Midterm 1 MAT 5313 Sp '14

1. Let $\theta : B \times A \to A \times B$ be the bijection: $\theta(a,b) = [b,a]$

Then $\otimes\theta : B \times A \to A \otimes B$ is bilinear:

$\otimes\theta(kb + k'b', a) = \otimes(a, kb + k'b') = k\, a\otimes b + k'\, a \otimes b'$

$= k \otimes\theta(b,a) + k' \otimes \theta(b',a)$, similarly in the other var. ☺

If $f : B \times A \to C$ is bilinear, so is $f\theta^{-1}$.

$$\begin{tikzcd} B\times A \arrow[d, "\theta"] \arrow[dr, "f"] & \\ A\times B \arrow[d, "\otimes"] & C \\ A\otimes B \arrow[ur, dashed, "\exists!\,\phi"'] & \end{tikzcd}$$

Proof is similar to above, since it only uses bilinearity of $\otimes$.

By universality of $\otimes$ $\exists!\, \phi$ $\phi\otimes = f\theta^{-1}$

Then $\phi \otimes \theta = f$, so $\otimes\theta$ is universal.

Since universal objects are isomorphic, $A \otimes B \cong B \otimes A$ ☺

2. Let $d = \gcd(m,n)$ and $f : \mathbb{Z}_m \times \mathbb{Z}_n \to V$ bilinear.

Bèzout $\Rightarrow \exists\, s,t \in \mathbb{Z}$ $d = sm + tn$.

Then $d\, f(1,1) = sm\, f(1,1) + tn\, f(1,1) =$

$= s f(m,1) + t f(1,n) = s f(0,1) + t f(1,0) = 0$ $(*)$

Suppose $\phi : \mathbb{Z}_m \times \mathbb{Z}_n \to \mathbb{Z}_d$ with $\phi(x,y) = xy \bmod d$.

If $x \equiv x' \bmod m$, then $x \equiv x' \bmod d$, so $xy \equiv x'y \bmod d$.

Similarly with y's, so ϕ is well defined and clearly bilinear.

Define $\theta : \mathbb{Z} \to V$ by $\theta(k) = k f(1,1)$. Then

θ is clearly linear. By $(*)$, $d\mathbb{Z} \subseteq \ker\theta$, so

$\exists!$ linear $\psi : \mathbb{Z}_d \to V$ with $\psi([k]_d) = k f(1,1)$

$\psi(\phi(x,y)) = \psi([xy]_d) = xy\, f(1,1) = f(x,y)$.

If $\psi' \circ \phi = f$, then $\psi'(1) = \psi'(\phi(1,1)) = f(1,1) = \psi(1)$.

$\therefore \psi' = \psi$, so ψ is unique, so ϕ is universal. ☺

3. Let b_1, b_2 be a basis for F and let $f \in \mathrm{Alt}_2 F$

Given $x_1, x_2 \in F$ $\exists!$ $k_{11}, k_{12}, k_{21}, k_{22} \in K$ with

$x_1 = k_{11} b_1 + k_{12} b_2$, $x_2 = k_{21} b_1 + k_{22} b_2$

Since f is bilinear, $f(x_1, x_2) = k_{11} k_{21} f(b_1, b_1) +$

$k_{11} k_{22} f(b_1, b_2) + k_{12} k_{21} f(b_2, b_1) + k_{12} k_{22} f(b_2, b_2)$

Since f is alternating, $f(b_1, b_1) = f(b_2, b_2) = 0$ and

$f(b_1, b_2) = - f(b_2, b_1)$, so $f(x_1, x_2) = (k_{11} k_{22} - k_{12} k_{21}) f(b_1, b_2)$

$\therefore$ Any $f \in \mathrm{Alt}_2 F$ is a scalar multiple of det. ☺

4. Let $i : \mathbb{Z}_2 \to \mathbb{Z}_4$ be the injection $x \mapsto 2x$

Apply the functor $\_ \otimes \mathbb{Z}_2$, then

$(i \otimes 1)(1 \otimes 1) = 2 \otimes 1 = 2(1 \otimes 1) = 1 \otimes 2 = 1 \otimes 0 = 0$

$\therefore$ $i \otimes 1$ is not injective. ☺

5. a) $\mathbb{Z}^2 \otimes \mathbb{Z}^4 = (\mathbb{Z}^2)^4 = \mathbb{Z}^8$ $\quad (A \otimes K^n \cong A^n)$

b) $\mathbb{Z}^2 \otimes \mathbb{Z}_4 \cong (\mathbb{Z}_4)^2$

c) $\mathbb{Z}_2 \otimes \mathbb{Z}_4 \cong \mathbb{Z}_2$ $\quad$ (by 2, $\mathbb{Z}_2 \times \mathbb{Z}_4 \to \mathbb{Z}_2$ given by $[x, y] \mapsto xy$ is universal bilinear)

d) $\mathbb{Z}^3 \otimes \mathbb{Q} \cong \mathbb{Q}^3$

e) $\mathbb{Z}_3 \otimes \mathbb{Q} \cong 0$ $\quad$ ($a \otimes b = a \otimes \frac{3b}{3} = 3(a \otimes \frac{b}{3}) =$

$= (3a) \otimes (\frac{b}{3}) = 0 \otimes \frac{b}{3} = 0$ ☺)