A new approach to turbulent internal flows with high Reynolds numbers

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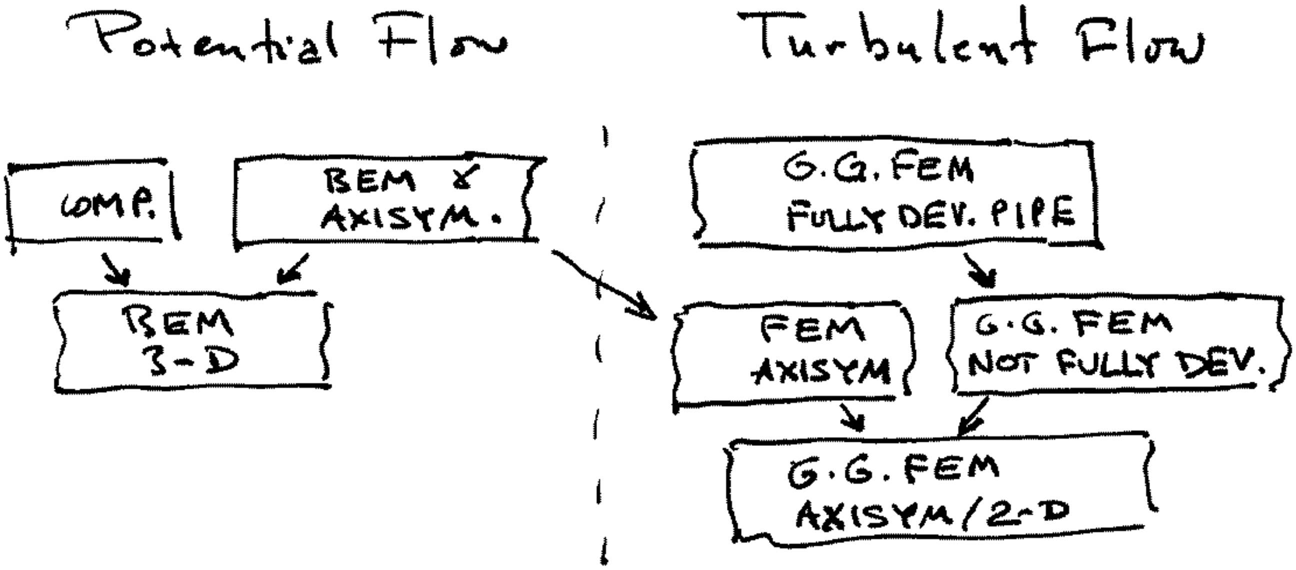
The authors have developed a method for fast and extremely accurate computation of axisymmetric or two-dimensional internal turbulent flows with Reynolds number between 10^7 and 10^8. The main ingredients are a modification of the Galerkin method based on splining (to overcome difficulties of the standard Galerkin method at the boundary) and a curvilinear orthogonal coordinate system following the boundary and approximately following the flow lines. I will describe the two techniques and present some applications.



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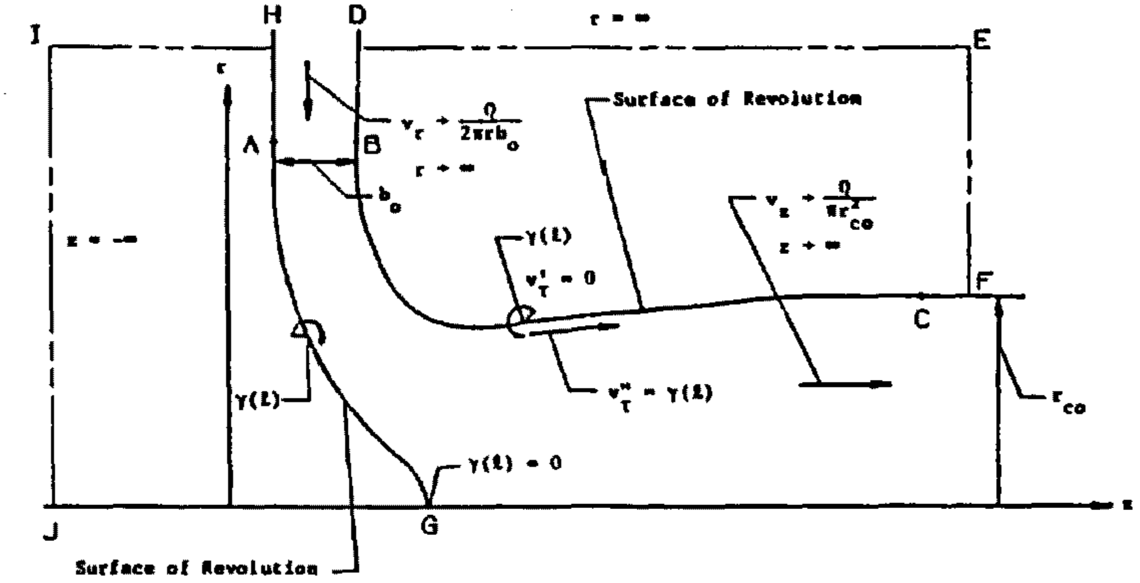
Boundary Element Method for Axisymmetric Flow



Governing equation: $\nabla^2\varphi = 0$

$$\frac{\partial^2\varphi}{\partial z^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{\partial^2\varphi}{\partial r^2}=0, \qquad v_r=\frac{\partial\varphi}{\partial r}, \quad v_z=\frac{\partial\varphi}{\partial z}, \quad v_\theta = 0$$

Boundary conditions:

(a) Radial with distance between the planes b_0: $v_z \prec v_r \sim \pm\dfrac{Q}{2\pi r b_0}$ as $r\to\infty$

(b) Cylinder with radius r_0: $v_r \to 0$, $v_z \to \pm\dfrac{Q}{\pi r_0^2}$ as $z \to \pm\infty$,

(c) Cone with vertex $(0, z_0)$ and generatrix $r = \tan\alpha\,(z - z_0)$:
$\mathbf{v}(r,z) \sim \pm\dfrac{Q}{2\pi(1-\cos\alpha)\,|\mathbf{n}|^3}\,\mathbf{n}$, as $|\mathbf{n}|\to\infty$, where $\mathbf{n} = (r, z - z_0)$.

$$\frac{\gamma}{2}=\frac{dr}{d\ell}u_r+\frac{dz}{d\ell}u_z \qquad \begin{aligned} u_r &= \frac{1}{4\pi}\int_{\mathcal{L}}\gamma(\ell)r(\ell)(z_c - z(\ell))\int_0^{2\pi}\frac{\sin\theta}{(r(\ell)^2+r_c^2-2r_c r(\ell)\sin\theta+(z_c-z(\ell))^2)^{\frac{3}{2}}}\,d\theta\,d\ell \\ u_z &= \frac{1}{4\pi}\int_{\mathcal{L}}\gamma(\ell)r(\ell)\int_0^{2\pi}\frac{r(\ell)-r_c\sin\theta}{(r(\ell)^2+r_c^2-2r_c r(\ell)\sin\theta+(z_c-z(\ell))^2)^{\frac{3}{2}}}\,d\theta\,d\ell \end{aligned}$$

(a) Radial: $\gamma \sim \pm\dfrac{Q}{2\pi b_0 r}$ as $r\to\infty$ with opposite signs on the two planes.

(b) Cylinder: $\gamma \to \pm\dfrac{Q}{\pi r_0^2}$ as $z\to\pm\infty$

(c) Cone: $\gamma \sim \pm\dfrac{Q}{2\pi(1-\cos\alpha)\,|\mathbf{n}|^2}$ as $|\mathbf{n}|\to\infty$, where $\mathbf{n} = (r, z - z_0)$

We can compute the integrals with respect to $d\ell$ exactly using the following table integrals

$$
\begin{aligned}
T_n &= \int \frac{\ell^n \, d\ell}{\left(\ell^2 + b\ell + a\right)^{\frac{3}{2}}} \qquad [n = 0, 1, -1] \\
T_0 &= \frac{2\,(2\ell + b)}{(4a - b^2)\,(\ell^2 + b\ell + a)^{\frac{1}{2}}} \qquad T_1 = -\frac{2\,(2a + b\ell)}{(4a - b^2)\,(\ell^2 + b\ell + a)^{\frac{1}{2}}} \\
T_2 &= -2\frac{(3a - b^2)\ell - ab}{(4a - b^2)\,(4a - b^2)\,(\ell^2 + b\ell + a)^{\frac{1}{2}}} + \ln\left(2\left(\ell^2 + b\ell + a\right)^{\frac{1}{2}} + 2\ell + b\right) \\
T_3 &= \frac{(4a - b^2)\ell^2 + b(10a - 3b^2)\ell + a(8a - 3b^2)}{(4a - b^2)\,(\ell^2 + b\ell + a)^{\frac{1}{2}}} - \frac{3b}{2}\ln\left(2\left(\ell^2 + b\ell + a\right)^{\frac{1}{2}} + 2\ell + b\right) \\
T_{-1} &= -\frac{2\,(b\ell - 2a + b^2)}{a\,(4a - b^2)\,(\ell^2 + b\ell + a)^{\frac{1}{2}}} - \frac{1}{a^{\frac{3}{2}}}\ln\left(\frac{2a + b\ell + 2a^{\frac{1}{2}}\,(\ell^2 + b\ell + a)^{\frac{1}{2}}}{\ell}\right)
\end{aligned}
$$

and their definite forms

$$
\begin{aligned}
T_n^* &= \int_{\ell_0}^{\infty} \frac{\ell^n \, d\ell}{\left(\ell^2 + b\ell + a\right)^{\frac{3}{2}}} = \lim_{\ell \to \infty} T_n(\ell) - T_n(\ell_0) \qquad [n = 0, 1, -1] \\
T_0^* &= \frac{2}{4a - b^2}\left(2 - \frac{2\ell_0 + b}{(\ell_0^2 + b\ell_0 + a)^{\frac{1}{2}}}\right) \qquad T_1^* = \frac{2}{4a - b^2}\left(\frac{2a + b\ell_0}{(\ell_0^2 + b\ell_0 + a)^{\frac{1}{2}}} - b\right) \\
T_{-1}^* &= \frac{2}{a\,(4a - b^2)}\left(\frac{b\ell_0 - 2a + b^2}{(\ell_0^2 + b\ell_0 + a)^{\frac{1}{2}}} - b\right) + \frac{1}{a^{\frac{3}{2}}}\ln\left(\frac{2a + b\ell_0 + 2a^{\frac{1}{2}}\,(\ell_0^2 + b\ell_0 + a)^{\frac{1}{2}}}{\left(b + 2a^{\frac{1}{2}}\right)\ell_0}\right)
\end{aligned}
$$

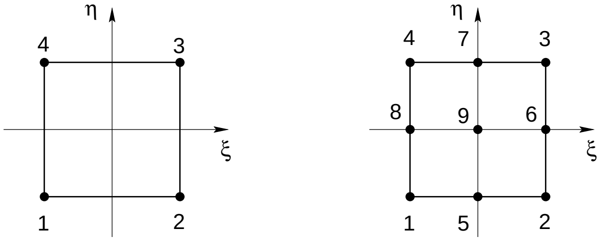
Radial: $r(\ell) = \ell$, $\ell > \ell_0$, $z = z_0$

$$
a = r_c^2 + (z_c - z_0)^2, \qquad b = -2r_c \sin\theta
$$

$$
\begin{aligned}
u_r &= \pm\frac{Q}{8\pi^2 b_0}\int_0^{2\pi}\!\!\int_{\ell_0}^{\infty} \frac{(z_c - z_0)\sin\theta}{(\ell^2 + r_c^2 - 2r_c\ell\sin\theta + (z_c - z_0)^2)^{\frac{3}{2}}}\,d\ell\,d\theta \\
&= \pm\frac{Q}{8\pi^2 b_0}\int_0^{2\pi} (z_c - z_0)\sin\theta\, T_0^*\,d\theta \\
&= \pm\frac{Q}{8\pi^2 b_0}\int_0^{2\pi} \frac{(z_c - z_0)\sin\theta\,d\theta}{(z_c - z_0)^2 + r_c^2\cos^2\theta}\left(1 - \frac{\ell_0 - r_c\sin\theta}{(\ell_0^2 + r_c^2 - 2r_c\ell_0\sin\theta + (z_c - z_0)^2)^{\frac{1}{2}}}\right) \\
u_z &= \pm\frac{Q}{8\pi^2 b_0}\int_0^{2\pi}\!\!\int_{\ell_0}^{\infty} \frac{\ell - r_c\sin\theta}{(\ell^2 + r_c^2 - 2r_c\ell\sin\theta + (z_c - z_0)^2)^{\frac{3}{2}}}\,d\ell\,d\theta \\
&= \pm\int_0^{2\pi} (T_1^* - r_c\sin\theta\, T_0^*)\,d\theta \\
&= \pm\int_0^{2\pi} \frac{d\theta}{(\ell_0^2 + r_c^2 - 2r_c\ell_0\sin\theta + (z_c - z_0)^2)^{\frac{1}{2}}}
\end{aligned}
$$

Finite elements with closed form Galerkin integrals

To implement the finite element scheme we will use the following master elements in dimensionless variables ξ and η: 4-node element with bilinear tensor Lagrangian interpolating shape functions for pressure and 9-node element for biquadratic tensor Lagrange interpolating shape functions for velocity as illustrated in the following figure and tables.



Master elements in dimensionless variables ξ and η

Deg.	Nodes	Polynomials		
n	$\{t_k\colon 1 \le k \le n\}$	$\ell_{ni} = \prod_{1\le k\le n,\, k\ne i} \frac{x - t_k}{t_i - t_k}$		
1	$\{-1, 1\}$	$\ell_{11}(x) = \frac{1}{2}(1 - x)$	$\ell_{12}(x) = \frac{1}{2}(1 + x)$	
2	$\{-1, 0, 1\}$	$\ell_{21}(x) = \frac{1}{2}x(x - 1)$	$\ell_{22}(x) = 1 - x^2$	$\ell_{23}(x) = \frac{1}{2}x(x + 1)$

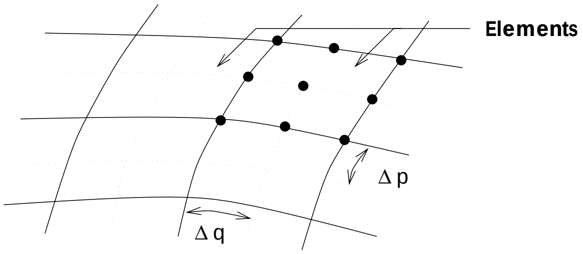
Lagrange interpolating polynomials

$$
\begin{aligned}
\widehat{\varphi}_1(\xi,\eta) &= \ell_{11}(\xi)\ell_{11}(\eta) & \widehat{\varphi}_2(\xi,\eta) &= \ell_{12}(\xi)\ell_{11}(\eta) \\
\widehat{\varphi}_3(\xi,\eta) &= \ell_{12}(\xi)\ell_{12}(\eta) & \widehat{\varphi}_4(\xi,\eta) &= \ell_{11}(\xi)\ell_{12}(\eta)
\end{aligned}
$$

Bilinear tensor Lagrange shape functions for the 4-node element

$$
\begin{aligned}
\widehat{\psi}_1(\xi,\eta) &= \ell_{21}(\xi)\ell_{21}(\eta) & \widehat{\psi}_2(\xi,\eta) &= \ell_{23}(\xi)\ell_{21}(\eta) & \widehat{\psi}_3(\xi,\eta) &= \ell_{23}(\xi)\ell_{23}(\eta) \\
\widehat{\psi}_4(\xi,\eta) &= \ell_{21}(\xi)\ell_{23}(\eta) & \widehat{\psi}_5(\xi,\eta) &= \ell_{22}(\xi)\ell_{21}(\eta) & \widehat{\psi}_6(\xi,\eta) &= \ell_{23}(\xi)\ell_{22}(\eta) \\
\widehat{\psi}_7(\xi,\eta) &= \ell_{22}(\xi)\ell_{23}(\eta) & \widehat{\psi}_8(\xi,\eta) &= \ell_{21}(\xi)\ell_{22}(\eta) & \widehat{\psi}_9(\xi,\eta) &= \ell_{22}(\xi)\ell_{22}(\eta)
\end{aligned}
$$

Biquadratic tensor Lagrange shape functions for the 9-node element



Transformed elements

$$
\begin{aligned}
p_i &= p_{i-1} + \Delta p & p_{i+1} &= p_i + \Delta p \\
q_i &= q_{i-1} + \Delta q & q_{i+1} &= q_i + \Delta q \\
\xi &= \frac{p - p_i}{\Delta p} \quad (-1 \le \xi \le 1) & \eta &= \frac{q - q_i}{\Delta q} \quad (-1 \le \eta \le 1)
\end{aligned}
$$

The relationship between p, q and ξ, η

The governing equations for incompressible steady two-dimensional flow are the conservation of mass (continuity) equation

$$\nabla \cdot \mathbf{v} = 0$$

and the balance of momentum (Navier-Stokes) equation

$$\frac{1}{2}\nabla\left(|\mathbf{v}|^2\right) - \mathbf{v} \times (\nabla \times \mathbf{v}) = \mathbf{F} - \frac{1}{\rho}\nabla P + (\nu + \tau)\nabla^2 \mathbf{v},$$

where

ρ	mass density
$\mathbf{v}$	velocity
$\mathbf{F}$	body force density
P	pressure
ν	Newtonian kinematic viscosity
τ	turbulent kinematic viscosity

The governing equations written in curvilinear orthogonal coordinates p and q in the two-dimensional are

$$\frac{\partial\left(h_q v_p\right)}{\partial p} + \frac{\partial\left(h_p v_q\right)}{\partial q} = 0 \tag{1}$$

$$\begin{aligned}\frac{1}{h_p}\left[v_p\frac{\partial v_p}{\partial p} + v_q\frac{\partial v_q}{\partial p}\right] - \frac{v_q}{h_p h_q}\left[\frac{\partial\left(h_q u_q\right)}{\partial p} - \frac{\partial\left(h_p u_p\right)}{\partial q}\right] = F_p - \frac{1}{\rho h_p}\frac{\partial P}{\partial p}\\ + \frac{\nu + \tau}{h_p h_q}\left[\frac{\partial}{\partial p}\left(\frac{h_q}{h_p}\frac{\partial v_p}{\partial p}\right) + \frac{\partial}{\partial q}\left(\frac{h_p}{h_q}\frac{\partial v_p}{\partial q}\right)\right],\end{aligned} \tag{2}$$

$$\begin{aligned}\frac{1}{h_q}\left[v_q\frac{\partial v_q}{\partial q} + v_p\frac{\partial v_p}{\partial q}\right] + \frac{v_p}{h_p h_q}\left[\frac{\partial\left(h_p u_p\right)}{\partial q} - \frac{\partial\left(h_q u_q\right)}{\partial p}\right] = F_q - \frac{1}{\rho h_q}\frac{\partial P}{\partial q}\\ + \frac{\nu + \tau}{h_p h_q}\left[\frac{\partial}{\partial p}\left(\frac{h_q}{h_p}\frac{\partial v_q}{\partial p}\right) + \frac{\partial}{\partial q}\left(\frac{h_p}{h_q}\frac{\partial v_q}{\partial q}\right)\right],\end{aligned} \tag{3}$$

where v_p and v_q are the components of $\mathbf{v}$, F_p and F_q are the components of $\mathbf{F}$, and h_p and h_q are the metrical coefficients (Lamé functions).

$$\widetilde{P} = \sum_{k=1}^{4} \widehat{\varphi}_k P_k \qquad \widetilde{v}_p = \sum_{k=1}^{9} \widehat{\psi}_k v_{pk} \qquad \widetilde{v}_q = \sum_{k=1}^{9} \widehat{\psi}_k v_{qk}$$

Continuity equation residual:

$$R_c(v_{pk}, v_{qk}) = \sum_k \left(\frac{\partial h_q}{\partial p} \psi_k + h_q \frac{\partial \psi_k}{\partial p} \right) v_{pk} + \left(\frac{\partial h_p}{\partial q} \psi_k + h_p \frac{\partial \psi_k}{\partial q} \right) v_{qk}$$

$$\langle R_c, \varphi_j \rangle = \sum_k \left(\left\langle \frac{\partial h_q}{\partial p} \psi_k, \varphi_j \right\rangle + \left\langle h_q \frac{\partial \psi_k}{\partial p}, \varphi_j \right\rangle \right) v_{pk} + \left(\left\langle \frac{\partial h_p}{\partial q} \psi_k, \varphi_j \right\rangle + \left\langle h_p \frac{\partial \psi_k}{\partial q}, \varphi_j \right\rangle \right) v_{qk}$$

where

$$\langle f, g \rangle = \iint f g \, h_p h_q \, dp \, dq$$

Spline, e.g.:

$$\frac{\partial h_q}{\partial p} h_p h_q = s(\lambda, \mu) = \sum_{i=1}^{12} s_i \, \widehat{\omega}_i = s(0,0) \, \widehat{\omega}_1 + s(1,0) \, \widehat{\omega}_2 + s(1,1) \, \widehat{\omega}_3 + s(0,1) \, \widehat{\omega}_4$$

$$+ \frac{\partial s}{\partial p}(0,0) \Delta p \, \widehat{\omega}_5 + \frac{\partial s}{\partial p}(1,0) \Delta p \, \widehat{\omega}_6 + \frac{\partial s}{\partial p}(1,1) \Delta p \, \widehat{\omega}_7 + \frac{\partial s}{\partial p}(0,1) \Delta p \, \widehat{\omega}_8$$

$$+ \frac{\partial s}{\partial q}(0,0) \Delta q \, \widehat{\omega}_9 + \frac{\partial s}{\partial q}(1,0) \Delta q \, \widehat{\omega}_{10} + \frac{\partial s}{\partial q}(1,1) \Delta q \, \widehat{\omega}_{11} + \frac{\partial s}{\partial q}(0,1) \Delta q \, \widehat{\omega}_{12}$$

where

$$\begin{array}{lll}
\widehat{\omega}_1(\lambda, \mu) = H_{01}(\lambda) H_{01}(\mu) & \widehat{\omega}_2(\lambda, \mu) = H_{02}(\lambda) H_{01}(\mu) & \widehat{\omega}_3(\lambda, \mu) = H_{02}(\lambda) H_{02}(\mu) \\
\widehat{\omega}_4(\lambda, \mu) = H_{01}(\lambda) H_{02}(\mu) & \widehat{\omega}_5(\lambda, \mu) = H_{11}(\lambda) H_{01}(\mu) & \widehat{\omega}_6(\lambda, \mu) = H_{12}(\lambda) H_{01}(\mu) \\
\widehat{\omega}_7(\lambda, \mu) = H_{12}(\lambda) H_{02}(\mu) & \widehat{\omega}_8(\lambda, \mu) = H_{11}(\lambda) H_{02}(\mu) & \widehat{\omega}_9(\lambda, \mu) = H_{01}(\lambda) H_{11}(\mu) \\
\widehat{\omega}_{10}(\lambda, \mu) = H_{02}(\lambda) H_{11}(\mu) & \widehat{\omega}_{11}(\lambda, \mu) = H_{02}(\lambda) H_{12}(\mu) & \widehat{\omega}_{12}(\lambda, \mu) = H_{01}(\lambda) H_{12}(\mu)
\end{array}$$

where λ and μ are shifts of ξ and η and H's are Hermite interpolant polynomials:

$$\begin{array}{ll}
H_{01}(x) = 1 - 3x^2 + 2x^3 & H_{02}(x) = 3x^2 - 2x^3 \\
H_{11}(x) = x - 2x^2 + x^3 & H_{12}(x) = -x^2 + x^3
\end{array}$$

Top left: $\lambda = \xi + 1$ and $\mu = \eta$

$$\iint \widehat{\omega}_7 \widehat{\psi}_2 \widehat{\varphi}_1 \, dp \, dq = \frac{\Delta p \Delta q}{16} \int_0^1 \int_{-1}^0 \left(-\xi^6 - 2\xi^5 + 2\xi^3 + \xi^2 \right) \left(2\eta^6 - 7\eta^5 + 8\eta^4 - 3\eta^3 \right) d\xi \, d\eta = -\frac{13 \Delta p \Delta q}{282240}$$

The Galerkin–Gokhman method

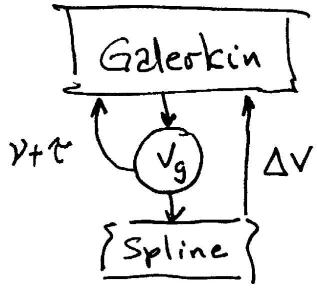
$$v_{pg} = \sum_k \psi_k v_{pk} \qquad v_{qg} = \sum_k \psi_k v_{qk} \qquad \Delta v_p = v_{ps} - v_{pg} \qquad \Delta v_q = v_{qs} - v_{qg}$$

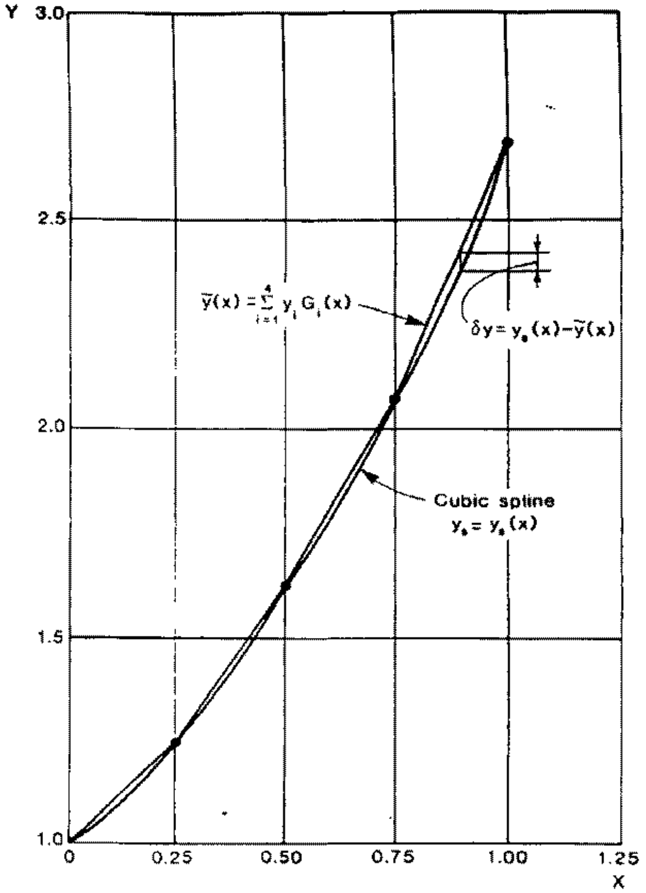
where v_{ps} and v_{qs} are splined.

$\vec{v} + \Delta\vec{v} \longrightarrow$ eqs.

Modified continuity eq:

$$\frac{\partial\left(h_q v_p\right)}{\partial p} + \frac{\partial\left(h_p v_q\right)}{\partial q} = -\frac{\partial\left(h_q \Delta v_p\right)}{\partial p} - \frac{\partial\left(h_p \Delta v_q\right)}{\partial q}$$





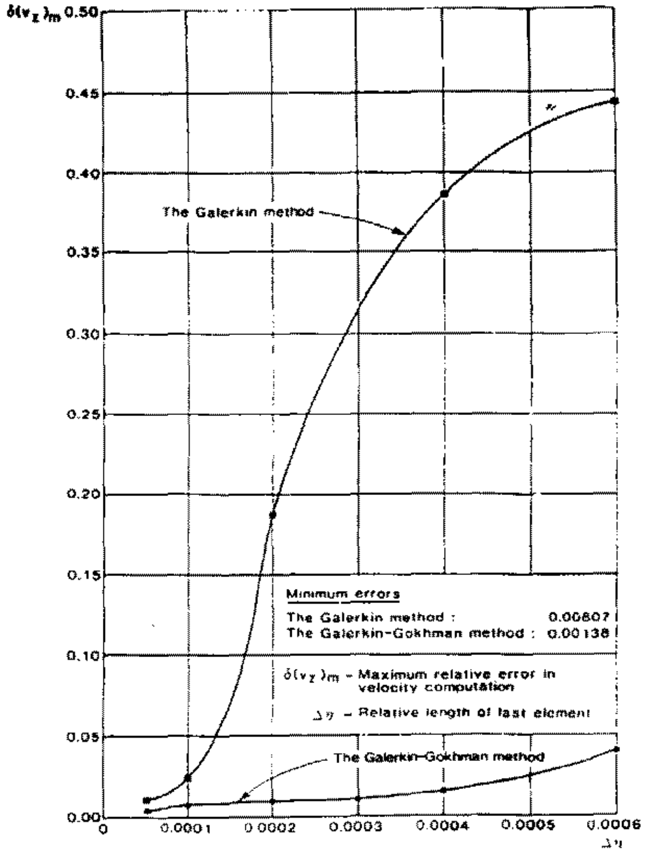


Fig. 2 Parametric study of accuracy of solution for equation describing fully developed flow in a pipe using Galerkin and Galerkin-Gokhman methods at Re = 10[illegible]