Newton's and Ritt's polygons

[from notes by Maxwell Rosenlicht]

Let k be an algebraically closed field and $F(t,y) \in k\,[[t,y]]$ such that $F(0,0)=0, F(0,y) \neq 0$. We want to show the existence of $\overline{y}(t) \in \bigcup_{\nu=1}^{\infty} k\left[\left[t^{1/\nu}\right]\right]$ such that $\overline{y}(0)=0$ and $F(t,\overline{y}(t))=0$.

Write $F(t,y)=\sum_{i,j\geq 0} a_{ij}t^iy^j$, each $a_{ij} \in k$. Then $a_{00}=0$ and there exists an $r>0$ such that $a_{0r} \neq 0$. If $\overline{y}(t)$ is as above, $\overline{y}(t)=ct^u+...$, with $u \in \mathbf{Q}, u>0, c \in k, c \neq 0$, then if $\gamma = \min_{a_{ij}\neq 0} ci+uj$, we have $\gamma \leq i+uj$ for all i,j such that $a_{ij} \neq 0$, $\gamma = i+uj$ for at least two (i,j) such that $a_{ij}\neq 0$, say $i_1+uj_1=i_2+uj_2$, $(i_1,j_1)\neq(i_2,j_2)$, $a_{i_1j_1}, a_{i_2j_2} \neq 0$, or $u=(i_2-i_1)/(j_1-j_2)$. Thus, u is the negative inverse of the slope of one of the edges of the lower part of the convex hull of the points $\{(i,j)\}_{a_{ij}\neq 0}$. Note that if there is no point on the x axis, i.e. all $a_{...}=0$, i. e. $F(t,0)=0$, we can take $\overline{y}(t)=0$.

Now suppose that $F(t,0)\neq 0$ and pick $u>0$ such that $\gamma=i+uj$ for at least two (i,j) such that $a_{ij}\neq 0$. Let $\varphi(t,y)=\sum_{i+uj=\gamma} a_{ij}t^iy^j$. The degree of $\varphi(1,y)\leq r$ and there exists $c\in k, c\neq 0$ such that $\varphi(1,c)=0$. If c is a zero of $\varphi(1,y)$ of multiplicity q, then $q\geq 1$ and $\varphi(1,y)=(y-c)^q\psi(y)$, where $\psi(y)\in k\,[\overline{y}]$ has degree $\partial^0\varphi(y)-q+\psi(c)\neq 0$. Set $y=t^u(c+u)$; we will want to find $\overline{u}(t)\in\bigcup_{\nu=1}^{\infty} k\left[\left[t^{1/\nu}\right]\right]$ such that $\overline{u}(0)=0$ and $\overline{y}(t)=t^u(c+\overline{u}(t))$ is such that $F(t,\overline{y}(t))=0$. Now

$$F(t,y)=F(t,t^u(c+u))=\sum_{i+uj=\gamma} a_{ij}t^i[t^u(c+u)]^j+\sum_{i+uj>\gamma} a_{ij}t^i[t^u(c+u)]^j=$$

$$=t^\gamma[\varphi(1,c+u)+\sum_{i+uj>\gamma} t^{i+uj-\gamma}a_{ij}(c+u)^j]=t^\gamma F_1(t^{1/\Upsilon},u),$$

where Υ is the denominator of u and $F_1(t^{1/\Upsilon},u)=u^q\psi(c+u)+t^{1/\Upsilon}\cdot\Phi$, where $\Phi\in k\left[t^{1/\Upsilon},u\right]$.

Finding $\overline{y}(t)$ reduces to finding $\overline{u}(t)$ such that $F_1(t^{1/\overline{\Upsilon}},\overline{u})=0$. Note that $F_1(0,0)=0$, $F_1(0,u)=u^q\psi(c+u)$ and $\gamma>0$ is a multiple of $1/\Upsilon$. Thus, we have the same kind of problem as before with r replaced by $q\leq r$. Suppose we apply this process repeatedly, to reduce r to a minimum. Then $q=r=\partial^0\varphi(1,y)$ and $\varphi(1,y)=(y-c)^r\psi$, where $\psi\in k, \psi\neq 0 (\psi=\psi(y))$. Since $\psi(1,y)\in k(y-c)$ has terms in $1,y,y^2,...,y^r$, the j's for which there is an i such that $a_{ij}\neq 0$ and $i+uj=\gamma$ include $0,1,...,r$ and therefore $\Upsilon=1$, i.e. $F_1(t^{1/\Upsilon},u)\in k\,[t,u]$. Thus after r reaches its minimum, no more fractional powers are introduced and, therefore, we can find our $\overline{y}(t)$ by continuing indefinitely. If we get to the point where $r=1$, then $\partial F/\partial y(0,0)\neq 0$, so we can invert our series to get directly $\overline{y}(t)\in k\,[[t]]$.

If, always, r remains <1, then taking above $\psi(y)=1$, we have $F(t,y)=(y-ct^u)^r+\Psi$, where $\Psi\in k\,[[t,y]]$ consists of terms of order $>ru$. If $\overline{y}(t)\sim c_1t^{u_1}+c_2t^{u_2}+...$, then

$$F(t,y)=(y-c_1t_1^u)^r+\Psi_1=(y-c_1t_1^u-c_2t_2^u)^r+\Psi_2=...,$$

where $\Psi_j\in k\,[[t,y]]$ consists of terms of order $>ru_j$ and $u_1<u_2<...$, each $u_i\in\mathbf{Z}$. Thus, we get $(y-\sum_{i=1}^{\infty}c_it^{u_i})^r \mid F(t,y)$, so F and $\partial F/\partial y$ have a common zero, which could have been excluded from the beginning. Note that by multiplying $F(t,y)$ at the beginning by a suitable unit in $k\,[[t,y]]$, we could have got $F(t,y)=\sum_{j=0}^{r}\sigma_jy^j$, where $\sigma_j\in k\,[[t]]$ and $\sigma_r\in k, \sigma_r\neq 0$.

In the Ritt polygon process [Ritt DA, pp. 57–62] we have the same sort of set-up, with F this time $\in k\,[[t,y,y',...]]$, topologized by powers of the ideal $(t,y_1,y_2,...)$ with k a differentially closed field; $t,y_1,y_2,...$ differentially independent; $t'=0, y_i'=y_{i+1}$ for all $i\geq 0$ (i. e. t is a transcendental constant over $k\{y\}$). Here we write $F=\sum at^i\Pi(y_k)$, where $a\in k$ and $\Pi(y_k)$ is a power product of the y_k, so j is the total y-degree. Ritt makes special choices of u (i. e. edge of polygon) and of the zero c of the differential polynomial $\varphi(1,y)$ (which may or may not be necessary) to make the process go through (it does!). Note that the same holds for *partial* differential fields.

Let k be a differentially closed partial differential field, t a transcendental constant over k, y a differential indeterminate over $k(t)$ and $F(t,y)\in k\,[[t,y,y',...]]$ topologized by powers of the ideal $(t,y_1,y_2,...)$ such that $F(0,0)=0, F(0,y)\neq 0$. We want to show the existence of $\overline{y}(t)\in\bigcup_{\nu=1}^{\infty}k\left[\left[t^{1/\nu}\right]\right]$ such that $\overline{y}(0)=0$ and $F(t,\overline{y}(t))=0$.

Write $F(t,y)=\sum_{i,j\geq 0}t^iA_{ij}(y)$, where $A_{ij}(y)\in k\,\{y\}$ is a form of homogeneous degree j in y and its various derivatives. Then $A_{00}=0$ and there exists an $r>0$ such that $A_{0r}\neq 0$. If all $A_{i0}=0$, then $F(t,0)=0$ and we can take $\overline{y}(t)=0$, so suppose not all $A_{i0}=0$. Then we can consider the lower part of the convex hull of the points $\{(i,j)\}_{A_{ij}\neq 0}$ to get a $u>0$ and $\gamma>0$ such that $\gamma=i+uj$ if $A_{ij}\neq 0$, with equality for at least two (i,j), say (i_1,j_1), (i_2,j_2). Note that $\gamma\leq u\gamma$ and $j_1,j_2\leq r$. Let $\varphi(t,y)=\sum_{i+uj=\gamma}t^iA_{ij}(y)$. The degree of $\varphi(1,y)\leq r$ and there exists $c\in k, c\neq 0$ such that $\varphi(1,c)=0$.