FUNCTIONS IN A HARDY FIELD NOT ULTIMATELY $\mathcal{C}^\infty$

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Abstract

I construct a class of functions, whose germs belong to Hardy fields and all of whose derivatives *a fortiori* ultimately exist, but the functions are not ultimately $\mathcal{C}^\infty$. The existence of such functions, while counterintuitive at first glance, is explained by the fact that the higher order derivatives exist in progressively smaller neighborhoods of $+\infty$. A function not a Hardy field satisfying the required smoothness properties was given in [1]. I provide a proof of the required smoothness properties of this function (omitted in [1]), and then use this function in the present construction.

A Hardy field is a differential field of germs of continuous real valued functions at $+\infty$. The reader is referred to [5] for an introduction to Hardy fields. Any function whose germ is in a Hardy field must have derivatives of any order. However, it is not necessarily true that there is a single (one-sided) neighborhood of $+\infty$, where all these derivatives exist. We prove this result by explicitly constructing a function (actually many functions), whose germ belongs to a Hardy field containing $\mathbf{R}$, but which is not $\mathcal{C}^\infty$ in any neighborhood of $+\infty$.

Remark 0.1 To show that a function generates a Hardy field, it is sufficient to show that a nonzero element of $\mathbf{R}[f, f'...]$, i.e. any non-trivial differential polynomial of this function, is ultimately [1] zero free and, thus, invertible.

We begin with a fairly standard lemma proved here for convenience.

Lemma 0.1 (cf. Theorems 5.15, 5.16, [4]) *Suppose $x_0 \in (a, b) \subseteq \mathbf{R}$, f is a continuous function on (a, b) which is differentiable on $(a, x_0) \cup (x_0, b)$ and $L = \lim_{x \to x_0} f'(x)$ exists. Then f is differentiable at x_0 and, thus, on (a, b) with $L = f'(x)$.*

Proof: By definition $f'(x_0) = \lim_{h \to 0} \left(f(x_0 + h) - f(x_0)\right)/h$. Without loss of generality assume that $0 < h < b - x_0$. Since f is differentiable on $(x_0, x_0 + h)$, by the Mean Value Theorem there exists $\xi(h) \in (x_0, x_0 + h)$ such that $f'(\xi(h)) = \left(f(x_0 + h) - f(x_0)\right)/h$. Since $x_0 < \xi(h) < x_0 + h$ and $\lim_{h \to 0} x_0 + h = x_0$, we have $\lim_{h \to 0} \xi(h) = x_0$. Therefore, $f'(x_0) = \lim_{h \to 0} \left(f(x_0 + h) - f(x_0)\right)/h = \lim_{h \to 0} f'(\xi(h)) = \lim_{x \to x_0} f'(x) = L$ ■

The function $(\sin^2 x)^x$ was given in [1] as an example of a not ultimately infinitely smooth function, all of whose derivatives ultimately exist. We provide a proof in the next four lemmas.

[1] A property is said to hold *ultimately* exactly when it holds in a neighborhood of $+\infty$.

Lemma 0.2 *Let $f(x) = (\sin^2 x)^x$. Then $f \in \mathcal{C}^\infty[(0,\pi) \cup (\pi, 2\pi) \cup ...]$ and its derivatives are of the form*

$$f^{(n)}(x) = \left(\sin^2 x\right)^{x-\frac{n}{2}} \sum_\alpha d_\alpha \log^{l_\alpha}\left(\sin^2 x\right) x^{i_\alpha} \cos^{j_\alpha} x \sin^{n-k_\alpha} x,$$

where the sum is finite; d_α, i_α, j_α, k_α, $l_\alpha \in \mathbf{Z}$; $d_\alpha \neq 0$; $i_\alpha, l_\alpha \geq 0$; $j_\alpha \in \{0,1\}$; $0 \leq k_\alpha \leq n$; there exists α_0 such that $k_{\alpha_0} = n$; and for all α if $l_\alpha > 0$, then $k_\alpha < n$.

Proof: Note that $f(x) = e^u$, where $u = x \log(\sin^2 x)$. Then

$$u' = \log\left(\sin^2 x\right) + \frac{2\cos x}{\sin x}, \qquad u'' = \frac{4\cos x}{\sin x} - \frac{2x}{\sin^2 x}.$$

We see that u'' is a sum of terms of the form $a_k x^i \cos^j x \sin^{-k} x$, where $a_k, i \in \mathbf{Z}$; $j \in \{0,1\}$; and $k \in \{1,2\}$. The derivative of any such term is

$$\left(a_k x^i \cos^j x \sin^{-k} x\right)' = a_k \left(i x^{i-1} \cos^j x \sin^{-k} x + j x^i \cos^{j-1} x(-\sin x) \sin^{-k} x + \right.$$

$$\left.(-k) x^i \cos^j x \sin^{-k-1} x \cos x\right) = a_k i \frac{x^{i-1}\cos^j x}{\sin^k x} - a_k j \frac{x^i \cos^{j-1} x}{\sin^{k-1} x} - a_k k \frac{x^i \cos^{j+1} x}{\sin^{k+1} x}.$$

Therefore, by induction $u^{(n)}$ is a finite sum of such terms:

$$u^{(n)} = \sum_\alpha a_{k_\alpha} \frac{x^{i_\alpha}\cos^{j_\alpha} x}{\sin^{k_\alpha} x}. \tag{1}$$

Note that the largest k_α such that $a_{k_\alpha} \neq 0$ is n. Also $i_\alpha \in \{0,1\}$ and, since $\cos^2 x = 1 - \sin^2 x$, we may assume that $j_\alpha \in \{0,1\}$. Since $f = e^u$, $f^{(n)} = e^u p_n\left(u, u', ...u^{(n)}\right)$, where p_n is a differential polynomial in u of the form

$$p_n\left(u, u', ...u^{(n)}\right) = \sum_{\sum_{\gamma=1}^{l} i_\gamma + j_\gamma = n} c_\beta \left(u^{(i_1)}\right)^{j_1} \left(u^{(i_2)}\right)^{j_2} ... \left(u^{(i_l)}\right)^{j_l} = u^{(n)} + ... + \left(u'\right)^n, \tag{2}$$

where $c_\beta \in \mathbf{Z}$. Therefore,

$$f^{(n)} = \left(\sin^2 x\right)^x \sum_\alpha d_\alpha \frac{\log^{l_\alpha}(\sin^2 x)\, x^{i_\alpha} \cos^{j_\alpha} x}{\sin^{k_\alpha} x}. \tag{3}$$

Again we may assume that $j_\alpha \in \{0,1\}$. Note that in (1) there is only one term with the largest $k_\alpha = n$ and it contains x in the numerator. Therefore in (2) there is exactly one term with x in the numerator and $\sin^n x$ in the denominator, so the largest k_α appearing in (2) is n.

Bringing the sum in (3) to a common denominator gives

$$f^{(n)} = \left(\sin^2 x\right)^x \frac{1}{\sin^n x} \sum_\alpha d_\alpha \log^{l_\alpha}\left(\sin^2 x\right) x^{i_\alpha} \cos^{j_\alpha} x \sin^{n-k_\alpha} x,$$

Since $\log\left(\sin^2 x\right)$ appears only in u', we have $k_\alpha < n$ whenever $l_\alpha \neq 0$ in (2) ■

Lemma 0.3 *If $f(x) = \left(\sin^2 x\right)^x$, then $f \in \mathcal{C}^n\left[\left(\frac{n}{2}, +\infty\right)\right]$.*

Proof: From lemma 0.2 we see that $f^{(n)}$ exists for $x > 0$ which are not integral multiples of π. Suppose $x_0 > \frac{n}{2}$ is an integral multiple of π. Then $\lim_{x \to x_0} f^{(n)}(x) = P_1 P_2$, where

$$P_1 = \lim_{x \to x_0} \left(\sin^2 x\right)^{x - \frac{n}{2}}, \qquad P_2 = \lim_{x \to x_0} \sum_\alpha d_\alpha \log^{l_\alpha}\left(\sin^2 x\right) x^{i_\alpha} \cos^{j_\alpha} x \sin^{n-k_\alpha} x.$$

Since $x_0 > \frac{n}{2}$, P_1 exists and equals 0. Since $\lim_{x \to x_0} \sin x = 0$, $k_\alpha < n$ whenever $l_\alpha > 0$ and $\lim_{\xi \to 0^+} \xi^\epsilon \log^\delta \xi = 0$ (think of ξ as $\sin^2 x$, the relevant terms in P_2 form a polynomial in x with coefficients of the form $A \cos^j x$, where $A \in \mathbf{Z}$, $A \neq 0$, and $j \in \{0, 1\}$. Therefore, P_2 exists and, thus, $\lim_{x \to x_0} f^{(n)}(x)$ exists as well, so by lemma 0.1 we are done ■

Lemma 0.4 *If $f(x) = \left(\sin^2 x\right)^x$, then $\left|f^{(n)}\right|$ is ultimately bounded by a polynomial function of x (possibly dependent of n).*

Proof: Clearly $\left(\sin^2 x\right)^{x - \frac{n}{2}} \log^{l_\alpha}\left(\sin^2 x\right)$ is bounded for sufficiently large x. Since in (3) there is only a finite number of terms, there exist $\xi, M \in \mathbf{R}^+$ such that for all α we have $\left|\left(\sin^2 x\right)^{x - \frac{n}{2}} \log^{l_\alpha}\left(\sin^2 x\right)\right| \leq M$. Therefore, $\left|f^{(n)}\right| \leq M \sum_\alpha |d_\alpha| x^i$ for $x > \xi$ ■

Lemma 0.5 *If $f(x) = \left(\sin^2 x\right)^x$, then f is not ultimately $\mathcal{C}^\infty$.*

Proof: It is sufficient to show that for any $x_0 > 0$ there exists $n \in \mathbf{Z}^+$ such that $f^{(n)}$ does not exist at some $x_1 \in (x_0, +\infty)$. Given such x_0 choose x_1 to be an integral multiple of π such that $x_1 > x_0$. Now choose n such that $\frac{n}{2} - x_1 > 2\pi$. Following lemma 0.3, consider the polynomial in x with coefficients of the form $A \cos^j x$, where $A \in \mathbf{Z}$, consisting of the relevant terms of P_2. There are the following cases

Case 1. All j's are zero.

Since the polynomial is non-trivial and has integer coefficients, its value at an integral multiple of π, a real number transcendental over $\mathbf{Q}$, must be non-zero. Therefore, since $x_1 < \frac{n}{2}$, $\lim_{x \to x_1} f^{(n)}(x)$ does not exist.

Case 2. Not all j's are zero.

Without loss of generality assume that $\cos(x_1) = 1$. If the above polynomial is non-zero when $\cos x$ is replaced by 1 in all appropriate coefficients, then we are back to case 1. Otherwise, the polynomial must consist of pairs of terms of the form $Ax^i - A\cos(x)x^i$. If we replace x_1 by $x_1 + \pi$, then $\cos x_1 = -1$. The polynomial is non-zero when $\cos x$ is replaced by -1 in all appropriate coefficients, so since the new x_1 is still less that $\frac{n}{2}$, we are back to case 1 ∎

Remark 0.2 We follow the notation of [5]. Let $G(x) = \sum_{i=0}^{\infty} a_i x^i$ be an analytic function at infinity such that a_i are algebraically independent over $\mathbf{Q}$. Since G is analytic, so is any differential polynomial of G. If the latter is not identically zero, then its zeros are isolated, so there is a neighborhood of ∞, where it is zero free. Therefore, in view of Remark 0.1, the germ of G at $+\infty$ is contained in some Hardy field extension of $\mathbf{R}$ (cf. Proposition 7.1 [2]). In addition, G is differentially transcendental over $\mathbf{R}$ with respect to the derivation $D_x = \frac{d}{dx}$ (see [3], cf. Proposition 7.4 [2]).

Theorem 0.1 *There exists a continuous function* $g\colon (0,+\infty) \to \mathbf{R}$ *that is not ultimately* $\mathcal{C}^{\infty}$ *and whose germ at* $+\infty$ *is contained in a Hardy field extension of* $\mathbf{R}$.

Proof: Let $g(x) = G(x) + e^{-x} f(x)$, where G is an analytic function at infinity such that a_i are algebraically independent over $\mathbf{Q}$. and $f(x) = \left(\sin^2 x\right)^x$. Since $f(x) = e^x \left(g(x) - G(x)\right)$, and $G, e^{\pm x}$ are $\mathcal{C}^{\infty}$, g has the same degree of differentiability as f on any interval $(\epsilon, +\infty)$. Therefore, g is not ultimately $\mathcal{C}^{\infty}$, but is ultimately $\mathcal{C}^{n}$ for any n. To show that the germ of g at $+\infty$ is contained in a Hardy field extension of $\mathbf{R}$ it is sufficient to choose an arbitrary non-trivial differential polynomial q over $\mathbf{R}$ with respect to the derivation $D_x = \frac{d}{dx}$ and prove that $q\left(g, g', ...g^{(m)}\right)$ ultimately has a definite sign.

By successively differentiating g we see that, where defined,

$$g^{(n)}(x) = G^{(n)}(x) + e^{-x} P_n\left(f(x), f'(x), ...f^{(n)}(x)\right),$$

where P_n is a differential polynomial over $\mathbf{Q}$ with respect to the derivation $D_x = \frac{d}{dx}$. Then

$$q\left(g, g', ...g^{(m)}\right) = q\left(G, ...G^{(m)}\right) + \sum_{j=1}^{\overline{m}} e^{-jx} Q_j\left(G, f, G', f'...G^{(m)}, f^{(m)}\right),$$

where Q_j are differential polynomials in G and f. By lemma 0.4 $\left|f^{(n)}(x)\right|$ is ultimately bounded by a polynomial in x. Since $\lim_{x\to+\infty} G(x) = a_0$, $|G(x)|$ and, similarly, $\left|x^{(n)}\right|$ are ultimately bounded by constants. Therefore, each $|Q_j|$ is ultimately bounded by a polynomial in x. Let $\sum_{i=i_0}^{\infty} d_i x^{-i}$ be the power series representing the analytic function at

infinity
$q\left(G(x), G'(x), ...G^{(m)}(x)\right)$ with $d_{i_0} \neq 0$. Then

$$\lim_{x\to+\infty} x^{i_0} q\left(g, g', ...g^{(m)}\right) - d_{i_0} = \lim_{x\to+\infty} x^{i_0}\left[q\left(g, g', ...g^{(m)}\right) - q\left(G, G', ...G^{(m)}\right)\right] =$$

$$\lim_{x\to+\infty} \sum_{j=0}^{\overline{m}} e^{-jx} x^{i_0} Q_j = 0.$$

Therefore, $x^{i_0} q\left(g, g', ...g^{(m)}\right)$ and, thus, $q\left(g, g', ...g^{(m)}\right)$ ultimately has the same sign as d_{i_0} ■

References

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