TOPOLOGIES FOR HYBRID SOLUTIONS

Dmitry Gokhman
Department of Mathematics
The University of Texas at San Antonio
San Antonio, Texas 78249-0664
gokhman@math.utsa.edu

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Abstract

The Tavernini metric for hybrid solutions the Skorohod J_1 topology.

1 INTRODUCTION

> *The Skorohod topology, used in stochastic processes for the space of cadlag functions, is mathematically appealing, but tends to be very cumbersome to work with in practice.* — Lygeros et al. [7]

Hybrid solutions (trajectories), while discontinuous, have some regularity properties. Typically hybrid trajectories belong to the space D of *cadlag* (right continuous with left limits) functions (see §3.3 [12]). Cadlag functions are uniformly continuous on intevals of continuity, have only jump discontinuities, and only finitely many discontinuities, where the jump exceeds an arbitrary positive threshold (12.2.1 [12]).

Numerical simulation and qualitative analysis of hybrid systems was carried out by Tavernini (1987), who introduced the concept of a *differential automaton* and a metric on the space of solutions incorporating the uniform distances between extended continuous parts of the solutions and the distances between *switching points* (as well as the distances between the discrete states) (see [9], §4 [10],[11]).

Broucke [3, 4] (1998), Collins [5] and others adopted for spaces of hybrid solutions the Skorohod topology, originally introduced in 1956 in the context of convergence of sample paths of stochastic processes, e.g. IID processes with infinite variance. For an excellent introduction to Skorohod topology the reader is referred to [2, 12]. The four Skorohod topologies (J_1, J_2, M_1, and M_2) can be characterized via graph distance (the M topologies use completed graphs) where subscript 1 refers to parametric distance and subscript 2 to Hausdorff distance (see §11.5.2 [12]).

Skorohod topologies are *coarser* (see §2.1 [8]) than the compact open topology and *finer* than the Lebesgue topologies. For example among functions on the unit interval, the indicator function of $[0, \frac{1}{2})$ is δ close to the indicator function of $[0, \frac{1}{2}+\delta)$ in the Skorohod J_1 metric, whereas in the uniform metric the distance between these functions is 1. Meanwhile, the indicator function of $[0, \delta)$ is δ close to zero in any of the Lebesgue metrics, but the Skorohod J_1 distance between them is 1. The J_2 and M_1 topologies are not comparable, and are both coarser than J_1 and finer than M_2 (see Fig. 11.2 [12]).

Definition 1.1 The Skorohod J_1 topology is defined on the space D of cadlag functions on a compact interval by

$$d(\xi, \eta) = \inf_{\lambda \in \Lambda} \max \left\{ \|\xi \circ \lambda - \eta\|_\infty , \|\lambda - e\|_\infty \right\},$$

where Λ is the group of all order preserving homeomorphisms of I and $e \in \Lambda$ is the identity.

It is worth noting that metrics defined for functions on a compact interval can be extended to functions on the positive real half-line using any of several similar techniques based on normalizing the metric for functions restricted to intervals (§2.2 [1], §2.2 [5], Pf. XII.8.2 (1) [6], (3.3.6) [12]) and combining them, often using a weighted (for convergence) sum (XII.8.5 [6], §2.2 [1], (3.3.6) [12]).

The Skorohod J_1 metric is incomplete (see Example 12.2 [2]), but a topologically equivalent complete metric can be defined by replacing Λ in the definition by a subset of reparametrizations with average speeds bounded away from 0 and ∞ (see Th. 12.1 [2]).

Our main result is that the Tavernini metric, appealing for its simplicity, induces the Skorohod J_1 topology.

2 Differential automata and the Tavernini metric

In this section we follow [11] to briefly introduce the concept of a differential automaton, hybrid solutions to an initial value problem, and the Tavernini metric.

Definition 2.1 A *differential automaton* $A = (Q, \nu, \lambda, E)$ over $\mathbf{R}^m$ consists of

- a discrete finite set Q,
- a set of discrete state transition operators, i.e. self-maps $\nu(x)$ on Q for each $x \in \mathbf{R}^m$,
- a set of leap operators, i.e., self-maps $\lambda(q)$ on $\mathbf{R}^m$ for each $q \in Q$, and
- a family $E = \{E_q \colon q \in Q\}$ of one-parameter continuous semigroups E_q of continuous operators on $\mathbf{R}^m$ such that for each ball $B \in \mathbf{R}^m$ and $T > 0$ there exists a constant L with $|E_q(s)x - E_q(t)y| \leq L(|s-t| + |x-y|)$ for all $x, y \in B$ and $s, t \in [0, T]$.

The *immediate successors* of $q \in Q$ is the (possibly empty) set $I_q = \nu(\mathbf{R}^m)q \setminus \{q\}$. For $p \in I(q)$ define $M_{qp} = \{x \in \mathbf{R}^m \colon \nu(x)q = p\}$. It is required that for each $q \in Q$, unless $I(q)$ is a singleton and $M_{qp} = \mathbf{R}^m$, there exists a smooth function g_{qp} on $\mathbf{R}^m$ with 0 a regular value, such that $M_{qp} = g_{qp}^{-1}(\mathbf{R}^+)$.

Definition 2.2 Each boundary $\partial M_{qp} = g_{qp}^{-1}(0)$ is called a *switching manifold* of A. For each discrete state $q \in Q$ the set

$$\delta(q) = \mathbf{R}^m \setminus [\bigcup_{p \in I(q)} M_{qp}]$$

is called the *domain of capture* of q. The set of all *capturing states* is the set $S_0 = \{(x, q) \in S \colon x \in \delta(q)\}$. The set of all *capturing discrete states* is the set $Q_0 = \{q \colon \delta(q) \neq \varnothing\}$.

Definition 2.3 The set of *admissible states* S^* is the union of S_0 and the sets S_k, $k \geq 1$ consisting of states (x_0, q_0) for which there exists a sequence $(x_j, q_j), j \leq k$ such that $x_{j+1} = \lambda_{q_j}(x_j), q_{j+1} = \nu_{x_j}(q_j), (x_j, q_j) \notin S_0$ for $j < k$ and $(x_k, q_k) \in S_0$. Define a self map $+$ on S^* by $(x, q)^+ = (x_k, q_k)$.

Definition 2.4 A *hybrid solution* with an admissible initial state (x_0, q_0) is a pair of maps (ξ, σ) on an interval $[0, T)$ to $\mathbf{R}^m$ and Q and the unique partition (t_j) of $[0, T]$ with $t_0 = 0$ and $t_k = T$ such that ξ is uniformly continuous and σ is constant on each subinterval

$[t_j, t_{j+1})$; $(\xi(0), \sigma(0)) = (x_0, q_0)^+$; $\xi(t) = E_{\sigma(t)}(t - t_i)\xi(t_i)$, $\sigma(t) = \nu(\xi(t))\sigma(t)$, $t \in [t_j, t_{j+1})$; $\xi(t_{j+1}^-) \notin \delta(\sigma(t_{j+1}^-)$; $(\xi(t_{j+1}), \sigma(t_{j+1})) = (\xi(t_{j+1}^-), \sigma(t_{j+1}^-))^+$. The points t_j, $0 < j < k$ are called the *switching points* of the solution.

Definition 2.5 Suppose $[a, b) \subset [0, T)$. Define the continuous extension $\widetilde{\varphi} \in C([0, T], \mathbf{R}^m)$ of $\varphi \in C_0([a, b), \mathbf{R}^m)$ — the space of continuous functions on $[a, b)$ having a left limit at b — by

$$\widetilde{\varphi}(t) = \begin{cases} \varphi(a), & t < a \\ \varphi(t), & a \le t < b \\ \lim\limits_{\tau \to b^-} \varphi(\tau), & t \ge b \end{cases}$$

Definition 2.6 For a hybrid solution (ξ, σ) define projections π_0 to $\ell_\infty(\mathbf{R})$, π_j, $j \ge 1$ to $C([0, T], \mathbf{R}^m)$, and π_* to $\ell_\infty(\mathbf{R})$ by

$$\pi_0(\xi) = (t_1, \ldots t_k, 0, \ldots)$$

$$\pi_j(\xi) = \begin{cases} \widetilde{\xi}_j, & 1 \le j \le k \\ 0, & j > k \end{cases}$$

$$\pi_*(\sigma) = (\sigma(t_0), \ldots \sigma(t_{k-1}), 0, \ldots)$$

where t_j, $j = 0, \ldots k$ is the partition given by the switching points of the solution.

Definition 2.7 Given two hybrid solutions (ξ, σ) and (η, μ) define

$$\rho[(\xi, \sigma), (\eta, \mu)] = \rho_1(\xi, \eta) + \rho_2(\sigma, \mu)$$

where

$$\rho_1(\xi, \eta) = \|\pi_0(\xi) - \pi_0(\eta)\|_\infty + \max_{j \ge 1} \|\pi_j(\xi) - \pi_j(\eta)\|_\infty$$

$$\rho_2(\sigma, \mu) = \|\pi_0(\sigma) - \pi_0(\mu)\|_\infty + \|\pi_*(\sigma) - \pi_*(\mu)\|_\infty$$

Note 2.1 Given a solution (ξ, σ), an ε neighborhood of (ξ, σ) in the Tavernini metric with $\varepsilon < 1$ contains only solutions with the same σ (and the same number of switching points). Therefore, when exploring the topological properties of this metric we may assume that neighborhoods are sufficiently small so that the discrete part σ remains the same and may be omitted from the discussion.

The Tavernini metric ρ_1 is defined using the switching points of solutions, so can be thought of as a metric on the space F of cadlag functions ξ on $[0, T)$ to $\mathbf{R}^m$ paired with cadlag functions σ on $[0, T)$ to Q and with sequences of switching points $\pi_0(\xi)$. The switching points of ξ include all points of discontinuity of ξ, but there may be switching points where ξ remains continuous. As can be seen from the following example, the Tavernini metric is incomplete.

Example 2.1 (cf. Example 12.2 [2]) Suppose $t_k \to 0$ in $[0, T)$ and consider the indicator functions ξ_k of $[0, t_k)$ on $[0, T)$ with $\pi_0(\xi_k) = \{t_k, 0, \ldots\}$. For simplicity, choose Q to be a singleton, which allows us to ignore the discrete parts. Then $\rho_1(\xi_k, \xi_m) = |t_k - t_m|$. Since (t_k) is a Cauchy sequence in $[0, T)$, (ξ_k) is a Cauchy sequence in F. For $t > 0$ $\xi_k(t) \to 0$, but $\rho_1(\xi_k, 0) \ge 1$, irrespective of the choice of $\pi_0(0)$.

3 MAIN RESULT

Theorem 3.1 The Tavernini metric on the set of hybrid solutions to a differential automaton with finitely many switching points on each bounded interval induces the Skorohod J_1 topology.

Proof: Extensions to functions defined on $\mathbf{R}^+$ being the same for both topologies (see the introduction), it is enough to consider what happens on a bounded interval. Suppose ξ is a solution with finitely many switching points: $t_1, \dots t_k$. Let $\varepsilon > 0$ be sufficiently small as in Note 2.1 and with $\varepsilon < \frac{1}{2} \min_i \left|\xi(t_i^+) - \xi(t_i^-)\right|$.

We want to show that an ε neighborhood of ξ in the Tavernini metric contains a δ neighborhood of ξ in the Skorohod J_1 metric d. Suppose η is a solution with k switching points s_i and $d(\xi, \eta) < \delta \le \frac{\varepsilon}{4}$. Then there exists $\lambda \in \Lambda$ with $|\lambda - e| < \delta$ such that for all $x \in [0, 1]$ we have $|\xi(\lambda(t)) - \eta(t)| < \delta$. Note that if δ is smaller than half the smallest jump discontinuity of ξ, then $t_i = \lambda(s_i)$ and in particular $|t_i - s_i| < \delta$ for all i.

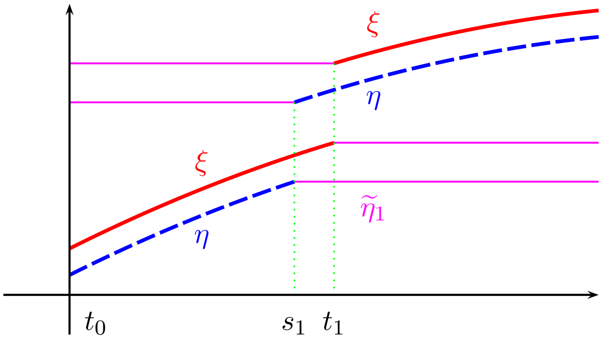

Figure 1

For concreteness let us first look at what happens on the interval $I = [0, t_1)$ assuming for the moment that $s_1 \le t_1$ (see Fig. 1). Since ξ has a left limit at t_1, ξ is uniformly continuous on I, i.e. the modulus of continuity $w_\xi(\delta) \to 0$ as $\delta \to 0$ (see §2.7 [2]). Thus, by choosing δ sufficiently small we can ensure that $w_\xi(\delta) < \frac{\varepsilon}{4}$. For $t \in [0, s_1)$, $\lambda(t) \in I$, so

$$|\xi(t) - \eta(t)| \le |\xi(t) - \xi(\lambda(t))| + |\xi(\lambda(t)) - \eta(t)| < \frac{\varepsilon}{4} + \delta \le \frac{\varepsilon}{2}. \tag{1}$$

If $t \in [s_1, t_1)$, then $\widetilde{\eta}(t) = \eta(s_1^-)$ and $|t - s_1| < \delta$, so

$$|\xi(t) - \widetilde{\eta}(t)| = \left|\xi(t) - \eta(s_1^-)\right| \le \left|\xi(t) - \xi(s_1^-)\right| + \left|\xi(s_1^-) - \eta(s_1^-)\right| < \frac{\varepsilon}{4} + \delta \le \frac{\varepsilon}{2}.$$

Finally, for $t \ge t_1$

$$\left|\widetilde{\xi}(t) - \widetilde{\eta}(t)\right| = \left|\xi(t_1^-) - \eta(s_1^-)\right| \le \left|\xi(t_1^-) - \xi(s_1^-)\right| + \left|\xi(s_1^-) - \eta(s_1^-)\right| < \frac{\varepsilon}{4} + \delta \le \frac{\varepsilon}{2}.$$

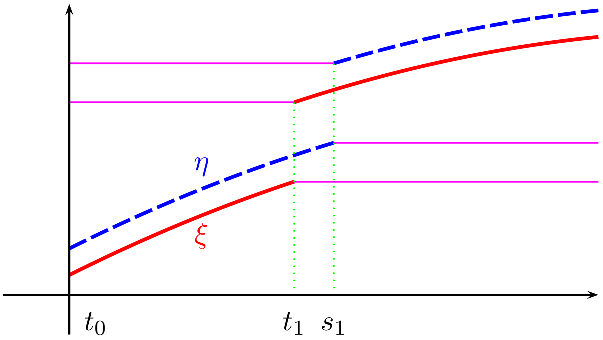

Figure 2

If $s_1 > t_1$ (see Fig. 2), then for $t \in [0, t_1)$ we have $\lambda(t) < t$, so $\lambda(t) \in [0, t_1)$ and (1) holds. If $t \in [t_1, s_1)$, then $\lambda(t) < t_1$ and $|\lambda(t) - t_1| \le |\lambda(t) - t| < \delta$, so

$$\left|\widetilde{\xi}(t) - \eta(t)\right| = \left|\xi(t_1^-) - \eta(t)\right| \le \left|\xi(t_1^-) - \xi(\lambda(t))\right| + |\xi(\lambda(t)) - \eta(t)| < \frac{\varepsilon}{4} + \delta \le \frac{\varepsilon}{2}.$$

Analysis of other intervals must include the consideration of left endpoints as well, but is similar.

Since the discrete parts of the solutions must coincide for sufficiently small distances (see Note 2.1) we have $\rho_2(\xi, \eta) = 0$. Since $|s_i - t_i| < \delta$

$$\rho(\xi, \eta) = \rho_1(\xi, \eta) = \|\pi_0(\xi) - \pi_0(\eta)\|_\infty + \max_{j \ge 1} \|\pi_j(\xi) - \pi_j(\eta)\|_\infty < \delta + \frac{\varepsilon}{2} \le \varepsilon.$$

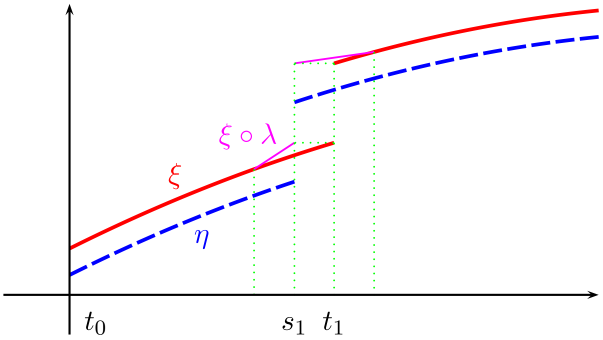

Figure 3

To prove that an ε neighborhood of ξ in the Skorohod J_1 metric contains a δ neighborhood of ξ in the Tavernini metric suppose that η is a solution with $\rho(\xi, \eta) < \delta$. Again let us look at the first interval $[0, t_1)$ assuming for now that $s_1 \le t_1$ (see Fig. 3). Choose $a \in (s_1 - \delta, s_1)$ and $b \in (t_1, t_1 + \delta)$ reasonably far away from other t_i. Define a reparametrization λ by

$$\lambda(t) = \begin{cases} t & \text{if } t \le a \\ a + \dfrac{a - s_1}{a - t_1}(t - a) & \text{if } a < t \le t_1 \\ b + \dfrac{b - s_1}{b - t_1}(t - b) & \text{if } t_1 < t \le b \\ t & \text{if } b < t \le \ldots \\ \text{etc.} & \end{cases}$$

Then since $|t_1 - s_1| \le \rho_1(\xi, \eta) \le \rho(\xi, \eta) < \delta$ and similarly for other intervals, we have $\|\lambda - e\|_\infty < \delta$. For $t \in [0, a)$ we have

$$|\xi(\lambda(t)) - \eta(t)| = |\xi(t) - \eta(t)| \le \|\xi - \eta\|_\infty < \delta < \frac{\varepsilon}{4}$$

For $t \in [a, s_1)$ we have

$$|\xi(\lambda(t)) - \eta(t)| \le |\xi(\lambda(t)) - \xi(t)| + |\xi(t) - \eta(t)| < \frac{\varepsilon}{4} + \delta \le \frac{\varepsilon}{2}$$

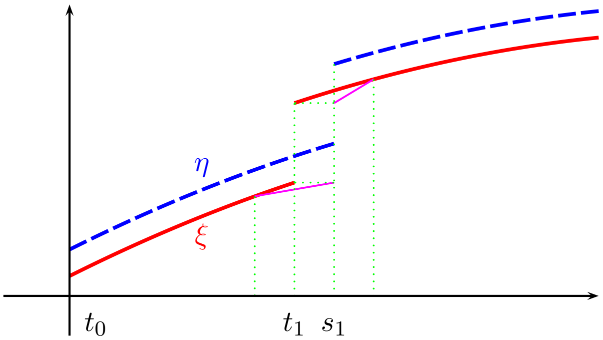

Figure 4

If $s_1 > t_1$ (see Fig. 4), choose $a \in (t_1 - \delta, t_1)$ and $b \in (s_1, s_1 + \delta)$ sufficiently far away from other t_i. Define a reparametrization λ as before. For $t \in [0, a)$ we have

$$|\xi(\lambda(t)) - \eta(t)| = |\xi(t) - \eta(t)| \le \|\xi - \eta\|_\infty < \delta < \frac{\varepsilon}{4}$$

If $t \in [a, s_1)$, then

$$|\xi(\lambda(t)) - \eta(t)| \le \left|\xi(\lambda(t)) - \widetilde{\xi}(t)\right| + \left|\widetilde{\xi}(t) - \eta(t)\right| < \frac{\varepsilon}{4} + \delta \le \frac{\varepsilon}{2}$$

Analysis of other intervals involves left endpoints, but is similar. We see that

$$d(\xi, \eta) \le \sup |\xi(\lambda(t)) - \eta(t)| \le \frac{\varepsilon}{2} < \varepsilon$$

and the proof is concluded. ■

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