the Académie received three entries and accepted those by Gaston Julia and Samuel Lattès. Fatou refrained from entering the competition, and the book offers some interesting speculations about the role of the aforementioned priority dispute on his decision. Another interesting contribution of the book is the disclosure of the third Prize entrant. (Should I name him? The answer is in page 126 of the book.)

Both Fatou and Julia realized that the key to understanding global dynamical properties is the distinction between points with "regular behavior" and "wild" points. Both used Montel's definition of normal families to formalize this idea: Assume $f : \overline{\mathcal{C}} \to \overline{\mathcal{C}}$ is a rational function of degree $d \geq 2$ on the Riemann sphere, and define $F_f \subset \overline{\mathcal{C}}$ as the set of points z for which the iterates $\{f^{\circ n}\}$ restricted to a neighborhood U of z form a normal family; that is, $\{f^{\circ n} : U \to \overline{\mathcal{C}}\}$ are locally equicontinuous in the spherical metric. The set J_f is defined as the complement $\overline{\mathcal{C}} \setminus F_f$. In modern parlance, F_f is the *Fatou set* of f, and J_f the *Julia set*. The imaginary axis in the case of Cayley's N, and the Cantor set of Fatou's f_k are examples of Julia sets.

The definition of J_f above is due to Fatou. Julia on the other hand defined it as the closure of the set of repelling periodic points of f. A central result in both works is the equivalence of the two definitions. Let us see how Fatou proved it. Pick a point z_* in the Julia set which is neither fixed nor critical, so that the preimage $f^{-1}(z_*)$ consists of d distinct points $z_1, \ldots, z_d$, all different from z_*. For each z_j there is a local inverse ψ_j of f such that $f \circ \psi_j = \mathrm{id}$ and $\psi_j(z_*) = z_j$. Let all ψ_j be restricted to a common domain U. Since $z_* \in J_f$, the family $\{g_n\}$ defined by

$$g_n(z) = \frac{\big(f^{\circ n}(z) - \psi_1(z)\big) \cdot \big(z - \psi_2(z)\big)}{\big(f^{\circ n}(z) - \psi_2(z)\big) \cdot \big(z - \psi_1(z)\big)}$$

is not normal on U, so by Montel's Theorem the ranges cannot omit all three values 0, 1, and ∞: For some point $w \in U$ and iterate $n > 0$, we must have $g_n(w) \in \{0, 1, \infty\}$. Thus, $f^{\circ n}(w)$ is either w, $\psi_1(w)$, or $\psi_2(w)$, and w must be periodic. Since it is known that only finitely many periodic points are not repelling, the result follows.

This sample is just the tip of the iceberg. Although no one had a clear idea at the time of how a Julia set looks, the work of Fatou and Julia made a deep impression and was regarded as one of the earliest successful applications of the new concepts that formed what we now call analysis. To give you an idea of the high esteem given to their results at that time, I will just mention that Montel gave a full account in his famous 1927 book on normal families, praising them as the most accomplished application of his theory.

3. Interlude. One basic question remained open, and it was of a local nature: Is the function (1) linearizable when the multiplier is irrationally neutral; i.e., when $a_1 = \mathrm{e}^{2\pi i\theta}$ and $\theta \notin \mathbb{Q}$?

To be precise, let f be as in (1). We seek an analytic function $\varphi(z) = \sum_{n=1}^{\infty} c_n z^n$ such that

$$(2) \qquad\qquad\qquad \varphi(a_1 z) = f \circ \varphi(z).$$

Notice that if such a linearizing map exists, multiplication by a constant c before applying φ simply rescales the domain so $\varphi(cz)$ is also a linearizing map. By setting $c = 1/\varphi'(0)$, the coefficient c_1 can be assumed to be 1. It is an instructive exercise to substitute the series for f and φ in (2), and prove that the coefficients of φ are given by the recursion

$$c_1 = 1, \qquad c_n = \left(\frac{1}{a_1^n - a_1}\right) \cdot \left(\sum_{r=2}^{n} \sum_{\ell_1 + \cdots + \ell_r = n} a_r \cdot c_{\ell_1} \cdot \ldots \cdot c_{\ell_r}\right).$$

This formula defines φ explicitly, so it seems to solve the linearization problem. However, depending on θ, the factors $1/(a_1^n - a_1)$ can get very large too often. This threatens the convergence of the power series for φ, and, indeed, it is possible that the series solution is only formal.

This is an example of a *small divisor problem* and its solution is not easy. Small divisors were first encountered in celestial mechanics and gave Poincaré no end of trouble. The linearization problem just described can be viewed as a baby version in which no physical considerations obscure the convergence issue. Still, a quarter of a century after the publication of Poincaré's *Méthodes Nouvelles*, no one had an idea for how to deal with small denominators. Fatou and Julia tackled the linearization problem to no avail (Julia even published a wrong proof claiming that there are no solutions). In 1917, George A. Pfeiffer found some nonlinearizable examples, and in 1928 Hubert Cremer found a dense G_δ set of angles θ for which no rational function is linearizable.

The first positive solution came in 1942 in a six-page article by Carl L. Siegel [14]. He proved that any f whose multiplier at 0 is $a_1 = \mathrm{e}^{2\pi\mathrm{i}\theta}$ with θ Diophantine is linearizable, and (assuming a standard normalization on f) gave an explicit lower bound on the radius of convergence which depends only on the Diophantiness degree of θ. Since the Diophantine condition is of full measure, Siegel's result was as strong as could be desired, even though it did not fully settle the linearization problem. Because of the importance of the result and its influence on later developments (particularly KAM theory), the 1942 paper has been praised as one of the most instrumental of the 20th century; see [12] for details.

After Siegel, much progress was made on circle diffeomorphisms and *rotation domains* (Fatou components where the map is conjugate to a rotation). Of particular mention are the works of Alexander D. Brjuno who described in 1971 the largest class of numbers θ for which every analytic function f with fixed point 0 of multiplier $\mathrm{e}^{2\pi\mathrm{i}\theta}$ is linearizable, and Michael R. Herman culminating with his results on quasi-symmetric conjugacy of circle homeomorphisms/diffeomorphisms to rigid rotations (see his unpublished manuscripts [8] and Douady's Bourbaki exposition [6]).

It is important to emphasize that there was other work done in the field. The book makes a point of noting the work of Irvine N. Baker, who published extensively on the iteration of entire maps, and it mentions Hans Brolin, Thomas Cherry, and others.

4. Modern theory. Sometime in the late 1970s the stage was finally set for the dynamical study of the quadratic family $\mathcal{Q} = \big\{ f_c(z) = z^2 + c \big\}$. Preliminary developments include the Kneading Theory of Milnor and Thurston, the experimental discovery of renormalization in the logistic family by Feigenbaum, and even advances in computer graphics.

Seen as a map on the Riemann sphere, f_c has two critical points: one at 0 and one at ∞. The later one is a superattracting fixed point, and it is an instructive exercise to prove that the boundary of its basin of attraction coincides with the Julia set J_c of f_c. The behavior of the critical point at 0 determines the dynamical properties of f_c. In particular, computer experimentation shows that J_c is connected when the orbit of 0 is bounded, and is a Cantor set when the orbit of 0 is attracted to ∞.

In order to understand this dichotomy, one is led to define the set $M = \left\{ c \,\middle|\, \{ f_c^{on}(0) \} \nrightarrow \infty \right\}$. To avoid the controversy of who discovered first the Mandelbrot set, I will simply point to an eloquent short note in *Scientific American* [9], which seems to cover all bases.

In any case, Adrien Douady and John H. Hubbard started around 1980 a systematic study of $\mathcal{Q}$ that culminated with their proof that the Mandelbrot set is connected (Nessim Sibony gave an alternate proof around the same time). Their argument begins with the observation that when J_c is connected, the Böttcher coordinate of the superattracting point at ∞ furnishes a Riemann map from the basin of attraction to the unit disk. An extension of this idea to parameters c outside M allowed them to construct a Riemann map for $\overline{\mathcal{C}} \setminus M$, thus showing that M is connected. The next obvious step has proved to be a fertile ground for ideas, but is disconcertingly unassailable:

MLC Conjecture. *The Mandelbrot set is locally connected.*

Douady and Hubbard also introduced a notion of renormalization to explain the self-similarity of M, and developed a kneading theory in the form of *external rays.* These are pullbacks of straight rays in the unit disk by the Böttcher coordinate at ∞. If the initial angle is a rational fraction of a full turn, the corresponding external ray accumulates at one point w of J_c which turns out to be (pre-)periodic; we say that the ray *lands* at w. Many rays can land at the same point w, and the binary representation of their angles encodes the itinerary of the iterates $f_c^{on}(w)$ around J_c. These tools have proved invaluable in attacking MLC.

In the context of the quadratic family, a parameter c is *hyperbolic* if f_c has an attracting periodic point. If true, MLC would imply the density of hyperbolic parameters. This is without a question the most important open problem in the field and has been solved only partially. Jean-Christophe Yoccoz proved MLC at parameters $c \in M$ that are at most finitely renormalizable. His proof uses external rays to construct a *puzzle* of pieces nested around c. The finite renormalization condition ensures that the diameter of these pieces goes to 0, proving local connectivity. Mikhail Lyubich refined this construction, describing a subset of puzzle pieces, the *principal nest*, that allowed him to establish the truth of MLC for certain family of infinitely renormalizable parameters.

Besides this explosion of activity in understanding the quadratic family, other developments turned complex dynamics into a very active field. In 1985 Dennis Sullivan introduced quasi-conformal maps as a new, powerful tool to construct functions with prescribed dynamics. Using this idea he proved the famous No Wandering Domains Theorem, stating that every component of the Fatou set F of a rational function f is eventually periodic. The argument is quite elegant. If the successive iterates of a component $W \subset F$ never intersect, we can put an arbitrary Beltrami differential μ on W and use the forward and backward iterates of f to spread μ along all components of F that have a forward image in common with W (the *grand orbit* of W). Straightening the resulting complex structure gives a new rational map conjugate to f; a *deformation* of f. Now, μ is arbitrary, so we can construct an infinite-dimensional space of deformations; but this is impossible since the space of rational maps of a fixed degree has finite dimension. Sullivan's theorem completed the classification of Fatou components of rational functions; a problem originally considered by Fatou and Julia.

5. Current status. One would expect that after all the "easy" theorems have been proved, it only remains to wait for a lucky break to establish MLC and close the subject. Instead, complex dynamics has branched in new and varied directions. It would be unwise to try an enumeration of names is this short space; instead, let me refer you to the Preface of [10], the Proceedings of the 2006 Fields Institute Conference mentioned at the beginning of this review. Its description of the papers submitted is an accurate portrait of the state of the art in complex dynamics today. Another characteristic sample can be found in [13].

Some representative advances in the classical theory include the proof of the MLC Conjecture for further families of infinitely renormalizable parameters, p-adic dynamics, the combinatorial treatment of the dynamics of transcendental maps, the phenomenon of *intermingled basins* in several variables, and the celebrated discovery of polynomial Julia sets with positive area.

One example of a topic making new connections to complex dynamics is the theory of *self-similar groups* of Volodymyr Nekrashevych: Let X be a finite alphabet and X^* denote the set of finite words over X. A faithful action of a group G on X^* is *self-similar* if for every $g \in G$ and $x \in \mathsf{X}$ there exist $h \in G$ and $y \in \mathsf{X}$ such that for all words $w \in \mathsf{X}^*$,

$$g(xw) = yh(w).$$

Self-similar groups appear naturally as *iterated monodromy groups* of branched covers of the Riemann sphere $\overline{\mathcal{C}}$. If f is such a branched cover with degree $d \geq 2$, let the set P be the union of the orbits of the branch points and assume that P is finite. The alphabet X is represented by the d inverse images of a generic point x_*. If γ is a loop in $\overline{\mathcal{C}} \setminus P$ based at x_*, the n-fold lift of γ permutes the d^n preimages of x_*. The inverse limit of the quotients of $\pi_1\left(\overline{\mathcal{C}} \setminus P, x_*\right)$ by these monodromy actions is the iterated monodromy group $\mathrm{IMG}(f)$. It is a self-similar group.

This algebraic construction brings new dynamical insights, while creating interesting examples for group theorists. This two-way feedback is characteristic of successful interactions between fields. To illustrate the first direction, consider Thurston's topological characterization of branched covers that are equivalent to rational maps (see [7]). The language of self-similar groups is ideally suited to describe Thurston obstructions, and gave Nekrashevych the means for his solution with Laurent Bartholdi of the famous "Twisted Rabbit Problem" in [3].

In the opposite direction, dynamically defined IMGs have supplied geometric group theory with many new groups with desirable properties. This includes groups of intermediate growth [5], and the first example of an amenable group that is not subexponentially amenable [4].

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The authors' stated goal was "to tell the story of the development of complex dynamics in the first half of the 20th century", but they have done more than that. Their narrative traces the origins of the subject back to Schröder, and it explores the aftermath of Siegel's result. One consequence of their meticulous work is dispelling the myth that a few scattered papers set the foundation of complex iteration. In fact, the story is much more complicated than what I have sketched here, and there are multiple connections between the main characters. For instance, Lattès is best known today for the function

$$\frac{(z^2 + 1)^2}{4z(z^2 - 1)},$$

whose Julia set is the full Riemann sphere. Earlier, Böttcher had studied other functions with this property, but as noted before, his work was not well known. I find it a bit of a mystery how he did come to be eventually recognized and given due credit. The first to describe his work in detail was Joseph F. Ritt, who together with Pfeiffer led the community of American mathematicians interested in iteration before 1920.

Another group whose work is reviewed in detail is the Italian School. The book makes an interesting case for the claim that Pincherle was the first person to describe iteration diagrams, Julia sets, *and* the Mandelbrot set. Pincherle's was the third submission to the Academié's Grand Prix (there, I said it).

A good portion of the book is related directly or indirectly to the Grand Prix. The drama of this story makes for interesting reading, and it has already been explored in detail in books by Michéle Audin [2] and by the first author of the present book [1]. The focus here is different and complements well the earlier works. The authors discuss many curious sources, such as the letters that Julia deposited with the Academié to support his priority claims, or some correspondence that shows the concern to provide financial help to Lattès' wife after he died (six months before the Prix was awarded).

More than a third of the book is taken by sixteen appendices. Eleven of these are contributions written by other authors and are an attempt to "underscore the connections between current research and its history". The other appendices contain a translation of the Academié's Grand Prix report, extensive biographies of Fatou and Julia, biographical sketches of other nineteen mathematicians, and remarks on computer graphics.

This last point brings out what is perhaps the only major drawback of the book. Although the numerous portraits of mathematicians bring them to life and add charm to the story, other illustrations are not so fortunate. The color slides in particular leave much to be desired. Some screen-shots have the wrong aspect ratio; a Julia set that is totally disconnected fails to look so because of insufficient iterations, and some other figures simply do not illustrate the phenomenon they are supposed to.

I am also irked a bit by the discontinuity in the narrative which jumps too often between the mathematical and historical discussions. The presentation seems disconnected, and some facts are repeated in different sections, perhaps due to having been written by different authors. Nevertheless, this does not seriously affect the quality of the exposition. All in all this is an authoritative reference, and quite entertaining. I can only hope to read one day a history of complex dynamics after 1942 that is as thorough as this book.

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