Differential transcendence of the gamma function

[from notes by Maxwell Rosenlicht]

Theorem (Hölder) *The gamma function* $\Gamma(x)$ *is differentially transcendental over the field of rational functions* $\mathbf{R}(x)$.

Proof: We follow [3], also see [1], [2]. For $\varphi(x) = \Gamma'(x)/\Gamma(x)$ we have

$$\varphi^{(n)}(x+1) = \varphi^{(n)}(x) + (-1)^n n!\, x^{-n-1}.$$

If there is a polynomial $p \in \mathbf{R}(x)[X_1, ..., X_n]$ such that $p(x, \varphi'(x), ..., \varphi^{(n)}(x)) = 0$, we choose p so that n is minimal and further that the total degree of p in X_i is minimal (so that p is irreducible). Then we normalize one of the leading coefficients to 1 (so that p is uniquely determined) and shift the argument by one to obtain

$$p(x, X_1, ..., X_n) = p(x+1, X_1 - x^{-2}, ..., X_n + (-1)^n n!\, x^{-n-1}).$$

Since the set of zeros and poles of a rational function cannot be invariant under translation, $p \notin \mathbf{R}(x) - \mathbf{R}$. The set of all $q \in \mathbf{R}(x)[X_1, ..., X_n]$ satisfying the above formula for p is closed under partial differentiation with respect to X_i, so if p were not a constant, there would be a linear such polynomial $q = a(x)X_i + b(x)$. For this q we would get

$$a(x)X_i + b(x) = a(x+1)(X_i + (-1)^i i!\, x^{-i-1}) + b(x+1),$$

so $a(x+1) = a(x), a(x) = a \in \mathbf{R}^*$ and

$$b(x) = (-1)^i a i!\, x^{-i-1} + b(x+1).$$

In particular, $b(x)$ is not a polynomial. However, any $c \in \mathbf{C}^*$ is a pole of $b(x)$ if and only if $c+1$ is, which happens only for rational functions that are actually polynomials. This shows that p must have been a constant, so $\text{tr.deg}_{\mathbf{R}}\mathbf{R}(\varphi'(x), \varphi''(x), ...) = \infty$, thus $\text{tr.deg}_{\mathbf{R}}\mathbf{R}(x, \Gamma(x), \Gamma'(x), ...) = \infty$.

References

[1] S. B. Bank and R. B. Kaufman. *A note on Hölder's theorem concerning the gamma function.* Math. Ann., **232**:115–120, 1978.

[2] E. W. Barnes. *The theory of the Gamma-function.* Messenger of Mathematics, **XXIX**:64–128, 1900.

[3] M. Rosenlicht. *The rank of a Hardy field.* Trans. AMS, **280**:659–671, 1983.