Differentially transcendental formal power series

Dmitry Gokhman
Division of Mathematics and Statistics
University of Texas at San Antonio

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Abstract

We prove that a formal power series in $1/x$, whose coefficients are in a field extension of $\mathbf{Q}$ and are algebraically independent over $\mathbf{Q}$, is differentially transcendental (i.e. not differentially algebraic) over this field extension. This is stated without proof in [2]. This result provides a source of functions analytic at ∞ that are not differentially algebraic over $\mathbf{R}$. Such functions are of particular interest, because their germs belong to Hardy fields, but not to the class E of [1] — the intersection of all maximal Hardy fields.

Suppose F is a field extension of the field of rational numbers $\mathbf{Q}$. Let x be an indeterminate and let $u = 1/x$. Let $F[[u]] = F[[1/x]]$ denote the ring of formal power series in u with coefficients in F. Then its field of quotients $F((u))$ is a differential field with a least two possible derivations: formal differentiation with respect to x and u denoted by D_x and D_u.

Definition 0.1 (see e.g. §VI.1 [7]) *Suppose F is an extension field of K and $S \subseteq F$.*

(i) *S is algebraically dependent over K if a nonzero polynomial in finitely many variables with coefficients in K is annulled by elements of S.*

(ii) *S is a transcendence basis over K means that S is algebraically independent (i.e. not algebraically dependent) over K and is maximal with respect to this property (i.e. F is algebraic over $K(S)$).*

(iii) *A transcendence basis has unique cardinality (Theorem VI.1.9 [7]) which is called the transcendence degree of F over K and is denoted by* $\mathrm{tr.deg.}_K F$.

Definition 0.2 (see e.g. §I.6 [8]) *Suppose F is a field, K is a differential field, $F \subseteq K$, and $f \in K$. To say that f is differentially algebraic over F means that $\{f, f', f'', ...\}$ is algebraically dependent over K, i.e. f is a root of a nonzero differential polynomial with coefficients in F.*

Proposition 0.1 (Proposition 7.4, [2]) *If the set $\{a_i \in F, i = 0, 1, ...\}$ is algebraically independent over $\mathbf{Q}$, then*

$$f = \sum_{i=0}^{\infty} a_i x^{-i} \in F\left(\left(\frac{1}{x}\right)\right)$$

is not differentially algebraic over F with respect to D_x.

Proof: Suppose f is a root of a non-zero differential polynomial p over F with respect to D_x. Let $b_j \in F$ $(j = 0, 1, ...\overline{j})$ be the coefficients of p. Then f is differentially algebraic with respect to D_x over K, where $K = \mathbf{Q}(\{b_j\})$, so

$$\text{tr.deg.}_K K\left(f, D_x f, D_x^2 f, ...\right) < \infty.$$

Since $\text{tr.deg.}_{\mathbf{Q}} K < \infty$, we have

$$\text{tr.deg.}_{\mathbf{Q}} K\left(f, D_x f, D_x^2 f, ...\right) =$$

$$\text{tr.deg.}_{\mathbf{Q}} K + \text{tr.deg.}_K K\left(f, D_x f, D_x^2 f, ...\right) < \infty.$$

Therefore

$$\text{tr.deg.}_{\mathbf{Q}} \mathbf{Q}\left(f, D_x f, D_x^2 f, ...\right) =$$

$$\text{tr.deg.}_{\mathbf{Q}} K\left(f, D_x f, D_x^2 f, ...\right) - \text{tr.deg.}_{\mathbf{Q}(f, D_x f, D_x^2 f, ...)} K\left(f, D_x f, D_x^2 f, ...\right) < \infty.$$

Note that

$$D_x f = D_u f\left(-\frac{1}{x^2}\right) = D_u F\left(-u^2\right),$$

$$D_x^2 f = \left(D_u^2 f\left(-u^2\right) + D_u f(-2u)\right)\left(-u^2\right)$$

and so on, so $\mathbf{Q}(x, f, D_x f, D_x^2 f, ...) = \mathbf{Q}(u, f, D_u f, D_u^2 f, ...)$.

Now since $\text{tr.deg.}_{\mathbf{Q}} \mathbf{Q}(u) = 1$ and $\text{tr.deg.}_{\mathbf{Q}} \mathbf{Q}(f, D_x f, D_x^2 f, ...) < \infty$, we have $\text{tr.deg.}_{\mathbf{Q}} \mathbf{Q}(u, f, D_u f, D_u^2 f, ...) < \infty$ and $\text{tr.deg.}_{\mathbf{Q}} \mathbf{Q}(f, D_u f, D_u^2 f, ...) < \infty$. Therefore, f satisfies some non-zero differential polynomial q over $\mathbf{Q}$ with respect to D_u. Since the field $\mathbf{Q}$ is infinite, there exist $t_k \in \mathbf{Q}$ $(k = 0, 1, ...\overline{k})$ such that $q(t_0, t_1, ...t_{\overline{k}}) \neq 0$. Let

$$g(u) = \sum_{k=0}^{\infty} \frac{t_k}{k!} u^k.$$

Then $q(g, D_u g, ...) \neq 0$, because it is non-zero when evaluated at $u = 0$ (it equals $q(\{t_k\})$).

If h is a formal sum $\sum_{i=0}^{\infty} y_i u^i$ with y_i indeterminate, then $q(h, D_u h, ...)$ is a power series in u, whose coefficients are polynomials in y_i over $\mathbf{Z}$. Since evaluating y_i at $(t_0, t_1, ...t_{\overline{k}}, 0, 0...)$ produces a non-zero answer, one of the above coefficients is a non-zero polynomial in y_i over $\mathbf{Q}$. Let r denote this polynomial. Evaluating y_i at a_i gives $q(f, D_u f, ...) = 0$, so, in particular, $r(a_i) = 0$, which contradicts the assumption that a_i are algebraically independent over $\mathbf{Q}$ ■

The value of this result lies in providing a class of functions analytic at ∞ that are differentially transcendental over $\mathbf{R}$. The germs of such functions necessarily belong to Hardy fields (see the definition below), but not to Boshernitzan's class E [1] — the intersection of all maximal Hardy fields. [1]

Definition 0.3 (see [4]) *A differential field of continuous germs* [2] *of real functions at* $+\infty$*, where the derivation is ordinary differentiation, is called a Hardy field.*

Corollary 1 *Suppose* f *is a real function that is analytic at* ∞ *and the set of coefficients of its Taylor series at* ∞ *is algebraically independent over* $\mathbf{Q}$*. Then*

(i) f *is differentially transcendental over* $\mathbf{R}$*,*

(ii) $\mathbf{R}\,(f, f'...)$ *is a Hardy field,*

(iii) $f \notin E$*, where* E *is the intersection of all maximal Hardy fields.*

Proof: (i) is a special case of Proposition 0.1 with $F = \mathbf{R}$ and the series actually convergent. (ii) is a consequence of the fact that the zeros of an analytic function are necessarily isolated and, thus, cannot have ∞ as an accumulation point. This means that every nonzero element of $\mathbf{R}\,[f, f'...]$, i.e. a nonzero differential polynomial of f, is invertible in a (punctured) neighborhood of ∞ (see Theorem 7.1 [1]). (iii) is a consequence of the fact that E is a differentially algebraic extension of $\mathbf{R}$ (Theorem 14.4 [2]). ∎

The few other known examples of classes of functions satisfying the conditions of Corollary 1 include

(i) Euler's Γ-function, which is not differentially algebraic over $\mathbf{R}$ by Hölder's theorem [6] and generates a Hardy field [9];

(ii) Functions represented by a Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^x},$$

where $a_n \in \mathbf{R}$ are subpolynomial in n and the set of all prime divisors of n in the support of a_n [3] is infinite, e.g. the Riemann ζ-function on a positive half-line [9];

(iii) The function

$$\sum_{n=1}^{\infty} \frac{1}{e_n(x)},$$

where $e_1(x) = e^x$ and $e_n(x) = e^{e_{n-1}(x)}$ for $n > 1$ [9];

[1] E is an extension of Hardy's class L of logarithmico-exponential functions [5] and is the maximal scale for functions in Hardy fields (functions of *regular growth*).

[2] A *germ* is an equivalence class of functions, where two functions are equivalent exactly when they agree in a neighborhood of the point of interest, in our case $+\infty$.

[3] The *support* of a_n is the set of all n such that $a_n \neq 0$.

(iv) Certain fractional iterates of e^x [3];

(v) Certain ultimately [4] $\mathcal{C}^\infty$ transexponential solutions of two difference equations: $f(x+1) = e^{f(x)}$ and $f(x+1) = e^{f(x)} - 1$ [3].

References

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[9] M. Rosenlicht. *The rank of a Hardy field.* Trans. AMS, **280**:659–671, 1983.

[4] A property is said to hold *ultimately* exactly when it holds in a neighborhood of $+\infty$.