REGULAR GROWTH OF FUNCTIONS

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I have been doing what I guess you won't let me do when we are married, sitting up till 3 o'clock in the morning fighting hard against a mathematical difficulty.

— Sir George Gabriel Stokes, from a letter to a young lady, March 19, 1857.

Are you one of those tennis-shoed liberals who wants to regulate everything?
Let them grow any way they want.

— Lucio Tavernini, e-mail regarding the talk title, February 24, 1994.

Asymptotic approximations

- ☐ F. Carlini (planetary motion, 1817), J. Liouville (1837), G. Green (fluid motion in a canal, 1837): asymptotic approximation.
- ☐ G. Stokes (Airy's integral, 1856), H. Hankel (1868): approximate divergent series solutions.

  Stokes phenomenon: the correspondence between approximate solutions and actual solutions varies with the argument.

  - ➪ Consider $x^2w'' + xw' - (n^2 + x^2)w = 0$.

    Let $w = x^\rho e^{\mu x} \sum\limits_{k=0}^{\infty} A_k x^{-k}$.

    Formal substitution gives $w = x^{-1/2} e^{\pm x} \sum\limits_{k=0}^{\infty} A_k x^{-k}$,

    where $A_0 = 1, \quad A_k = \dfrac{(-1)^k}{8^k} \prod\limits_{j=1}^{k} \dfrac{(2j-1)^2 - 4n^2}{j}$
  - ➪ Dominant and recessive (subdominant) solutions.
  - ➪ Discontinuities in the coefficients on the Stokes lines: the positive and negative real axes.

- A. Cauchy (1833–1835), K. Weierstrass (1842), C. Briot and J.-C. Bouquet (1854), L. Fuchs (1866), J.A. Lappo-Danilevskij (1934): analytic theory for power series solutions, majorants.

  Consider $w^{(n)} + q_{n-1}(x)w^{(n-1)} + ... + q_0(x)w = 0$.

  - Regular points: radius of convergence, fundamental set.
  - Singular points.

- Regular singularity at x_0: finitely many negative powers of x in the Laurent series. The coefficients can be determined.

  - Indicial equation: $[\lambda]_n + q_{n-1}(x_0)[\lambda]_{n-1} + ... q_0(x_0) = 0$, where $[\lambda]_k = \lambda(\lambda-1)...(\lambda-k+1)$.

    If the roots are distinct modulo $\mathbf{Z}$, then

    $$w = (x-x_0)^{\lambda} \sum_{k=-\infty}^{\infty} b_k (x-x_0)^k,$$

    where the Laurent series are convergent.

    Otherwise

    $$w = (x-x_0)^{\lambda} \sum_{j=0}^{s-1} \varphi_j \left(\log(x-x_0)\right)^j ,$$

    where s is the multiplicity modulo $\mathbf{Z}$ and φ_j are convergent Laurent series.

  - Fuchs's criterion (at ∞):
    $w^{(n)} + x^{-1} f_{n-1}\left(x^{-1}\right) w^{(n-1)} + ... + x^{-n} f_0\left(x^{-1}\right) = 0$.

- L. Thomé (1877), E. Fabry (1885), H. Poincaré (1886): Irregular singularities.

  Consider $q_n w^{(n)} + q_{n-1}(x) w^{(n-1)} + ... + q_0(x) w = 0$.

  - Characteristic equation: $\sum_{j=0}^{n} a_j \alpha^j = 0$, where a_j are the leading coefficients of q_j.
  - Normal series (α distinct modulo $\mathbf{Z}$): $w = e^{Q(x)} x^{\rho} \sum_{j=0}^{\infty} c_j x^{-j}$ (order $= \deg Q$).
  - Logarithmic normal series (α distinct, but not modulo $\mathbf{Z}$): $w = e^{Q(x)} x^{\rho} \sum_{i=0}^{\mu} \psi_i(x^{-1})(\log x)^i$, where ψ_i are series in x^{-1}.
  - Anormal series (α has multiplicity): $w = e^{Q(x^{-1/n})} x^{\rho} \psi\left(x^{-1/n}\right)$ (order $= k$, where $(k-1)n < \deg Q < kn$).
  - Poincaré's rank: let $N = \max\{(\deg q_i - \deg q_n)/(n-i)\}$. Rank $k = -\mathrm{entier}(-N)$. All normal solutions have order k.

- Poincaré's definition of asymptotic meaning: $f(x) \sim \sum_{k=0}^{\infty} a_k x^{-k}$ means

  $$\forall n \; x^n \left| f(x) - \sum_{k=0}^{n} a_k x^{-k} \right| \to 0 \text{ as } x \to +\infty.$$

  - $a_n = \lim_{x \to +\infty} x^n \left(f(x) - \sum_{k=0}^{n-1} a_k x^{-k} \right)$
  - Operations — ok, except differentiation on a ray (ok in a sector, but may need to go to a proper subsector).
  - Convergent $\Rightarrow$ asymptotic.
  - If asymptotic validity holds in a neighborhood of a singular point, then the series is convergent.

- ☐ A. Kneser (1896), J. Horn (1897):
  sectors of validity, overlapping regions, uniform asymptotics.

  Consider $w'' + x^k F_1(x) w' + x^{2k} F_0 w = 0$, where F_i have convergent power series representations at ∞. Then ∞ is an irregular singularity of rank $k+1$. In each of the $2k+2$ sectors bounded by rays

  $$\operatorname{Re}\left\{e^{\mathbf{i}(k+1)\theta}(\alpha_1 - \alpha_2)\right\} = 0,$$

  where α_1, α_2 are distinct roots of the characteristic equation, there is a fundamental set of solutions represented asymptotically by the two normal series on any ray properly within the sector.
- ☐ C.Love (1914), W. Sternberg (1920): multiple characteristic roots.
- ☐ H. von Koch (1892): infinite determinants.
- ☐ G. Birkhoff (1909), M. Hukuhara (1937), J. Horn (1938), F. Gantmacher (1953), H. Turrettin (1953), P. Masani (1959): systems.
- ☐ J. Horn (1915), W. Trjitzinsky (1935): Convergent factorial series.
- ☐ A. Wiman, M. Matell (1924): algebroid functions y give approximate solutions $e^{\int y}$.
- ☐ Directions:
  - ➪ Error bounds
  - ➪ Initial value problems
  - ➪ Banach algebra valued functions
  - ➪ Stokes multipliers
  - ➪ Gevrey asymptotics
  - ➪ Not meromorphic coefficients
  - ➪ Hardy fields

Comparison of functional growth

E. Landau	G. Hardy		
$f = o\,(g)$	$f \prec g$	$\nu(f) > \nu(g)$	$f/g \to 0$
$f = \mathcal{O}\,(g)$			f/g ultim. bdd. away from $0, \infty$
$f = \mathcal{O}_s\,(g)$	$f \asymp g$	$\nu(f) = \nu(g)$	$f/g \to r \neq 0, \neq \infty$
$f \sim g$	$f \sim g$		$f/g \to 1$

Note that $f \sim g \Leftrightarrow f - g \prec f \Leftrightarrow f - g \prec g$.

Scales of growth/regular growth

☐ $\ldots \prec \varphi_k \prec \varphi_{k+1} \prec \ldots$

☐ Inadequate (P. du Bois-Reymond, ≈1875; J. Hadamard)

☐ E. Borel: $e^{x^{a-\delta}} < f < e^{x^{a+\delta}}$.

☐ P. Boutroux, E. Lindelöf: $e^{x^{\alpha_0}(\log(x))^{\alpha_1}\ldots(\log_k(x))^{\alpha_k-\delta}} < f < e^{x^{\alpha_0}(\log(x))^{\alpha_1}\ldots(\log_k(x))^{\alpha_k+\delta}}$.

☐ G. Hardy (≈1910): Scale of *L-functions* (the differential field of logarithmico-exponential functions is constructed from x by algebraic operations, powers, exponentiation, logarithm, and composition).

Regular growth = “comparable” to an L-function.

☐ M. Boshernitzan (1981): Scale of *E-functions* (the “extended” scale is the intersection of all maximal Hardy fields). Note that $L \subset E$.

Regular growth = “comparable” to an E-function, i.e. the germ belongs to a Hardy field.

Definitions

- A property $P(x)$ is said to hold *ultimately* exactly when it holds for all sufficiently large x, i.e. $\exists x_0 \, \forall x \geq x_0 \; P(x)$.
- A *germ* (at $+\infty$) is an equivalence class of functions, where two functions are equivalent (have the same germ) exactly when they ultimately agree.
- A *differential ring* is a ring with a derivation $'$, i.e. an endomorphism satisfying the Leibniz rule $(ab)' = a'b + ab'$.
- N. Bourbaki, 1961: A *Hardy field* is a differential field of (continuous) germs with derivation $=$ ordinary differentiation.
- M. Boshernitzan, 1983: Given a Hardy field H, a function f is H*-regular* exactly when there exists a Hardy field extension of H containing the germ of f.

Elementary properties

- If f belongs to a Hardy field, then it has *definite sign*, i.e. ultimately $f > 0$, $f < 0$, or $f \equiv 0$.
- If f belongs to a Hardy field, then it has a limit in the extended reals.
- If H is a Hardy field, then f is H-regular exactly when $H(f, f'...)$ is a Hardy field, i.e. for any differential polynomial p over H the germ $p(f, f'...)$ has definite sign.
- A function analytic at ∞ is $\mathbf{R}$-regular.

Limits and algebra

❑ An *ordered abelian group* is an abelian group Γ with a subset Γ^+ of "positive" elements such that

- ➪ $a, b \in \Gamma^+ \Rightarrow a + b \in \Gamma^+$
- ➪ $\Gamma^+ \cap (-\Gamma^+) = \varnothing$
- ➪ $\Gamma^+ \cup (-\Gamma^+) \cup \{0\} = \Gamma$

❑ W. Krull, 1932: A *valuation* (logarithmic nonarchimedean norm) on a field F is a group homomorphism $\nu : F^* \to \Gamma$ from the multiplicative group of units $F^* = F \setminus \{0\}$ onto an ordered abelian group Γ (written additively) such that, where defined,

- ➪ $\nu(ab) = \nu(a) + \nu(b)$
- ➪ $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$

We can think of $\nu(0) = \infty$.

❑ $R = \nu^{-1}(\{0\} \cup \Gamma^+)$ is a *valuation subring*, i.e. R is a local ring (a ring with a unique maximal ideal $M = \nu^{-1}(\Gamma^+)$) such that $\forall x \in F\ x \in R$ or $1/x \in R$.

❑ M. Rosenlicht, 1980: A valuation on a differential field is *differential* exactly when

- ➪ $a' = 0 \Rightarrow \nu(a) = 0$
- ➪ $\forall a \in F^*\ \nu(a) \geq 0\ \exists b, c \in F^*\ b' = 0, \nu(c) > 0, a = b + c$
- ➪ $\nu(a) \geq 0, \nu(b) > 0 \Rightarrow \nu(a'b) > \nu(b')$ (where defined)

This is an algebraic version of the two cases of l'Hôpital's rule. Indeed, let $a = c/b$. Then

$$\frac{c}{b} - \frac{c'}{b'} = a - \frac{a'b + ab'}{b'} = -\frac{a'b}{b'}$$

- P. du Bois-Reymond, 1875; A. Lightstone and A. Robinson, 1975: Suppose H is a Hardy field. Then $H_1 = \{f \in H\colon f \asymp 1\}$ is a multiplicative subgroup of H^*. Let $\Gamma = H^*/H_1$ and $\nu : H^* \to \Gamma$ be the natural projection. Then ν is the canonical differential valuation for H.
- $\nu(f) \geq 0 \Rightarrow \nu(f') > 0$. (l'Hôpital's rule for f/x)
- $\nu(f) \leq 0 \Rightarrow \nu(f'/f) > \nu\left(|f|^{\epsilon}\right)$. (consider $|f|^{-\epsilon}$)

Examples of Hardy fields

- $\mathbf{R}$: $\forall a\ a' = 0, \nu(a) = 0$.
- $\mathbf{R}(x)$: $\nu(p/q) = \deg q - \deg p$.
- Hardy's L-functions.

Extension theorems

- M. Rosenlicht, 1980: A valuation on a field extends to the algebraic closure.
- A Hardy field has a canonical real algebraic closure.

  Proof: Suppose H is a Hardy field and $F(y) = 0$, where F is a polynomial over H.

    - Since relatively prime factors of $F(y)$ cannot be simultaneously zero, connectedness lets us assume that F is irreducible.
    - (F) is a nonzero prime ideal of $H[X]$, so $H[y] \simeq H[Y]/(F)$ is field.
    - Since F and $F' = \partial F/\partial Y$ are relatively prime, $F'(y)$ is ultimately nonzero.
    - By the Implicit Function Theorem y is ultimately differentiable.
    - Since $(F(y))' = F'(y)y' + \partial F/\partial x = 0$ and $F'(y) \neq 0$, $y' \in H[y]$.
    - $H[y]$ is closed under differentiation, so is a Hardy field.

- M. Singer, 1976; et al.: Suppose H is a Hardy field, F is a polynomial in two variables over H and y is a solution of $F(y, y') = 0$. Then $H(y)$ is a Hardy field, i.e. y is H-regular.

  Proof of a special case $y' = F(y)/G(y)$, where F, G are polynomials over H:

  - Given a differential polynomial over H we can rewrite it using the equation as a polynomial over H and factor it into linear and quadratic factors (with negative discriminant) over the real algebraic closure of H.
  - Suppose a linear factor $y - c$ does not have definite sign.
  - Since $y \neq \text{ const}$, $F \not\equiv 0$, so rewrite $F/G = (y - c)^r P/Q$, where $r \geq 0$, $P(c) \neq 0$ and $Q(y)$ is ultimately zero free.
  - Positive r violates uniqueness (consider y and c), so $r = 0$.
  - $P(c)/Q(c)|_x \in \mathbf{R}(x)$, so has definite sign.
  - If x_1, x_2 are sufficiently large consecutive zeros of $y - c$, then $y'(x_1) = P(c)/Q(c)|_{x_1}$ and $y'(x_2) = P(c)/Q(c)|_{x_2}$ have the same sign, a contradiction.

- M. Boshernitzan, 1982: second order linear homogeneous equation $u'' + p(x)u' + q(x)u = 0$ with p, q in a Hardy field H:

  - $u(x) = y(x)e^{-\frac{1}{2}\int_x p(t)\,dt}$ gives $y'' = \varphi(x)y$, where $\varphi = p^2/4 - q$.
  - Via $w = y'/y$ convert to the Riccati equation $w' + w^2 = \varphi(x)$.
  - Oscillation criterion for $\varphi \in E$: for all $n \geq 1$, ultimately

    $$\varphi > \frac{1}{4x^2} \sum_{k=0}^{n-1} \left(\prod_{j=1}^{k} \log_j x \right)^{-2} .$$

  - No oscillations: solutions are H-regular.
  - Oscillations: the Riccati equation has no solutions at ∞.

    Conjecture: the general solution is of the form $C_1 f(x) \cos(g(x) + C_2)$ where f, g are H-regular.

- M. Boshernitzan, 1982: For first order linear systems with regular coefficients in a Hardy field H, if homogeneous solutions are H-regular, then nonhomogeneous solutions are H-regular.

Comparability classes

Comparability classes in a Hardy field H correspond to archimedean components of the value group Γ. Specifically two germs f and g belong to the same comparability class. i.e. $[f] = [g]$, exactly when $\exists n, m \in \mathbf{Z}\ |\nu(f)| < n\,|\nu(g)|\,, |\nu(g)| < m\,|\nu(f)|$.

- ❑ Comparability classes are linearly ordered.
  The number of comparability classes is called the *rank* of H.
- ❑ $[f] > [g] \Rightarrow [fg] = [f]$
- ❑ $\nu(f), \nu(g) < 0 \Rightarrow [f] \leq [g] \Leftrightarrow \nu(f'/f) \geq \nu(g'/g)$.
- ❑ $\nu(f) < 0, [f] < [x] \Rightarrow \nu(f'/f) > \nu(1/x)$.
- ❑ $\nu(f) < 0, [f] \leq [x] \Rightarrow [f'/f] \leq [f]$.
- ❑ $\nu(f) < 0, [f] \leq [x] \Rightarrow \nu(f') < 0, [f'] = [f]$.
- ❑ $\nu(f) < 0, [\log|f|] > [x] \Rightarrow \nu(f'/f) < 0, [\log|f|] = [f'/f]$.
- ❑ $\nu(f), \nu(g) < 0, [f] \leq [g] \Rightarrow \left[e^f\right] \leq [e^g]$.

F. Olver's problem: $y'' = \varphi(x)y, \quad \varphi(x) \to +\infty$

❑ Assume that $\varphi(x)$ is in a Hardy field H.

❑ Existence and uniqueness for the positive initial value problem to ∞.
$y > 0$. (if not, $\exists$ positive local maximum, so $y'' < 0$, contr.)
$y' > 0$. (since $y'' > 0$)

❑ Let $f(x) = \sqrt{\varphi(x)}$. The corresp. Riccati equation is $w' + w^2 = f(x)^2$.

❑ $w \sim \pm f$.

➪ $\nu(w) < 0$. (if $\nu(w) \geq 0$, then $\nu(w') < 0$, so $\nu(f^2) \geq 0$, contr.)

➪ $w'/w \prec w$, so $w' = f^2 - w^2 \prec w^2$, so $w^2 \sim f^2$

❑ $\exists w \sim f$. Then $w = f\sqrt{1 - w'/f^2} = fS_\infty\left(w'/f^2\right)$
(makes sense since $\nu(w'/f^2) > 0$)

❑ $w_1 = f, \quad w_{k+1} = fS_k\left(w_k'/f^2\right)$, where $S_k(t) = \sum\limits_{i=0}^{k} \frac{t^i}{i!} \prod\limits_{j=0}^{i-1} \left(j - \frac{1}{2}\right)$

❑ $w_k \sim f$

❑ $\nu\left(w - f\sqrt{1 - w_k'/f^2}\right) = \nu\left(\dfrac{w^2 - f^2 + w_k'}{w + f\sqrt{1 - w_k'/f^2}}\right) = \nu\left((w_k - w)'/f\right)$.

❑ $\nu\left(f\sqrt{1 - w_k'/f^2} - w_{k+1}\right) \geq (k+1)\nu(g) - k\nu(f)$, where $g = f'/f$.

➪ $\left[\sqrt{1-t} - S_k(t)\right]\left[\sqrt{1-t} + S_k(t)\right] = 1 - t - S_k^2(t) = \sum\limits_{j=k+1}^{2k} q_j t^j$,
where $q_j \in \mathbf{Q}$.

➪ $\nu\left(\sqrt{1 - w_k'/f^2} - S_k(w_k'/f^2)\right) \geq (k+1)\nu(w_k'/f^2) = (k+1)\nu(g/f)$

- $\nu\left(w - w_{k+1}\right) \geq \min\left\{\nu\left((w - w_k)'/f\right), (k+1)\nu(g) - k\nu(f)\right\}$
- $\nu\left(w - w_{k+1}\right) > \nu\left(f^{\epsilon - k + 1}\right)$
  - $\nu\left(g\right) > \nu\left(f^{\epsilon}\right)$.
  - $(k+1)\nu(g) - k\nu(f) > \nu\left(f^{(k+1)\epsilon - k}\right)$
  - By induction assume $\nu(w - w_k) > \nu\left(f^{\epsilon - k + 2}\right)$,
    then by l'Hôpital's rule $\nu\left((w - w_k)'\right) > \nu\left(f^{\epsilon - k + 1} f'\right) = \nu\left(f^{\epsilon - k + 2} g\right)$,
    so $\nu\left((w - w_k)'/f\right) > \nu\left(f^{\epsilon - k + 1} g\right)$
- $\nu\left(w - w_k\right) > \nu\left(1/x^n\right)$ for any $n > 0$ and sufficiently large k.
- $\exists$ linearly independent solutions $y_{\pm}$ of $y' = f^2 y$ such that
  $\nu(y_-) > \nu(y_+)$, so $w_{\pm} \sim \pm f$.
  - If $\nu(y_-) = \nu(y_+)$, replace y_- with a linear combination,
    so that $\nu(y_-) > \nu(y_+)$.
  - $\nu(y_-) \neq \nu(y_+) \Rightarrow w_- \not\sim w_+$
- $w_k = f + \sum\limits_{j=0}^{n(k)} a_j f^{-j}$, where a_j can be found by formal substitution:
  - Let $w = f + \sum\limits_{j=0}^{\infty} a_j f^{-j}$
  - $w' = f' + a_0' + \sum\limits_{j=1}^{\infty}\left(a_j' f^{-j} - j a_j f^{-j-1} f'\right) = fg + a_0' + \sum\limits_{j=1}^{\infty}\left(a_j' - j a_j g\right) f^{-j}$
  - $w^2 = f^2 + 2a_0 f + \sum\limits_{j=0}^{\infty} f^{-j}\left(2a_{j+1} + \sum\limits_{i=0}^{j} a_i a_{j-i}\right)$
  - $g + 2a_0 = 0, \quad a_0' + 2a_1 + a_0^2 = 0, \quad a_j' - j a_j g + 2a_{j+1} + \sum_{i=0}^{j} a_i a_{j-i} = 0$

☐ Solve for a_j:

- ⇨ $a_0 = -\frac{1}{2}g$
- ⇨ $a_1 = -\frac{1}{2}a_0^2 - \frac{1}{2}a_0' = -\frac{1}{8}g^2 + \frac{1}{4}g'$
- ⇨ $a_2 = -\frac{1}{2}a_1' + \frac{1}{2}a_1 g - a_0 a_1 = -\frac{1}{8}g^3 + \frac{3}{8}gg' - \frac{1}{8}g''$
- ⇨ $a_3 = -\frac{1}{2}a_2' + a_2 g - a_0 a_2 - \frac{1}{2}a_1^2$
  $= -\frac{25}{128}g^4 + \frac{25}{32}g^2 g' - \frac{3}{8}gg'' - \frac{7}{32}(g')^2 + \frac{1}{16}g'''$
- ⇨ $a_4 = -\frac{1}{2}a_3' + \frac{3}{2}a_3 g - a_0 a_3 - a_1 a_2$
  $= -\frac{13}{32}g^5 + \frac{65}{32}g^3 g' - \frac{37}{32}g^2 g'' - \frac{21}{16}g(g')^2 + \frac{5}{16}gg''' + \frac{7}{16}g'g'' - \frac{1}{32}g''''$

☐ Examples:

- ⇨ $f = e^x \Rightarrow w \sim e^x - \frac{1}{2} - \frac{1}{8}e^{-x} - \frac{1}{8}e^{-2x} - \frac{25}{128}e^{-3x} - \frac{13}{32}e^{-4x} + \ldots$
- ⇨ $f^2 = x^x \Rightarrow w \sim x^{x/2} - \frac{1+\log x}{4} - \left(\frac{1}{8x} - \frac{(1+\log x)^2}{32}\right) x^{-x/2} + \ldots$

The maximal regular scale

- M. Boshernitzan, 1982: Let E be the intersection of all maximal Hardy fields. Functions in E are differentially algebraic.

- O. Hölder, 1887: Euler's Γ function ($\mathbf{R}$-regular) is differentially transcendental.

- If a_i are algebraically independent over $\mathbf{Q}$, then the formal power series
$$\sum_{i=0}^{\infty} a_i x^{-i}$$
is differentially transcendental over $\mathbf{R}$. If the series converges, then it is a $\mathbf{R}$-regular function not in E.

- Functions of the form
$$\sum_{i=0}^{\infty} a_i x^{-i} + e^{-x} \left(\sin^2 x\right)^x ,$$
where the series converges and a_i are algebraically independent over $\mathbf{Q}$, are $\mathbf{R}$-regular, yet not $\mathcal{C}^\infty$.

- M. Boshernitzan, 1984: Transexponential functions do not belong to E. Certain $\mathcal{C}^\infty$ solutions of $f(x+1) = e^{f(x)}$ are transexponential and $\mathbf{R}$-regular.

Complex Hardy fields

- Trivial examples
  - $\mathbf{C}$ on $\mathbf{C}$, $\nu \equiv 0$.
  - $\mathbf{C}\left(z\right)$ on $\mathbf{C}$, $\nu(p/q) = \deg q - \deg p \in \mathbf{Z}$.
  - $\mathbf{C}\left(e^z\right)$ on any proper horizontal substrip of $-\pi/2 + n\pi < y < \pi/2 + n\pi$.
- Asymptotic existence for $w' + w^2 = f(z)^2$.
  - Curvilinear coordinates p, q with $g_{11} \asymp 1$
  - Assume $1 \prec |f|$, $|f|_p \prec |f|^2$, $(\arg f)_p \prec |f|$, $\cos\left(\arg f + \tan^{-1}\left(y_p/x_p\right)\right) \asymp 1$.
  - Approximate solutions $\sim \pm f$. Let $\varphi = u + \mathbf{i}v = \int f$.
  - $w = f(1+\alpha) \Rightarrow \alpha' + (2f + f'/f)\,\alpha + f\alpha^2 + f'/f = 0$.
  - $\alpha_0 = 0, \quad \alpha_{k+1} = \dfrac{1}{fe^{2\varphi}}\left(C - \int\left(f' + f^2\alpha_k^2\right)e^{2\varphi}\right)$
  - If $u \to -\infty$, assume that $e^{2u}\left|f\right|^2$ and $e^{2u}\left|f'\right|$ are integrable to ∞.
  - $\alpha_k \to 0 \Rightarrow \alpha_{k+1} \to 0$
  - $\alpha_k \sim \alpha_1$
  - $\alpha_k \to \alpha \sim \alpha_1$
- Examples
  - $f \sim$ meromorphic, $(p, q) = (\rho, \theta)$.
  - $f = e^z$, $(p, q) = (x, y)$.
  - $f = e^{e^z}$, $(p, q) = \left(\sqrt{y^2\left(\log y - 1/2\right) + x^2}, -x/\log y\right)$.

- Sol's to $w' + w^2 = e^{2z}$ are $\mathbf{C}\left(e^z\right)$-regular between $y = \pm K e^{-x-2\epsilon e^x}$.
  - Let P be a differential polynomial in w. Using the equation we may assume that P is a polynomial and by going to the algebraic closure of $\mathbf{C}\left(e^z\right)$ we may assume further that $u = P(w) = w - s$.
  - $\dfrac{u}{f-s} = 1 + \dfrac{f\alpha}{f-s} = 1 + \dfrac{\alpha}{1 - s/f}$
  - May assume that $s \sim f = e^z$
  - $u' + 2su + u^2 - e^{2z} + s^2 + s' = 0$
  - Puiseux expansion: $s \sim \sum\limits_{k=0}^{\infty} a_k e^{r_k z}, \quad r_k \in \mathbf{Q}$
  - $u' + 2su + u^2 + g = 0$, where $g \asymp e^{rz}$, $r \in \mathbf{Q}$, $r < 1$.
  - Apply a procedure similar to the one for α.
  - After many estimates $u \asymp e^{(r-1)x}$

Limits

- If H is a field of holomorphic germs in a sector and $f \in H$ has a limit ($\in \mathbf{C} \cup \{\infty\}$) on a ray, then the limit is uniform in any proper subsector.

  If not, by Lindelöf-Montel theorem a value ω is attained by f infinitely often, so $f - \omega$ is not invertible.
  - $-\log z$ takes the sector $\rho < r, -a < \theta < a$ to a horizontal strip $x > -\log r, -a < y < a$
  - $\log z$ takes this to a neighborhood of ∞ between $y = \pm \tan^{-1}\left(a/\sqrt{e^{2x} - a^2}\right) \sim \pm a e^{-x}$

❑ The complex Hardy field $\mathbf{C}\,(e^z)$ has a canonical valuation.

Restricted to the real axis $p(e^z)/q(e^z)$ has a limit, since
$\mathrm{Re}\ (p(e^z)/q(e^z)) = \mathrm{Re}\,(p(e^x)\overline{q(e^x)})/\left|q(e^x)\right|^2$,
$\mathrm{Im}\ (p(e^z)/q(e^z)) = \mathrm{Im}\,(p(e^x)\overline{q(e^x)})/\left|q(e^x)\right|^2$
are $\mathbf{R}$-regular.

❑ If w is solution of $w' + w^2 = e^{2z}$ which is real on the positive real axis, then the complex Hardy field $\mathbf{C}\,(e^z, w)$ has a canonical valuation.

l'Hôpital's rule

❑ Rosenlicht's criterion for the differentiability of the valuation:

For each $f \prec 1\ \exists M$ such that for all sufficiently large ρ_1, ρ_2
$\int_{\rho_1}^{\rho_2} \left|f_\rho(\rho e^{\mathbf{i}\theta}\right|\, d\rho \leq M \left|f(\rho_1 e^{\mathbf{i}\theta}) - f(\rho_1 e^{\mathbf{i}\theta})\right|$

➪ $z^p e^{-az^q}$, where $p \geq 0, q > 0, a > 0$

➪ $Ai(z)$ — a solution of Airy's equation $y'' = zy$

Questions

❑ Higher order equations

❑ Difference equations

❑ Fractional composition

❑ Regularity of complex functions

❑ Existence and differentiability of the valuation

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