(1) $$x(t) = x_0 + \int_0^t f(x(s))\,ds$$

$$x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} ds$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} ds =$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ t \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} ds$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} -s \\ 1 \end{bmatrix} ds =$$

$$= \begin{bmatrix} 1 - t^2/2 \\ t \end{bmatrix}$$

$$x_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 - s^2/2 \\ s \end{bmatrix} ds$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} -s \\ 1 - s^2/2 \end{bmatrix} ds$$

$$= \begin{bmatrix} 1 - t^2/2 \\ t - \frac{t^3}{3!} \end{bmatrix} \begin{matrix} \to \cos t \\ \cdots \to \sin t \end{matrix}$$

(2) a) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \underbrace{\begin{bmatrix} -x_1 \\ -x_2 + x_1^3 \end{bmatrix}}_{f}$

c) ☺

$Df = \begin{bmatrix} -1 & 0 \\ 3x_1^2 & -1 \end{bmatrix}$

$x_1 = y_1 e^{-t}$

$x_2' + x_2 = y_1^3 e^{-3t}$

$e^t x_2' + e^t x_2 = y_1^3 e^{-2t}$

$\quad (e^t x_2)'$

$e^t x_2 = y_1^3 \dfrac{e^{-2t}}{-2} + C$

$x_2 = -\dfrac{y_1^3}{2} e^{-3t} + Ce^{-t}$

$y_2 = -\dfrac{y_1^3}{2} + C$

$x_2 = -\dfrac{y_1^3}{2} e^{-3t} + \left(y_2 + \dfrac{y_1^3}{2}\right) e^{-t}$

$$\boxed{u(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 e^{-t} \\ -\dfrac{y_1^3}{2} e^{-3t} + \left(y_2 + \dfrac{y_1^3}{2}\right) e^{-t} \end{bmatrix}}$$

$$\Phi = \frac{\partial u}{\partial y} = \begin{bmatrix} e^{-t} & 0 \\ -\dfrac{3y_1^2}{2} e^{-3t} + \dfrac{3y_1^2}{2} e^{-t} & e^{-t} \end{bmatrix}$$

$\Phi(0, y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ☺

Want: $\Phi' = Df\, \Phi$

d)

$$Df\,\bar{\Phi} = \begin{bmatrix} -1 & 0 \\ 3x_1^2 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ -\frac{3y_1^2}{2}e^{-3t} + \frac{3y_1^2}{2}e^{-t} & e^{-t} \end{bmatrix}$$

$$3y_1^2 e^{-2t}$$

$$\begin{bmatrix} -e^{-t} & 0 \\ \underbrace{3y_1^2 e^{-3t} + 3\frac{y_1^2}{2}e^{-3t}}_{\frac{9}{2}y_1^2 e^{-3t}} - \frac{3y_1^2}{2}e^{-t} & -e^{-t} \end{bmatrix}$$

$$\Phi = \frac{\partial u}{\partial y} = \begin{bmatrix} e^{-t} & 0 \\ -\frac{3y_1^2}{2}e^{-3t} + \frac{3y_1^2}{2}e^{-t} & e^{-t} \end{bmatrix}$$

$$\Phi' = \begin{bmatrix} -e^{-t} & 0 \\ \frac{9}{2}y_1^2 e^{-3t} - \frac{3y_1^2}{2}e^{-t} & -e^{-t} \end{bmatrix}$$

③ $x' = x^3 \qquad x(0) = y$

$$\int \frac{dx}{x^3} = \int dt$$

$$-\frac{1}{2x^2} = t + C \qquad -\frac{1}{2y^2} = C$$

$$= t - \frac{1}{2y^2}$$

$$-2x^2 = \frac{1}{t - \frac{1}{2y^2}} = \frac{2y^2}{2y^2 t - 1}$$

$$x^2 = \frac{y^2}{1 - 2y^2 t} \qquad x = \frac{y}{\sqrt{1-2y^2 t}}$$

for $y \neq 0$

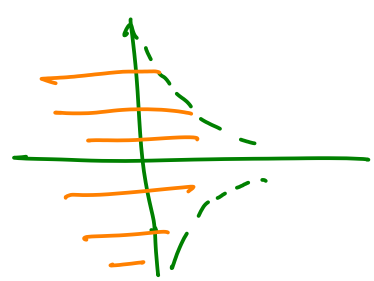
$x \equiv 0$ for $y = 0$.

$$J(y) = \{ t : 1 - 2y^2 t > 0 \}$$

$$1 - 2y^2 t = 0$$

$$\Rightarrow \quad y^2 = \frac{1}{2t}$$

$$y = \pm \frac{1}{\sqrt{2t}}$$



④ $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -x_1 \\ 2x_2 + x_1^2 \end{bmatrix}$$

$$x_1 = y_1 e^{-t}$$

$$x_2' - 2x_2 = y_1^2 e^{-2t}$$

$$\underbrace{e^{-2t} x_2' - 2e^{-2t} x_2}_{(e^{-2t} x_2)'} = y_1^2 e^{-4t}$$

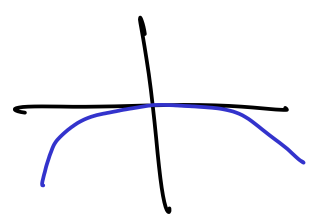
$$e^{-2t} x_2 = -\frac{1}{4} y_1^2 e^{-4t} + c$$

$$x_2 = -\frac{1}{4} y_1^2 e^{-2t} + c e^{2t}$$

$$= -\frac{1}{4} y_1^2 e^{-2t} + \left(\frac{1}{4} y_1^2 + y_2\right) e^{2t}$$

$$\boxed{\underset{\varphi_t(y)}{\underset{\shortparallel}{u(t,y)}} = \begin{bmatrix} y_1 e^{-t} \\ -\frac{1}{4} y_1^2 e^{-2t} + \left(\frac{1}{4} y_1^2 + y_2\right) e^{2t} \end{bmatrix}}$$

b) $S = \{x \in \mathbb{R}^2 : x_2 = -x_1^2/4\}$



Assume $x_2 = -\frac{x_1^4}{4}$

Compute u_2 and $-\frac{u_1^4}{4} = -\frac{y_1^2 e^{-2t}}{4^2}$

$$-\frac{1}{4} y_1^2 e^{-2t} + \left(\frac{1}{4} y_1^2 + \underbrace{y_2}_{-y_1^2/4}\right) e^{2t} = -\frac{1}{4} y_1^2 e^{-2t}$$

(5) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \underbrace{\begin{bmatrix} x_1^2 - x_2^2 - 1 \\ 2x_2 \end{bmatrix}}_{f}$

a) Equi. pts: $2x_2 = 0 \Rightarrow x_2 = 0$

$x_1^2 - x_2^2 - 1$ becomes $x_1^2 - 1$

$x_1 = \pm 1$

$\boxed{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}}$

$Df = \begin{bmatrix} 2x_1 & -2x_2 \\ 0 & 2 \end{bmatrix}$

$Df\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ source

$Df\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$ saddle