

# Interpolant splines in 1-d

## Local polynomial bases

### Linear

$$h_0(t) = 1 - t, \quad h_1(t) = t.$$

$$h'_0(t) = -1, \quad h'_1(t) = 1.$$

$$H_0(t) = t - \frac{1}{2}t^2, \quad H_1(t) = \frac{1}{2}t^2.$$

$$\Delta H_0 = \frac{1}{2}, \quad \Delta H_1 = \frac{1}{2}.$$

### Cubic

$$h_{00}(t) = 1 - 3t^2 + 2t^3, \quad h_{01}(t) = 3t^2 - 2t^3, \quad h_{10}(t) = t - 2t^2 + t^3, \quad h_{11}(t) = -t^2 + t^3.$$

$$h'_{00}(t) = -6t + 6t^2, \quad h'_{01}(t) = 6t - 6t^2, \quad h'_{10}(t) = 1 - 4t + 3t^2, \quad h'_{11}(t) = -2t + 3t^2.$$

$$h''_{00}(t) = -6 + 12t, \quad h''_{01}(t) = 6 - 12t, \quad h''_{10}(t) = -4 + 6t, \quad h''_{11}(t) = -2 + 6t.$$

$$h''_{00}(0) = -6, \quad h''_{00}(1) = 6, \quad h''_{01}(0) = 6, \quad h''_{01}(1) = -6, \quad h''_{10}(0) = -4, \quad h''_{10}(1) = 2, \quad h''_{11}(0) = -2, \quad h''_{11}(1) = 4.$$

$$H_{00}(t) = t - t^3 + \frac{1}{2}t^4, \quad H_{01}(t) = t^3 - \frac{1}{2}t^4, \quad H_{10}(t) = \frac{1}{2}t^2 - \frac{2}{3}t^3 + \frac{1}{4}t^4, \quad H_{11}(t) = -\frac{1}{3}t^3 + \frac{1}{4}t^4.$$

$$\Delta H_{00} = \frac{1}{2}, \quad \Delta H_{01} = \frac{1}{2}, \quad \Delta H_{10} = \frac{1}{12}, \quad \Delta H_{11} = -\frac{1}{12}.$$

### Quintic

$$q_{00}(t) = 1 - 10t^3 + 15t^4 - 6t^5, \quad q_{01}(t) = 10t^3 - 15t^4 + 6t^5, \quad q_{10}(t) = t - 6t^3 + 8t^4 - 3t^5,$$

$$q_{11}(t) = -4t^3 + 7t^4 - 3t^5, \quad q_{20}(t) = 0.5t^2 - 1.5t^3 + 1.5t^4 - 0.5t^5, \quad q_{21}(t) = 0.5t^3 - t^4 + 0.5t^5.$$

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$$q'_{00}(t) = -30t^2 + 60t^3 - 30t^4, \quad q'_{01}(t) = 30t^2 - 60t^3 + 30t^4, \quad q'_{10}(t) = 1 - 18t^2 + 32t^3 - 15t^4,$$

$$q'_{11}(t) = -12t^2 + 28t^3 - 15t^4, \quad q'_{20}(t) = t - 4.5t^2 + 6t^3 - 2.5t^4, \quad q'_{21}(t) = 1.5t^2 - 4t^3 + 2.5t^4.$$

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$$q''_{00}(t) = -60t + 180t^2 - 120t^3, \quad q''_{01}(t) = 60t - 180t^2 + 120t^3, \quad q''_{10}(t) = -36t + 96t^2 - 60t^3,$$

$$q''_{11}(t) = -24t + 84t^2 - 60t^3, \quad q''_{20}(t) = 1 - 9t + 18t^2 - 10t^3, \quad q''_{21}(t) = 3t - 12t^2 + 10t^3.$$

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$$q'''_{00}(t) = -60 + 360t - 360t^2, \quad q'''_{01}(t) = 60 - 360t + 360t^2, \quad q'''_{10}(t) = -36 + 192t - 180t^2,$$

$$q'''_{11}(t) = -24 + 168t - 180t^2, \quad q'''_{20}(t) = -9 + 36t - 30t^2, \quad q'''_{21}(t) = 3 - 24t + 30t^2.$$

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$$q'''_{00}(0) = -60, \quad q'''_{00}(1) = -60, \quad q'''_{01}(0) = 60, \quad q'''_{01}(1) = 60, \quad q'''_{10}(0) = -36, \quad q'''_{10}(1) = -24,$$

$$q'''_{11}(0) = -24, \quad q'''_{11}(1) = -36, \quad q'''_{20}(0) = -9, \quad q'''_{20}(1) = -3, \quad q'''_{21}(0) = 3, \quad q'''_{21}(1) = 9.$$

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$$q''''_{00}(t) = 360 - 720t, \quad q''''_{01}(t) = -360 + 720t, \quad q''''_{10}(t) = 192 - 360t,$$

$$q''''_{11}(t) = 168 - 360t, \quad q''''_{20}(t) = 36 - 60t, \quad q''''_{21}(t) = -24 + 60t.$$

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$$q''''_{00}(0) = 360, \quad q''''_{00}(1) = -360, \quad q''''_{01}(0) = -360, \quad q''''_{01}(1) = 360, \quad q''''_{10}(0) = 192, \quad q''''_{10}(1) = -168,$$

$$q''''_{11}(0) = 168, \quad q''''_{11}(1) = -192, \quad q''''_{20}(0) = 36, \quad q''''_{20}(1) = -24, \quad q''''_{21}(0) = -24, \quad q''''_{21}(1) = 36.$$

## Grid and local coordinates

$n$  nodes:  $t_0, t_1, \dots, t_{n-1}$

$$\Delta t_i = t_i - t_{i-1}, \quad \tau_i(t) = \frac{t - t_{i-1}}{\Delta t_i}, \quad i = 1, \dots, n-1$$

$$\tau_i' = \frac{1}{\Delta t_i}, \quad d\tau_i = \frac{dt}{\Delta t_i}, \quad dt = \Delta t_i d\tau_i$$

## Interpolating functions

### Linear

$$f(t) = h_0(\tau_i)x_{i-1} + h_1(\tau_i)x_i, \quad \tau_i = \tau_i(t), \quad t \in [t_{i-1}, t_i]$$

$$f'(t) = [h_0'(\tau_i)x_{i-1} + h_1'(\tau_i)x_i] \frac{1}{\Delta t_i} = [x_i - x_{i-1}] \frac{1}{\Delta t_i}$$

$$F(t) dt = [H_0(\tau_i)x_{i-1} + H_1(\tau_i)x_i] \Delta t_i + C$$

$$\int_{t_{i-1}}^{t_i} f(t) dt = [\Delta H_0 x_{i-1} + \Delta H_1 x_i] \Delta t_i$$

$$\int_{t_{i-1}}^t f(t) dt = [H_0(\tau_i)x_{i-1} + H_1(\tau_i)x_i] \Delta t_i$$

### Cubic

$$f(t) = h_{00}(\tau_i)x_{i-1} + h_{01}(\tau_i)x_i + [h_{10}(\tau_i)d_{i-1} + h_{11}(\tau_i)d_i] \Delta t_i$$

$$f'(t) = [h_{00}'(\tau_i)x_{i-1} + h_{01}'(\tau_i)x_i] \frac{1}{\Delta t_i} + h_{10}'(\tau_i)d_{i-1} + h_{11}'(\tau_i)d_i$$

$$f''(t) = [h_{00}''(\tau_i)x_{i-1} + h_{01}''(\tau_i)x_i] \frac{1}{\Delta t_i^2} + [h_{10}''(\tau_i)d_{i-1} + h_{11}''(\tau_i)d_i] \frac{1}{\Delta t_i}$$

$$F(t) dt = [H_{00}(\tau_i)x_{i-1} + H_{01}(\tau_i)x_i] \Delta t_i + [H_{10}(\tau_i)d_{i-1} + H_{11}(\tau_i)d_i] \Delta t_i^2 + C$$

$$\int_{t_{i-1}}^{t_i} f(t) dt = [\Delta H_{00} x_{i-1} + \Delta H_{01} x_i] \Delta t_i + [\Delta H_{10} d_{i-1} + \Delta H_{11} d_i] \Delta t_i^2$$

$$\int_{t_{i-1}}^t f(t) dt = [H_{00}(\tau_i)x_{i-1} + H_{01}(\tau_i)x_i] \Delta t_i + [H_{10}(\tau_i)d_{i-1} + H_{11}(\tau_i)d_i] \Delta t_i^2$$

### Quintic

$$f(t) = q_{00}(\tau_i)x_{i-1} + q_{01}(\tau_i)x_i + [q_{10}(\tau_i)d_{i-1} + q_{11}(\tau_i)d_i] \Delta t_i + [q_{20}(\tau_i)s_{i-1} + q_{21}(\tau_i)s_i] \Delta t_i^2$$

$$f'(t) = [q_{00}'(\tau_i)x_{i-1} + q_{01}'(\tau_i)x_i] \frac{1}{\Delta t_i} + q_{10}'(\tau_i)d_{i-1} + q_{11}'(\tau_i)d_i + [q_{20}'(\tau_i)s_{i-1} + q_{21}'(\tau_i)s_i] \Delta t_i$$

$$f''(t) = [q_{00}''(\tau_i)x_{i-1} + q_{01}''(\tau_i)x_i] \frac{1}{\Delta t_i^2} + [q_{10}''(\tau_i)d_{i-1} + q_{11}''(\tau_i)d_i] \frac{1}{\Delta t_i} + q_{20}''(\tau_i)s_{i-1} + q_{21}''(\tau_i)s_i$$

$$f'''(t) = [q_{00}'''(\tau_i)x_{i-1} + q_{01}'''(\tau_i)x_i] \frac{1}{\Delta t_i^3} + [q_{10}'''(\tau_i)d_{i-1} + q_{11}'''(\tau_i)d_i] \frac{1}{\Delta t_i^2} + [q_{20}'''(\tau_i)s_{i-1} + q_{21}'''(\tau_i)s_i] \frac{1}{\Delta t_i}$$

$$f''''(t) = [q_{00}''''(\tau_i)x_{i-1} + q_{01}''''(\tau_i)x_i] \frac{1}{\Delta t_i^4} + [q_{10}''''(\tau_i)d_{i-1} + q_{11}''''(\tau_i)d_i] \frac{1}{\Delta t_i^3} + [q_{20}''''(\tau_i)s_{i-1} + q_{21}''''(\tau_i)s_i] \frac{1}{\Delta t_i^2}$$

## Matrix assembly for splining

### Cubic

$$f''(t_{i-}) = [h''_{00}(1)x_{i-1} + h''_{01}(1)x_i] \frac{1}{\Delta t_i^2} + [h''_{10}(1)d_{i-1} + h''_{11}(1)d_i] \frac{1}{\Delta t_i}$$

$$f''(t_{i+}) = [h''_{00}(0)x_i + h''_{01}(0)x_{i+1}] \frac{1}{\Delta t_{i+1}^2} + [h''_{10}(0)d_i + h''_{11}(0)d_{i+1}] \frac{1}{\Delta t_{i+1}}$$

$$f''(t_{i-}) = f''(t_{i+}) \Rightarrow a_{i,i+1}d_{i+1} + a_{i,i}d_i + a_{i,i-1}d_{i-1} = b_i, \quad i = 1, \dots, n-2$$

$$a_{i,i+1} = \frac{h''_{11}(0)}{\Delta t_{i+1}}, \quad a_{i,i} = \frac{h''_{10}(0)}{\Delta t_{i+1}} - \frac{h''_{11}(1)}{\Delta t_i}, \quad a_{i,i-1} = -\frac{h''_{10}(1)}{\Delta t_i}$$

$$b_i = -\frac{h''_{01}(0)}{\Delta t_{i+1}^2}x_{i+1} + \left[ -\frac{h''_{00}(0)}{\Delta t_{i+1}^2} + \frac{h''_{01}(1)}{\Delta t_i^2} \right]x_i + \frac{h''_{00}(1)}{\Delta t_i^2}x_{i-1}$$

### Quintic

$$f'''(t_{i+}) = [q'''_{00}(0)x_i + q'''_{01}(0)x_{i+1}] \frac{1}{\Delta t_{i+1}^3} + [q'''_{10}(0)d_i + q'''_{11}(0)d_{i+1}] \frac{1}{\Delta t_{i+1}^2} + [q'''_{20}(0)s_i + q'''_{21}(0)s_{i+1}] \frac{1}{\Delta t_{i+1}}$$

$$f'''(t_{i-}) = [q'''_{00}(1)x_{i-1} + q'''_{01}(1)x_i] \frac{1}{\Delta t_i^3} + [q'''_{10}(1)d_{i-1} + q'''_{11}(1)d_i] \frac{1}{\Delta t_i^2} + [q'''_{20}(1)s_{i-1} + q'''_{21}(1)s_i] \frac{1}{\Delta t_i}$$

$$f''''(t_{i+}) = [q''''_{00}(0)x_i + q''''_{01}(0)x_{i+1}] \frac{1}{\Delta t_{i+1}^4} + [q''''_{10}(0)d_i + q''''_{11}(0)d_{i+1}] \frac{1}{\Delta t_{i+1}^3} + [q''''_{20}(0)s_i + q''''_{21}(0)s_{i+1}] \frac{1}{\Delta t_{i+1}^2}$$

$$f''''(t_{i-}) = [q''''_{00}(1)x_{i-1} + q''''_{01}(1)x_i] \frac{1}{\Delta t_i^4} + [q''''_{10}(1)d_{i-1} + q''''_{11}(1)d_i] \frac{1}{\Delta t_i^3} + [q''''_{20}(1)s_{i-1} + q''''_{21}(1)s_i] \frac{1}{\Delta t_i^2}$$

$$f'''(t_{i+}) = f'''(t_{i-}), \quad f''''(t_{i+}) = f''''(t_{i-}) \Rightarrow$$

$$a_{i,i+1}d_{i+1} + a_{i,i}d_i + a_{i,i-1}d_{i-1} + a_{i,n+i+1}s_{i+1} + a_{i,n+i}s_i + a_{i,n+i-1}s_{i-1} = b_i, \quad i = 1, \dots, n-2$$

$$a_{n+i,i+1}d_{i+1} + a_{n+i,i}d_i + a_{n+i,i-1}d_{i-1} + a_{n+i,n+i+1}s_{i+1} + a_{n+i,n+i}s_i + a_{n+i,n+i-1}s_{i-1} = b_{n+i}, \quad i = 1, \dots, n-2$$

$$a_{i,i+1} = \frac{q'''_{11}(0)}{\Delta t_{i+1}^2}, \quad a_{i,i} = \frac{q'''_{10}(0)}{\Delta t_{i+1}^2} - \frac{q'''_{11}(1)}{\Delta t_i^2}, \quad a_{i,i-1} = -\frac{q'''_{10}(1)}{\Delta t_i^2}$$

$$a_{i,n+i+1} = \frac{q'''_{21}(0)}{\Delta t_{i+1}}, \quad a_{i,n+i} = \frac{q'''_{20}(0)}{\Delta t_{i+1}} - \frac{q'''_{21}(1)}{\Delta t_i}, \quad a_{i,n+i-1} = -\frac{q'''_{20}(1)}{\Delta t_i}$$

$$b_i = -\frac{q'''_{01}(0)}{\Delta t_{i+1}^3}x_{i+1} + \left[ -\frac{q'''_{00}(0)}{\Delta t_{i+1}^3} + \frac{q'''_{01}(1)}{\Delta t_i^3} \right]x_i + \frac{q'''_{00}(1)}{\Delta t_i^3}x_{i-1}$$

$$a_{n+i,i+1} = \frac{q''''_{11}(0)}{\Delta t_{i+1}^2}, \quad a_{n+i,i} = \frac{q''''_{10}(0)}{\Delta t_{i+1}^2} - \frac{q''''_{11}(1)}{\Delta t_i^2}, \quad a_{n+i,i-1} = -\frac{q''''_{10}(1)}{\Delta t_i^2}$$

$$a_{n+i,n+i+1} = \frac{q''''_{21}(0)}{\Delta t_{i+1}}, \quad a_{n+i,n+i} = \frac{q''''_{20}(0)}{\Delta t_{i+1}} - \frac{q''''_{21}(1)}{\Delta t_i}, \quad a_{n+i,n+i-1} = -\frac{q''''_{20}(1)}{\Delta t_i}$$

$$b_{n+i} = -\frac{q''''_{01}(0)}{\Delta t_{i+1}^3}x_{i+1} + \left[ -\frac{q''''_{00}(0)}{\Delta t_{i+1}^3} + \frac{q''''_{01}(1)}{\Delta t_i^3} \right]x_i + \frac{q''''_{00}(1)}{\Delta t_i^3}x_{i-1}$$

$$s_i = d_{n+i}, \quad i = 0, \dots, n-1$$

## Computing cubic Hermite interpolant splines by iteration

First we find derivatives at the interior nodes using

$$x'_i = \frac{\Delta x_i \Delta t_{i-1}^2 + \Delta x_{i-1} \Delta t_i^2}{\Delta t_{i-1} \Delta t_i (\Delta t_{i-1} + \Delta t_i)} = a \frac{\Delta x_i}{\Delta t_i} + b \frac{\Delta x_{i-1}}{\Delta t_{i-1}}$$

where

$$a = \frac{\Delta t_{i-1}}{\Delta t_{i-1} + \Delta t_i}, \quad b = \frac{\Delta t_i}{\Delta t_{i-1} + \Delta t_i}$$

Now we re-compute the derivatives using

$$x'_i = \Delta \left[ \frac{3}{2} \left( \frac{\Delta x_{i-1}}{\Delta t_{i-1}^2} + \frac{\Delta x_i}{\Delta t_i^2} \right) - \frac{1}{2} \left( \frac{x'_{i-1}}{\Delta t_{i-1}} + \frac{x'_{i+1}}{\Delta t_i} \right) \right]$$

where

$$\Delta = \frac{1}{\frac{1}{\Delta t_{i-1}} + \frac{1}{\Delta t_i}} = \frac{\Delta t_{i-1} \Delta t_i}{\Delta t_i + \Delta t_{i-1}}$$

In other words

$$x'_i = p \frac{\Delta x_{i-1}}{\Delta t_{i-1}} + q \frac{\Delta x_i}{\Delta t_i} + r x'_{i-1} + s x'_{i+1}$$

where

$$p = \frac{3}{2} \frac{\Delta t_i}{\Delta t_{i-1} + \Delta t_i}, \quad q = \frac{3}{2} \frac{\Delta t_{i-1}}{\Delta t_{i-1} + \Delta t_i}, \quad r = -\frac{1}{2} \frac{\Delta t_i}{\Delta t_{i-1} + \Delta t_i}, \quad s = -\frac{1}{2} \frac{\Delta t_{i-1}}{\Delta t_{i-1} + \Delta t_i}$$