# Meromorphic differential systems

**Theorem:** *Let $A(z)$ be an $n \times n$ matrix function holomorphic for $|z| > x_0$, $z \in S$, where $S$ is a sector with vertex at the origin. Assume that $A(z)$ possesses an asymptotic series in powers of $z^{-1}$, as $z \to \infty$ in $S$. Then, corresponding to every sufficiently narrow subsector of $S$, the differential system $z^{-\tau}V' = A(z)V$ possesses a fundamental matrix solution of the form $V(z) = z^{G_1}e^{G_2(z)}U(z)$, where $G_1$ is a constant matrix, $G_2$ is a diagonal matrix whose entries are polynomials in $z^{\frac{1}{\varsigma}}$, $\varsigma \in \mathbf{Z}^+$, and $U(z)$ admits, as $z \to \infty$ in this subsector, an asymptotic series in powers of $z^{\frac{1}{\varsigma}}$.*

**Sketch of proof:** Suppose we have a linear system $z^{-q}Y' = AY$, with $A$ holomorphic at $\infty$. (or having asymptotic expansion in $z^{-1}$), such that $A(\infty)$ has not all the same eigenvalue. Block-diagonalize it. Then transform $Z = PY$, to get $z^{-q}Z' = BZ$, where not just $B(\infty)$, but $B$ is block-diagonal. One can get an asymptotic expansion for $P$ by formal considerations (plugging in and comparing terms).

To show that $P(z)$ exists, one writes a vector equation (first order quadratic, almost linear) for the entries and solves it with iterated variation of parameters. The integral in the variation of parameters is really a multiple one here, since we are dealing with matrices, and the paths are chosen differently each time to effect the various estimates. So far the width of the sector is $\pi/q + 1$ and position doesn't matter.

This reduces the problem by induction to the case where there is only one eigenvalue for $A(\infty)$. Now reduce to the case when $A(\infty)$ is nilpotent, i.e. the eigenvalue $= 0$. Note that we have gone back to calling the matrix $A$ and the vector $Y$. Transform $A(\infty)$ to Jordan canonical form. Then you need to find $P$ such that the transformation $Z = PY$ gives an asymptotic formula for $B = \sum B_k z^{-k}$, s.t. $B_0 = A_0$ (i.e. $P_0$ is the identity matrix) and the rest of $B_k$ have nonzero terms only in bottom rows corresponding to Jordan blocks. Now apply shearing transformations, $\mathrm{diag}(1, z^g, z^{2g}...)$. Pick $g$ according to lowest intersection of line with negative slope with the one with slope 1 out of lines expressing the dependence of the order of pole of entries of B as functions of $g$.

If you are lucky, you've reduced the problem. Otherwise there is even more mess to go through. In either case, $g$ will be a rational number, so we finally obtain solutions $Y = e^Q z^G W$, where $Q$ is diagonal, with coefficients polynomial in $z^{-1/p}$ ($p$ is the denominator of $g$), $G$ is a constant matrix, $W$ has an asymptotic expansion in $z^{-1/p}$ valid in some small sector whose size depends on $g$'s from one or more shearing transformations. ∎

**Example:** Suppose $F \in \mathbf{C}[z]$ and $D$ is a domain with polar coordinates $(p, q) = (\rho, \theta)$, $\rho > 1, -\alpha < \theta < \alpha$. We can transform the Riccati equation $W' + W^2 = F(z)^2$ to a $2 \times 2$ linear system as follows. First we change to a second order linear equation $Y'' = \Psi\, Y$, where $Y = e^{\int W}$ and $\Psi = F^2$. This is equivalent to

$$\begin{pmatrix} Y \\ Y' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ \Psi & 0 \end{pmatrix} \begin{pmatrix} Y \\ Y' \end{pmatrix}$$

If $F$ is a polynomial, $\Psi = F^2$ is a polynomial of degree $\tau$ which is twice the degree of $F$. i. e. $\Psi(z) = \sum_{k=0}^{\tau} a_k z^k$. Then

$$\begin{pmatrix} Y \\ Y' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ \sum_{k=0}^{\tau} a_k z^k & 0 \end{pmatrix} \begin{pmatrix} Y \\ Y' \end{pmatrix}$$

i. e.

$$z^{-\tau} \begin{pmatrix} Y \\ Y' \end{pmatrix}' = \begin{pmatrix} 0 & z^{-\tau} \\ \sum_{k=0}^{\tau} a_k z^{-\tau+k} & 0 \end{pmatrix} \begin{pmatrix} Y \\ Y' \end{pmatrix}$$

At infinity the matrix becomes

$$A(\infty) = \begin{pmatrix} 0 & 0 \\ a_\tau & 0 \end{pmatrix}$$

To convert to Jordan canonical form, we need a change of basis. Take a constant vector not in the kernel of $A(\infty)$, say $(1, 0)$ and apply the matrix $A(\infty)$ to obtain a second basis element $(0, a_\tau)$. Let

$$P = \begin{pmatrix} 0 & 1 \\ a_\tau & 0 \end{pmatrix}.$$

Then

$$P^{-1} = \begin{pmatrix} 0 & a_\tau^{-1} \\ 1 & 0 \end{pmatrix},$$

so

$$B = P^{-1}AP = \begin{pmatrix} 0 & a_\tau^{-1} \sum_{k=0}^{\tau} a_k z^{-\tau+k} \\ a_\tau z^{-\tau} & 0 \end{pmatrix}$$

and

$$B(\infty) = P^{-1}A(\infty)P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Finally,

$$z^{-\tau} P^{-1} \begin{pmatrix} Y \\ Y' \end{pmatrix}' = BP^{-1} \begin{pmatrix} Y \\ Y' \end{pmatrix},$$

i. e.

$$z^{-\tau} \begin{pmatrix} a_\tau^{-1} Y' \\ Y \end{pmatrix}' = B \begin{pmatrix} a_\tau^{-1} Y' \\ Y \end{pmatrix}.$$

It remains to work out the shearing transformation.

For example, in the special case of the polynomial being $z$ (Airy's equation), the system becomes

$$z^{-1} \begin{pmatrix} Y' \\ Y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ z^{-1} & 0 \end{pmatrix} \begin{pmatrix} Y' \\ Y \end{pmatrix}.$$

and by considering

$$\begin{pmatrix} 1 & 0 \\ 0 & z^g \end{pmatrix} \begin{pmatrix} 0 & 1 \\ z^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z^{-g} \end{pmatrix}$$

it's not too difficult to work out the following expansion:

$$\mathrm{Ai}(z) \sim \frac{z^{-\frac{1}{4}}}{2\sqrt{\pi}} e^{-\frac{2}{3} z^{\frac{3}{2}}} \sum_{k=0}^{\infty} a_k z^{-\frac{3}{2}k},$$

where

$$a_0 = 1, \quad a_k = \frac{(-1)^k}{k!\, 48^k} \prod_{j=1}^{k} (6j-5)(6j-1)$$

**Reference:**

[1] W. Wasow, *Asymptotic expansions for ordinary differential equations,* Wiley-Interscience, New York, 1965.