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BOUNDARY ELEMENT METHOD FOR INTERNAL AXISYMMETRIC FLOW

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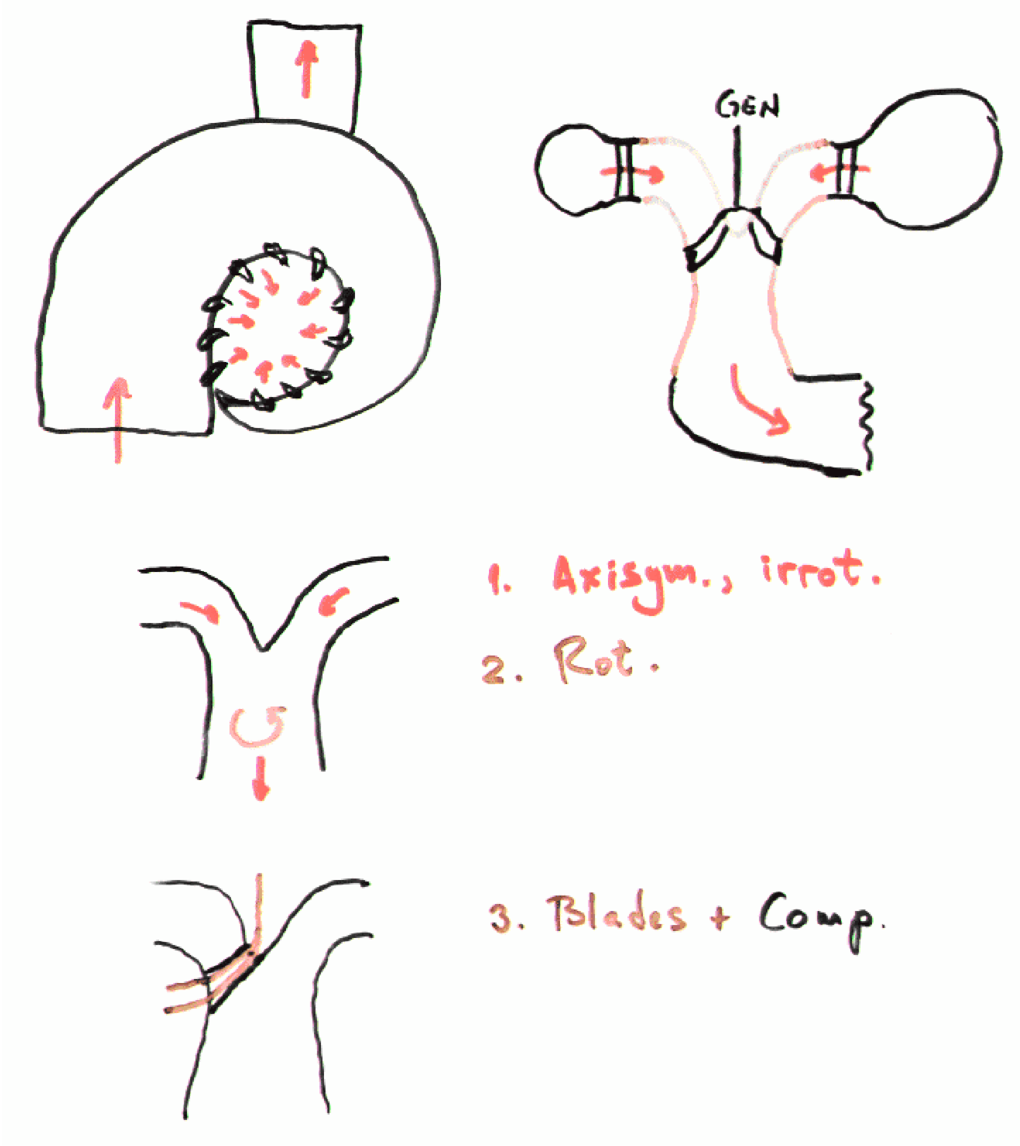
Superposition of flows in a hydroturbine



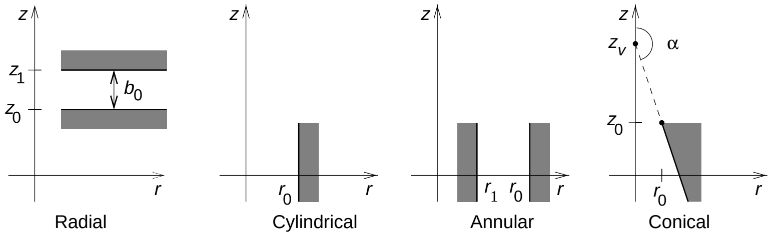
Formulation of the Problem

Hypotheses for the flow v:

(i) The flow is incompressible, i.e. $\nabla \cdot \overline{v} = 0$;
(ii) The flow is potential, i.e. there exists a scalar potential φ with $\overline{v} = \nabla\varphi$;
(iii) The flow is axisymmetric, i.e. $\overline{v} = \overline{v}(r, z)$;
(iv) The flow is irrotational, i.e. the circumferential component of velocity $v_\theta = 0$.

Hypotheses for the geometry of the passage:

(v) The passage is a domain of revolution in $\mathbf{R}^3$ generated by a simply connected domain $\Omega \subseteq \{(r, z) \in \mathbf{R}^2 \colon r \geq 0\}$;
(vi) The boundary of the passage is the union of two disjoint C^1 surfaces of revolution. The inner surface represents the crown and the outer surface represents the bend.
(vii) The inlet and the exit belong to the following types:



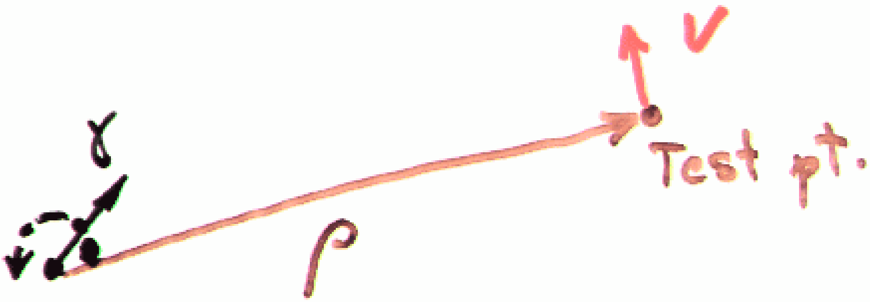
Governing equation $\nabla^2\varphi = 0$:

$$\frac{\partial^2\varphi}{\partial z^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial r^2} = 0, \qquad v_r = \frac{\partial\varphi}{\partial r}, \quad v_z = \frac{\partial\varphi}{\partial z}, \quad v_\theta = 0$$

Boundary conditions:

(i) At the wall we have the normal component of velocity $v_n = \overline{v} \cdot \widehat{n} = 0$.
(ii) At infinity we impose the following asymptotic conditions:
  (a) Radial with distance between the planes b_0: $v_z \prec v_r \sim \pm\dfrac{q}{2\pi r b_0}$ as $r \to \infty$ between the planes forming the passage;
  (b) Cylinder with radius r_0: $v_r \to 0$, $v_z \to \pm\dfrac{q}{\pi r_0^2}$ as $z \to \pm\infty$ inside the cylinder;
  (c) Annular cylinder with radii $0 < r_1 < r_0$: $v_r \to 0$, $v_z \to \pm\dfrac{q}{\pi\left(r_0^2 - r_1^2\right)}$ as $z \to \pm\infty$ inside the annulus;
  (d) Cone with vertex $(0, z_v)$ and generatrix $r = (z - z_v)\tan\alpha$: $v_\rho \sim \pm\dfrac{q}{\pi \sin^2\alpha\, \rho^2}$, as $\rho \to \infty$, where $\rho = |(r, z - z_v)|$ inside the cone.

Biot-Savart



$$V = \frac{1}{4\pi}\frac{\gamma \times \rho}{|\rho|^3}$$

$$R = |\rho|^2$$

Theory

Theorem A. The induced velocity at a test point $\overline{x}_c = (0, r_c, z_c)$ due to axisymmetric distributed surface vorticity $\gamma(\ell)$ on the bounding surfaces of revolution is

$$\begin{aligned} v_r &= \frac{1}{4\pi}\int_{\mathscr{L}} \gamma(\ell) r(\ell)(z_c - z(\ell)) \int_0^{2\pi} \frac{\sin\theta}{R(r(\ell), z(\ell), \theta)^{\frac{3}{2}}}\, d\theta\, d\ell, \\ v_z &= \frac{1}{4\pi}\int_{\mathscr{L}} \gamma(\ell) r(\ell) \int_0^{2\pi} \frac{r(\ell) - r_c \sin\theta}{R(r(\ell), z(\ell), \theta)^{\frac{3}{2}}}\, d\theta\, d\ell, \\ v_\theta &= 0, \end{aligned}$$

where

$$R(r, z, \theta) = r^2 + r_c^2 - 2 r_c r \sin\theta + (z_c - z)^2,$$

and $\mathscr{L}$ denotes the union of curves that generate the boundary and ℓ is arclength along these curves.

Theorem B. The boundary condition $v_n = 0$ is equivalent to $v_\tau = \gamma$, where v_τ is the tangential component of velocity inside the wall. (see next page)

Theorem C. Axisymmetric distributed surface vorticity $\gamma(\ell)$ on the bounding surfaces of revolution satisfies a Fredholm integral equation of the second kind

$$\frac{\gamma(\ell_c)}{2} = \frac{1}{4\pi}\int_{\mathscr{L}} \gamma(\ell) r(\ell) \int_0^{2\pi} \frac{r'(\ell_c)(z(\ell_c) - z(\ell))\sin\theta + z'(\ell_c)(r(\ell) - r(\ell_c)\sin\theta)}{(r(\ell)^2 + r(\ell_c)^2 - 2r(\ell_c)r(\ell)\sin\theta + (z(\ell_c) - z(\ell))^2)^{\frac{3}{2}}}\, d\theta\, d\ell,$$

where prime denotes differentiation with respect to arclength ℓ.

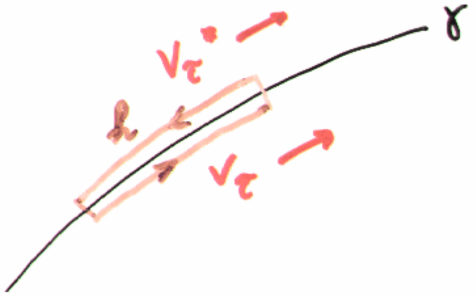
Theorem D. Let q be constant. If γ satisfies the following *a priori* conditions for the different cases of inlet/exit:

(a) $\gamma = \pm\frac{q}{2\pi b_0 r}$ on two horizontal planes, a distance b_0 apart with opposite signs on the two planes,
(b) $\gamma = \frac{q}{\pi r_0^2}$ on a vertical cylinder with radius r_0,
(c) $\gamma = \pm\frac{q}{\pi\,(r_0^2 - r_1^2)}$ on a vertical annular cylinder with radii $0 < r_1 < r_0$, with opposite signs at r_1 and r_0,
(d) $\gamma = \pm\frac{q}{\pi \sin^2\alpha\, \rho^2}$, where $\rho = |(r, z - z_v)|$, on a cone with generatrix $r = (z - z_v)\,\tan\alpha$,

then the induced velocity field has flux that is asymptotic to q and asymptotically satisfies the boundary conditions at infinity (inlet/exit).

Theorem E. At the intersection of the inner bounding surface with the z axis $\gamma = 0$.

Proof of Theorem B



Stokes $\Rightarrow$ $v_\tau h - v_\tau^* h = \gamma h = \gamma h$

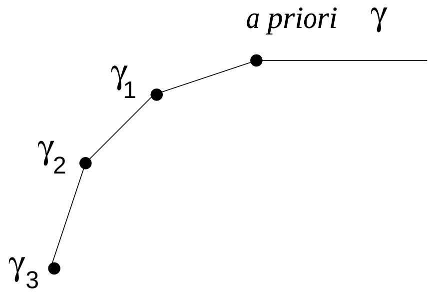
$$\Big\downarrow \lim_{h \to 0}$$

$$v_\tau - v_\tau^* = \gamma$$

Computation

Geometry + flow rate
⇓ Representation & integration
Coefficients
⇓ Linear solve (direct)
γ on the boundary

Test points: Located at the vertices of the boundary elements.



Representation and integration:

Type of boundary element	Geometrical representation of the generatrix	Representation of vortex density γ	Integration
General	linear spline	linear spline	r: "closed form" θ: numerical
Incident	cubic spline	linear spline	Split the integrand into regular and principal parts. Principal: r: "closed form" Regular: numerical
Semi-infinite	linear spline	*a priori* values	r: "closed form" θ: numerical

Linear system: We solve a linear system of equations for the unknown nodal values γ_i.

Boundary elements not incident to the test point

Geometry: $r(\ell) = a_r + b_r\ell$, $z(\ell) = a_z + b_z\ell$

where $a_r = r_i - m_{ri}\ell_i$, $b_r = m_{ri}$, $a_z = z_i - m_{zi}\ell_i$, $b_z = m_{zi}$,
$r(\ell_i) = r_i, z(\ell_i) = z_i$ is the starting point and $m_{ri}^2 + m_{zi}^2 = 1$.

Vorticity: $\gamma(\ell) = A + B\ell$,

where $A = \gamma_i - \dfrac{\ell_i(\gamma_{i+1} - \gamma_i)}{\ell_{i+1} - \ell_i}$, $B = \dfrac{\gamma_{i+1} - \gamma_i}{\ell_{i+1} - \ell_i}$,
$t = (\ell - \ell_i)/(\ell_{i+1} - \ell_i)$ and γ_i are the nodal values of γ.

Integration:

$$\begin{aligned}
u_r &= \frac{1}{4\pi}\int_0^{2\pi}\int_{\ell_i}^{\ell_{i+1}} \frac{\overline{a}_r + \overline{b}_r\ell + \overline{c}_r\ell^2 + \overline{d}_r\ell^3}{(a + b\ell + \ell^2)^{\frac{3}{2}}}\, d\ell\, d\theta = \frac{1}{4\pi}\int_0^{2\pi} \left(\overline{a}_r T_0^* + \overline{b}_r T_1^* + \overline{c}_r T_2^* + \overline{d}_r T_3^*\right) d\theta \\
u_z &= \frac{1}{4\pi}\int_0^{2\pi}\int_{\ell_i}^{\ell_{i+1}} \frac{\overline{a}_z + \overline{b}_z\ell + \overline{c}_z\ell^2 + \overline{d}_z\ell^3}{(a + b\ell + \ell^2)^{\frac{3}{2}}}\, d\ell\, d\theta = \frac{1}{4\pi}\int_0^{2\pi} \left(\overline{a}_z T_0^* + \overline{b}_z T_1^* + \overline{c}_z T_2^* + \overline{d}_z T_3^*\right) d\theta
\end{aligned}$$

where

$$\begin{aligned}
T_n^* &= T_n(\ell_{i+1}) - T_n(\ell_i) \\
T_0 &= \frac{2\,(2\ell + b)}{(4a - b^2)\,(\ell^2 + b\ell + a)^{\frac{1}{2}}} \\
T_1 &= -\frac{2\,(2a + b\ell)}{(4a - b^2)\,(\ell^2 + b\ell + a)^{\frac{1}{2}}} \\
T_2 &= -2\frac{(3a - b^2)\ell - ab}{(4a - b^2)\,(4a - b^2)\,(\ell^2 + b\ell + a)^{\frac{1}{2}}} + \ln\left(2\left(\ell^2 + b\ell + a\right)^{\frac{1}{2}} + 2\ell + b\right) \\
T_3 &= \frac{(4a - b^2)\ell^2 + b(10a - 3b^2)\ell + a(8a - 3b^2)}{(4a - b^2)\,(\ell^2 + b\ell + a)^{\frac{1}{2}}} - \frac{3b}{2}\ln\left(2\left(\ell^2 + b\ell + a\right)^{\frac{1}{2}} + 2\ell + b\right)
\end{aligned}$$

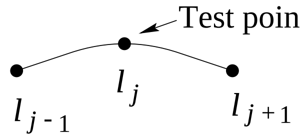
$$\begin{aligned}
a &= a_r^2 + r_c^2 - 2r_c a_r \sin\theta + (z_c - a_z)^2 \\
b &= 2(a_r b_r - r_c b_r \sin\theta - (z_c - a_z) b_z)
\end{aligned}$$

$$\begin{aligned}
\overline{a}_r &= A a_r (z_c - a_z) \sin\theta \\
\overline{b}_r &= (B a_r (z_c - a_z) + A b_r (z_c - a_z) - A a_r b_z) \sin\theta \\
\overline{c}_r &= (B b_r (z_c - a_z) - B a_r b_z - A b_r b_z) \sin\theta \\
\overline{d}_r &= -B b_r b_z \sin\theta
\end{aligned}$$

$$\begin{aligned}
\overline{a}_z &= A a_r (a_r - r_c \sin\theta) \\
\overline{b}_z &= B a_r (a_r - r_c \sin\theta) + A b_r (a_r - r_c \sin\theta) + A a_r b_r \\
\overline{c}_z &= B b_r (a_r - r_c \sin\theta) + B a_r b_r + A b_r^2 \\
\overline{d}_z &= B b_r^2
\end{aligned}$$

Note that the coefficient of ℓ^2 in the denominator of the integrands is $b_r^2 + b_z^2 = 1$.

Integrals over elements incident to the test point



Representation: $t = \dfrac{\ell - \ell_{j-1}}{\Delta\ell_{j-1}}$ for $\ell_{j-1} \leq \ell \leq \ell_j$, $\Delta\ell_{j-1} = \ell_j - \ell_{j-1}$, and $j = i, i+1$

$$\begin{aligned} r(t) &= r_{j-1}H_{01}(t) + r_j H_{02}(t) + \Delta\ell_{j-1}\left(r'_{j-1}H_{11}(t) + r'_j H_{12}(t)\right), \\ z(t) &= z_{j-1}H_{01}(t) + z_j H_{02}(t) + \Delta\ell_{j-1}\left(z'_{j-1}H_{11}(t) + z'_j H_{12}(t)\right), \\ \gamma(t) &= \gamma_{j-1}(1-t) + \gamma_j t, \end{aligned}$$

$$\begin{array}{ll} H_{01}(x) = 1 - 3x^2 + 2x^3, & H_{02}(x) = 3x^2 - 2x^3, \\ H_{11}(x) = x - 2x^2 + x^3, & H_{12}(x) = -x^2 + x^3, \end{array}$$

Integral: $s = \ell - \ell_i$

$$u_\tau = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\Delta\ell_{i-1}}^{\Delta\ell_i} \frac{r'_i\gamma r(z_i - z)\sin\theta + z'_i\gamma r\,(r - r_i\sin\theta)}{(r^2 + r_i^2 - 2r_i r\sin\theta + (z_i - z)^2)^{\frac{3}{2}}}\,ds\,d\theta$$

Taylor expansions: $\delta = \frac{\pi}{2} - \theta$

$$\begin{array}{ll} \sin\theta = 1 - \dfrac{1}{2}\delta^2 + \mathcal{O}\left(\delta^4\right), & r = r_0 + r'_0 s + \dfrac{1}{2}r''_0 s^2 + \dfrac{1}{6}r'''_{0\pm}s^3, \\ \gamma = \gamma_0 + \gamma'_{0\pm}s, & z = z_0 + z'_0 s + \dfrac{1}{2}z''_0 s^2 + \dfrac{1}{6}z'''_{0\pm}s^3. \end{array}$$

Theorem F. Let I denote the integrand. Then with the above notation

$$I = \frac{r'_0\gamma r(z_0 - z)\sin\theta + z'_0\gamma r\,(r - r_0\sin\theta)}{(r^2 + r_0^2 - 2r_0 r\sin\theta + (z_0 - z)^2)^{\frac{3}{2}}} = \frac{\gamma_0 r_0 A(\varphi)}{\rho} + T(\varphi) + \mathcal{O}(\rho),$$

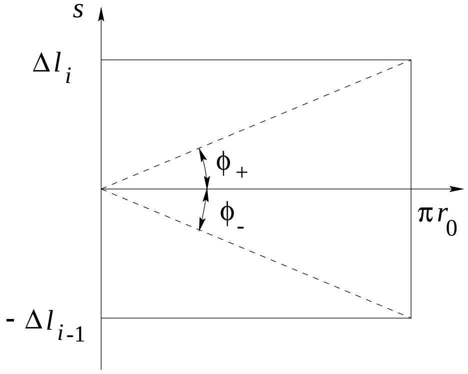
where $\rho\cos\varphi = r_0\delta$, $\rho\sin\varphi = s$, and

$$\begin{aligned} A(\varphi) &= \frac{1}{2}\left(z'_0 r''_0 - r'_0 z''_0\right)\sin^2\varphi + \frac{z'_0}{2r_0}\cos^2\varphi, \\ T(\varphi) &= \sin\varphi\left(\gamma'_{0\pm}r_0 + r'_0\gamma_0\right)A(\varphi) + \gamma_0 r_0\left[\frac{r'_0 z'_0}{2r_0^2}\cos^2\varphi\sin\varphi + \frac{1}{6}\left(z'_0 r'''_{0\pm} - r'_0 z'''_{0\pm}\right)\sin^3\varphi \right. \\ &\left. -\frac{3}{2}\left(\frac{r'_0}{r_0}\cos^2\varphi\sin\varphi + \frac{1}{2}\left(r'_0 r''_0 + z'_0 z''_0\right)\sin^3\varphi\right)A(\varphi)\right], \end{aligned}$$

Decomposition of the integrand into regular and principal parts:

$$I = \frac{\gamma_0 r_0 A(\varphi)}{\rho} + \left(I - \frac{\gamma_0 r_0 A(\varphi)}{\rho}\right)$$

$\lim_{\rho\to 0}\left(I - \frac{\gamma_0 r_0 A(\varphi)}{\rho}\right) = T(\varphi) \Rightarrow$ regular part → proper integral ⇒ numerical evaluation



Theorem G. Let P denote the contribution to u_τ of the principal part of I.

(a) If ℓ_i is an interior point of a cubic spline representing the boundary generatrix, then

$$P = \frac{\gamma_0 r_0}{2\pi}\left(-\Delta\ell_{i-1}\, J_1\Big|_{-\frac{\pi}{2}}^{-\varphi_-} + \pi r_0\, J_2\Big|_{-\varphi_-}^{\varphi_+} + \Delta\ell_i\, J_1\Big|_{\varphi_+}^{\frac{\pi}{2}}\right),$$

where $\varphi_- = \operatorname{atan2}(\pi r_0, \Delta\ell_{i-1})$, $\varphi_+ = \operatorname{atan2}(\pi r_0, \Delta\ell_i)$, and

$$J_1(\varphi) = -\frac{1}{2}\left(z_0' r_0'' - r_0' z_0''\right)\cos\varphi + \frac{z_0'}{2r_0}\left(\frac{1}{2}\log\left(\frac{1-\cos\varphi}{1+\cos\varphi}\right) + \cos\varphi\right),$$

$$J_2(\varphi) = \frac{1}{2}\left(z_0' r_0'' - r_0' z_0''\right)\left(\frac{1}{2}\log\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right) - \sin\varphi\right) + \frac{z_0'}{2r_0}\sin\varphi.$$

(b) If ℓ_i is the juncture of arcs with different radii or an arc and a straight line segment, then

$$P = \frac{\gamma_0 r_0}{2\pi}\left(-\Delta\ell_{i-1}\, J_1\Big|_{-\frac{\pi}{2}}^{-\varphi_-} + \pi r_0\, J_{2-}\Big|_{-\varphi_-}^{0} + \pi r_0\, J_{2+}\Big|_{0}^{\varphi_+} + \Delta\ell_i\, J_1\Big|_{\varphi_+}^{\frac{\pi}{2}}\right),$$

where $J_{2\pm}$ are constructed the same way as J_2, but using left or right second derivatives $r_{0\pm}''$ and $z_{0\pm}''$.

Improper integrals at the inlet and exit

Test point: (r_c, z_c).

Radial: (level z_0)

$$v_r = \frac{\gamma_0 h_c}{4\pi} \int_0^{2\pi} \frac{\sin\theta}{h_c^2 + r_c^2 \cos^2\theta} \left(1 - \frac{\ell_0 - r_c \sin\theta}{R(\ell_0, z_0, \theta)^{\frac{1}{2}}}\right) d\theta, \quad v_z = \frac{\gamma_0}{4\pi} \int_0^{2\pi} \frac{d\theta}{R(\ell_0, z_0, \theta)^{\frac{1}{2}}},$$

where $h_c = z_c - z_0$.

Cylindrical: (radius r_0)

$$\begin{aligned} v_r &= -\frac{\gamma_0 r_0}{4\pi} \int_0^{2\pi} \frac{\sin\theta \, d\theta}{R(r_0, \pm\ell_0, \theta)^{\frac{1}{2}}}, \\ v_z &= \frac{\gamma_0 r_0}{4\pi} \int_0^{2\pi} \frac{r_0 - r_c \sin\theta}{r_c^2 - 2 r_c r_0 \sin\theta + r_0^2} \left(\pm 1 - \frac{\pm\ell_0 - z_c}{R(r_0, \pm\ell_0, \theta)^{\frac{1}{2}}}\right) d\theta, \end{aligned}$$

Conical: (vertex $(0, z_v)$, angle α)

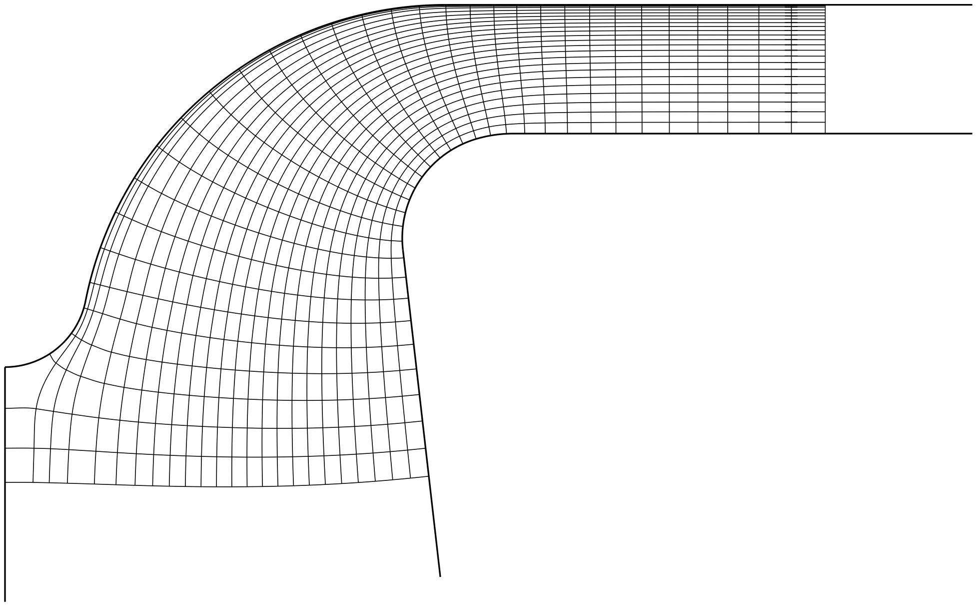
$$\begin{aligned} v_r &= \frac{\gamma_0 \sin\alpha}{4\pi s_c^2} \int_0^{2\pi} \left[\frac{1}{1-\sigma^2} \left(\tau_r - \frac{\tau_r \ell_0 + s_c \mu_r}{R_0^{\frac{1}{2}}}\right) + \lambda \cos\beta\right] \sin\theta \, d\theta, \\ v_z &= \frac{\gamma_0 \sin\alpha}{4\pi s_c^2} \int_0^{2\pi} \left[\frac{1}{1-\sigma^2} \left(\tau_z - \frac{\tau_z \ell_0 - s_c \mu_z}{R_0^{\frac{1}{2}}}\right) - \lambda \sin\beta \sin\theta\right] d\theta, \end{aligned}$$

where the distance and angle from the cone vertex to the test point are (s_c, β),

$$\sigma = \sin\alpha \sin\beta \sin\theta + \cos\alpha \cos\beta,$$

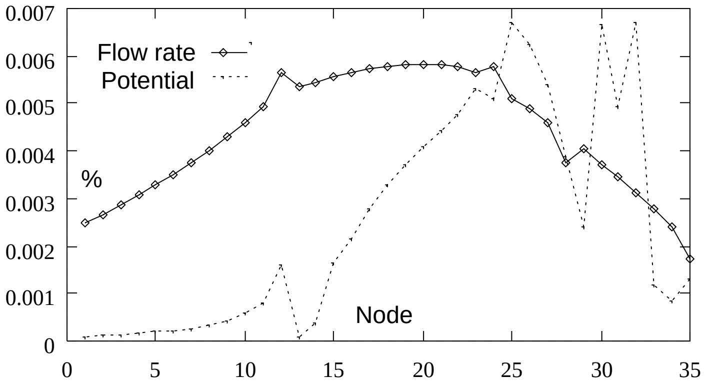
$$\begin{aligned} R_0 &= R(\ell_0 \sin\alpha, z_v + \ell_0 \cos\alpha, \theta) = s_c^2 - 2 s_c \ell_0 \sigma + \ell_0^2, \\ \lambda &= \log \frac{s_c - \sigma\ell_0 + R_0^{\frac{1}{2}}}{(1-\sigma)\ell_0}, \\ \tau_r &= \sigma \cos\beta - \cos\alpha, \\ \tau_z &= \sin\alpha - \sigma \sin\beta \sin\theta, \\ \mu_r &= \sigma \cos\alpha - \cos\beta (2\sigma^2 - 1), \\ \mu_z &= \sigma \sin\alpha - \sin\beta \sin\theta (2\sigma^2 - 1). \end{aligned}$$

Example: Butt Valley (Pacific Gas & Electric)



Streamlines and equipotentials, 35 nodes per surface (70 × 70 linear system)

Internal verification: Relative errors in flow rate and potential.



Speed: Not optimized FORTRAN 77 code.

CPU	OS	Compiler	Time
300 MHz Pentium II	Linux 2.0.30	g77 0.5.20	130.820u 0.030s 2:11.71 99.3%
296 MHz UltraSPARC-II	SunOS 5.6	g77 0.5.19	122.28u 0.59s 2:03.01 99.8%