Asymptotic solutions of second order linear ODE's in the complex plane.

http://www.math.utsa.edu/~gokhman/l_assol.html

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ABSTRACT

F. Olver posed the problem of systematically finding asymptotic solutions of equations of the form $y'' = f(x)y$, where $f \to \infty$ as $x \to \infty$. This problem was solved by M. Rosenlicht in 1983 in the case when f has "regular growth", i.e. every polynomial function of f and its derivatives is nonoscillatory.

Rosenlicht, in turn, asked: how far into the complex plane can one extend the asymptotic validity of the expansion? In the past, this question has been answered only for f with slow growth. We present an approach to the problem via value distribution theory and obtain a concrete answer for the example $f(z) = \exp(z)$.

The first, and somewhat difficult, step is to prove the regularity of growth of solutions in a region of the complex plane enclosing the positive real axis. Unlike the real case, there does not seem to be a general procedure for doing this.

The second step is an application of conformal mapping and Lindelöf's theorem on value distribution to prove the existence of uniform limits of rational functions of f and its derivatives in proper subsets of the region (we allow ∞ as a possible limit).

The hypotheses of Lindelöf's theorem require the existence of such limits on one ray. The fact that this requirement is satisfied along the positive real axis is shown to follow from the real theory of regular growth.

Together, the above two steps allow a direct application of Rosenlicht's method to prove the validity of the asymptotic expansion in a region of the complex plane containing the positive real axis.

F. Olver's problem

- Find asymptotic solutions for $y'' = f(x)y, \quad \lim_{x \to +\infty} f(x) = +\infty$
- Many special cases — F. Olver (unpublished)
- General solution, if f has *regular growth* —
  M. Rosenlicht, *Hardy fields*, J. Math. Anal. Appl. **93**:297–311 (1983) (Sec. 3)
- What is the domain of validity of the asymptotic solution in $\mathbf{C}$?

Regular growth

- **Def:** $f(x)$ is *non-oscillatory* whenever ultimately (for all sufficiently large x) f has definite sign ($f \equiv 0$ is allowed).
- **Def:** $f(x)$ has *regular growth* means for any multivariate polynomial P, the function $P(f, f', f'', \ldots)$ is non-oscillatory.
- If in the above definition the coefficients of P come from a field K of continuous germs at $+\infty$ that is closed under differentiation (i.e. a *Hardy field*), then we call f K*-regular*.

  This is equivalent to saying that $K(f, f', \ldots)$ is a Hardy field extension of K.

  Typical examples:

  - the field of rational functions $K = \mathbf{R}(x)$
  - $K = \mathbf{R}(x, e^x)$
- Regular growth ⇒ existence of limits (∞ ok) ⇒ asymptotic validity
  More details: ftp://sphere.math.utsa.edu/pub/gokhman/rgf.ps.gz

Domains of validity

- Meromorphic differential equations — sectors

  V.S. Varadarajan, *Linear Meromorphic Differential Equations: a Modern Point of View*, Bull. AMS, **33**:1-42 (1996)
  http://e-math.ams.org/journals/bull/1996-33-01/

  Y. Sibuya, *Linear Differential Equations in the Complex Domain: Problems of Analytic Continuation*, Translations of Mathematical Monographs **82**, AMS, 1990

- Algebraic differential equations with coefficients $\sim \prod_{i=0}^{n} \ell_i(x)^{\alpha_i}$, where $\ell_0(x) = x$ and $\ell_{i+1}(x) = \log \ell_i(x)$ — sectorial domains — W. Strodt, S. Bank et al.

  W. Strodt, R.K. Wright, *Asymptotic Behavior of Solutions and Adjunction Fields for Nonlinear First Order Differential Equations*, Memoirs AMS **109**, 1971

- Special cases —
  F. Olver, *Asymptotics and special functions*, Academic Press, 1974

- Coefficients of exponential, and higher, growth?

Asymptotic existence theorem

D. Gokhman, *An asymptotic existence theorem in* $\mathbf{C}$ *for the Riccati equation*, Complex Variables **24**:145–159 (1994)
http://www.math.utsa.edu/~gokhman/p_asex.html

- Let $w = y'/y$. We get the Riccati equation $w' + w^2 = f^2$.
- $\pm f$ are approximate solutions (additional conditions needed for $\mathbf{C}$).
- Let $w = f(1+\alpha)$. Then $\alpha' + \left(2f + \frac{f'}{f}\right)\alpha + f\alpha^2 + \frac{f'}{f} = 0$.
- Iterated variation of parameters (Newton's method):
  Let $T\alpha = \left(C - \int \left(f' + f^2\alpha^2\right) e^{2\varphi}\right) / \left(e^{2\varphi} f\right)$, where $\varphi' = f$.
  Let $\alpha_0 \equiv 0$, $\alpha_{i+1} = T\alpha_i$.
- $\alpha_i \to \alpha$, $T\alpha = \alpha$, $\alpha \sim \alpha_1$.
  - curvilinear coordinates p and q, $p \to +\infty$,
  - contraction mapping theorem, l'Hospital's rule w.r.t. p, uniform w.r.t q,
- Characterization of the domain as $p \to +\infty$
  - $g_{11} = (x_p)^2 + (y_p)^2$ is bounded away from 0 and ∞
  - $|f| \to +\infty$, $|f|^{-2} |f|_p \to 0$, $|f|^{-1} (\arg f)_p \to 0$
  - $\cos(\arg f + \theta)$ is bounded away from 0, where $\tan\theta = y_p/x_p$
  - if $\operatorname{Re}\varphi \to -\infty$, then $e^{2\operatorname{Re}\varphi} |f|^2$ and $e^{2\operatorname{Re}\varphi} |f'|$ are integrable to ∞
- Examples:
  - f polynomial/logarithmic: sectors, polar coordinates
  - $f = e^z$: strips, cartesian coordinates, e.g. $\{z\colon -\pi/2 < a \le \operatorname{Im} z \le b < \pi/2\}$
  - $f = e^{e^z}$: funnel, $p = \sqrt{y^2(\log y - 1/2) + x^2}$, $q = -x/\log y$

Regular growth

D. Gokhman, *Regular growth of solutions of the Riccati equation* $W' + W^2 = e^{2z}$ *in the complex plane*, Complex Variables **27**:365–382 (1995)
http://www.math.utsa.edu/~gokhman/p_rg.html

Solutions to $w' + w^2 = e^{2z}$ are $\mathbf{C}(e^z)$-regular between $y = \pm K e^{-x-2\epsilon e^x}$.

- Let P be a multivariate polynomial and $u = P(w, w', w''...)$. Using the equation we can elminate the derivatives of w from u, so we may assume that $u = P(w)$. By passing to the algebraic closure of $\mathbf{C}(e^z)$ we may assume further that $u = w - s$.
- $\dfrac{u}{f-s} = 1 + \dfrac{f\alpha}{f-s} = 1 + \dfrac{\alpha}{1 - s/f}$
- May assume that $s \sim f = e^z$
- $u' + 2su + u^2 - e^{2z} + s^2 + s' = 0$
- Puiseux expansion: $s \sim \sum_{k=0}^{\infty} a_k e^{r_k z}, \quad r_k \in \mathbf{Q}$
- $u' + 2su + u^2 + g = 0$, where $g \asymp e^{rz}$, $r \in \mathbf{Q}$, $r < 1$.
- Apply a procedure similar to the one for α.
- After many estimates $u \asymp e^{(r-1)x}$, i.e. $u \sim ce^{(r-1)x}$

Existence of limits

D. Gokhman, *Limits in differential fields of holomorphic germs*, Complex Variables **28**:27–36 (1995)
http://www.math.utsa.edu/~gokhman/p_lim.html

Theorem: (E. Lindelöf) [Theorem 15.4.4, E. Hille, *Analytic Function Theory*, Chelsea, 1987] Let $f(z)$ be holomorphic in the sector $S: \alpha < \arg z < \beta$, $0 < |z| < R$, and suppose that $f(z)$ omits two values when z is restricted to S. Suppose further that there is a γ such that $\alpha < \gamma < \beta$ and $\lim_{r\to 0} f\left(re^{\mathbf{i}\gamma}\right) \equiv c$ exists where c may be finite or infinite. Then $\lim_{r\to 0} f\left(re^{\mathbf{i}\theta}\right) = c$ for $\alpha < \theta < \beta$, uniformly in any fixed interior sector.

Theorem: If H is a field of holomorphic germs in a sector and $f \in H$ has a limit ($\in \mathbf{C} \cup \{\infty\}$) on a ray, then f has a uniform limit (w.r.t. θ) in any proper subsector.

Proof: If not, by Lindelöf's theorem a value ω is attained by f infinitely often in the sector near 0, so $1/(f-\omega)$ is not continuous near 0.

- Conformally map sectors to other regions, e.g.:
  - $-\log z$ takes the sector $\rho < r, -a < \theta < a$ to a horizontal strip $x > -\log r, -a < y < a$
  - $\log z$ takes this to a neighborhood of ∞ between $y = \pm\tan^{-1}\left(a/\sqrt{e^{2x}-a^2}\right) \sim \pm a e^{-x}$
- Example: elements of $\mathbf{C}\left(e^z\right)$ have uniform limits w.r.t. y on horizontal strips, e.g. $\{z\colon -\pi/2 < a \le \operatorname{Im} z \le b < \pi/2\}$.

  A typical element of $\mathbf{C}\left(e^z\right)$ is $P(e^z)/Q\left(e^z\right)$, where P and Q are polynomials. Restricted to the real axis $P(e^z)/Q(e^z)$ has a limit, since
  $\operatorname{Re}\left(P(e^z)/Q(e^z)\right) = \operatorname{Re}\left(P(e^x)\overline{Q(e^x)}\right)/\left|Q(e^x)\right|^2$,
  $\operatorname{Im}\left(P(e^z)/Q(e^z)\right) = \operatorname{Im}\left(P(e^x)\overline{Q(e^x)}\right)/\left|Q(e^x)\right|^2$
  are $\mathbf{R}$-regular.

Asymptotic validity example

Suppose w is solution of $w' + w^2 = e^{2z}$ that is real on the positive real axis. Then on the positive real axis

$$w \sim e^x - \frac{1}{2} - \frac{1}{8}e^{-x} - \frac{1}{8}e^{-2x} - \frac{25}{128}e^{-3x} - \frac{13}{32}e^{-4x} + ...$$

[see Sec. 3, M. Rosenlicht, *Hardy fields*, J. Math. Anal. Appl. **93**:297–311 (1983)]

- ❑ Let $H = \mathbf{C}(e^z, w)$. This field is closed under differentiation. By $\mathbf{C}(e^z)$-regularity of w every element of H is non-oscillatory on the domain bounded by $y = \pm K e^{-x-2\epsilon e^x}$.
- ❑ Elements of H have limits on the positive real axis. This is a consequence of the real Hardy field theory, since a typical element H is a rational function of w and e^z, so we rationalize and consider separately the real and imaginary parts.
- ❑ Thus, elements of H have uniform limits in a domain bounded by $y = \pm K e^{-x-2\epsilon e^x}$.
- ❑ The expansion above is valid in a domain bounded by $y = \pm K e^{-x-2\epsilon e^x}$.