Final

MAT 5313 Sp. '14

(1) Define $f : A \times [B_1 \oplus B_2] \to \bigoplus_{j=1}^{2} A \otimes B_j$ by $f(a,[b_1,b_2]) = [a\otimes b_1, a \otimes b_2]$

Then f is bilinear.

$$f(ma+m'a', [b_1,b_2]) = [(ma+m'a')\otimes b_1, (ma+m'a')\otimes b_2]$$
$$= [m(a\otimes b_1)+m'(a'\otimes b_1), m(a\otimes b_2)+m'(a'\otimes b_2)] =$$
$$= m[a\otimes b_1, a\otimes b_2] + m'[a'\otimes b_1, a'\otimes b_2] = m f(a,[b_1,b_2]) + m' f(a',[b_1,b_2])$$

$$f(a, m[b_1,b_2]+m'[b_1',b_2']) = f(a,[mb_1+m'b_1', mb_2+m'b_2']) =$$
$$= [a\otimes(mb_1+m'b_1'), a\otimes(mb_2+m'b_2')] = [m(a\otimes b_1)+m'(a\otimes b_1'), m(a\otimes b_2)+m'(a\otimes b_2')] =$$
$$= m[a\otimes b_1, a\otimes b_2] + m'[a\otimes b_1', a\otimes b_2'] = m f(a,[b_1,b_2]) + m' f(a,[b_1',b_2'])$$

$\therefore$ by the univ. prop. of $\otimes$ $\exists!$ linear map $\theta : A\otimes[B_1\oplus B_2] \to \bigoplus_{j=1}^{2} A\otimes B_j$

$$\begin{tikzcd} A\otimes[B_1\oplus B_2] \arrow[r, "\theta"] & \bigoplus_{j=1}^{2} A\otimes B_j \\ A\times[B_1\oplus B_2] \arrow[u, "\otimes"] \arrow[ur, "f"'] & \end{tikzcd}$$

By the univ. prop. of $\oplus$ $\exists!\,\psi$ $\psi e_i = \psi_i$,

$$\begin{tikzcd} A\otimes B_i \arrow[r, "e_i"] \arrow[dr, "\psi_i"'] & \bigoplus_{j=1}^{2} A\otimes B_j \arrow[d, dashed, "\exists!\,\psi"] \\ & A\otimes(B_1\oplus B_2) \end{tikzcd}$$

where $\psi_i = A\otimes \varepsilon_i$ and $\varepsilon_i : B_i \to B_1\oplus B_2$ are the usual inclusions, i.e.

$$\psi(a_1\otimes b_1, a_2\otimes b_2) = a_1\otimes[b_1,0] + a_2\otimes[0,b_2]$$

Claim: θ & ψ are inverses. As usual it is enough to check pure tensors.

$$\theta\psi(a_1\otimes b_1, a_2\otimes b_2) = [a_1\otimes b_1, 0] + [0, a_2\otimes b_2] = [a_1\otimes b_1, a_2\otimes b_2]$$

$$\psi\theta(a\otimes[b_1,b_2]) = a\otimes[b_1,0] + a\otimes[0,b_2] = a\otimes([b_1,0]+[0,b_2]) = a\otimes[b_1,b_2]$$

(2) Let b_1, b_2, b_3 be a basis for F and let $f \in Alt_2(F)$

Given $x_1, x_2 \in F$ $\exists!$ k_{ij}, $i=1,2$, $j=1,2,3$

$$x_1 = k_{11}b_1 + k_{12}b_2 + k_{13}b_3, \quad x_2 = k_{21}b_1 + k_{22}b_2 + k_{23}b_3$$

Since f is bilinear $f(x_1,x_2) = k_{11}k_{21}f(b_1,b_1) + k_{11}k_{22}f(b_1,b_2) +$
$+ k_{11}k_{23}f(b_1,b_3) + k_{12}k_{21}f(b_2,b_1) + k_{12}k_{22}f(b_2,b_2) + k_{12}k_{23}f(b_2,b_3) +$
$+ k_{13}k_{21}f(b_3,b_1) + k_{13}k_{22}f(b_3,b_2) + k_{13}k_{23}f(b_3,b_3)$.

Since f is alternating $f(x_1,x_2) = (k_{12}k_{23} - k_{13}k_{22})f(b_2,b_3) +$
$+ (k_{13}k_{21} - k_{11}k_{23})f(b_3,b_1) + (k_{11}k_{22} - k_{12}k_{21})f(b_1,b_2)$.

$\therefore$ The 3 determinant maps ($[x_1,x_2] \mapsto k_{12}k_{23} - k_{13}k_{22}$, etc.) form a basis for $Alt_2(F)$. ☺

③ Let $G < F \setminus \{0\}$, $n = \# G$. Basis of induction: $n=1$: $\{1\}$ is cyclic.
By the classification theorem for finite abelian groups,
$G \cong \mathbb{Z}_{p^m} \oplus \mathbb{Z}_{q_1^{n_1}} \oplus \ldots \oplus \mathbb{Z}_{q_k^{n_k}}$. By induction $H = \mathbb{Z}_{q_1^{n_1}} \oplus \ldots \oplus \mathbb{Z}_{q_k^{n_k}}$
is cyclic: $H \cong \mathbb{Z}_r$. Then $n = p^m \cdot r$. Let $\ell = \operatorname{lcm}(p^m, r)$.
If G is not cyclic, $p \mid r$, so $\ell < n$, but every element of G
satisfies $x^\ell - 1 = 0$, so $n \leq \ell$ ☹

④ $Z(S_5)$ is trivial. Conjugacy classes:

2-cycles $\binom{5}{2} = 10$

3-cycles $\binom{5}{3} \cdot 2 = 20$

5-cycles $5!/5 = 24$, 4-cycles $\binom{5}{4} 4!/4 = 30$

products of 2 disjoint 2-cycles $\binom{5}{4} \cdot 3 = 15$

products of a 3-cycle and a disj. 2-cycle: 20

$\boxed{1+10+20+30+24+15+20 = 120 = 5!}$

⑤ $GL(2, \mathbb{Z}_2) = \left\{ A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, A_4 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, A_5 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$

Center $= \{A_0\}$. For $i \neq 2, 5$ $A_i^{-1} = A_i$ and $A_2^{-1} = A_5$. More computations:

$A_2^{-1} A_1 A_2 = A_4$ $\quad A_4^{-1} A_1 A_4 = A_3$ $\quad A_1^{-1} A_2 A_1 = A_5$

$A_3^{-1} A_1 A_3 = A_4$ $\quad A_5^{-1} A_1 A_5 = A_3$

$\boxed{1+3+2=6}$

$\therefore$ conjugacy classes are $\{A_1, A_3, A_4\}$ and $\{A_2, A_5\}$

(6) $u^3 = u\cdot u^2 = u(-5u-1) = -5u^2 - u = -5(-5u-1) - u = \underline{24u+5}$

Let $w = 1-u$, then $u = 1-w$, $(1-w)^2 + 5(1-w) + 1 = 0 \Rightarrow$

$w^2 - 7w = -7 \Rightarrow w(w-7) = -7$, $\therefore (1-u)^{-1} = -\frac{1}{7}(w-7) = \underline{\frac{1}{7}u + \frac{6}{7}}$

(7) Let $w = 24u+5$, then $u = \frac{1}{24}w - \frac{5}{24}$, so

$\left(\frac{1}{24}w - \frac{5}{24}\right)^2 + 5\left(\frac{1}{24}w - \frac{5}{24}\right) + 1 = 0 \Rightarrow \underline{w^2 + 110w + 1 = 0}$

Let $w = \frac{1}{7}u + \frac{6}{7}$, then $u = 7w - 6$, so

$(7w-6)^2 + 5(7w-6) + 1 = 0 \Rightarrow \underline{w^2 - w + \frac{1}{7} = 0}$

(8) Let $p(x) = x^3 - 2$. Roots: $2^{1/2}\omega^k$, $k = 0, 1, 2$,
where $\omega = e^{i2\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Splitting field: $L = \mathbb{Q}(2^{1/2}, \omega)$.

(It's easy to see roots $\in L$. Conversely $2^{1/2}$ is a root & $\omega = 2^{1/2}\omega / 2^{1/2}$)

Since $[L:\mathbb{Q}] = [L, \mathbb{Q}(2^{1/2})][\mathbb{Q}(2^{1/2}):\mathbb{Q}] = 3\cdot 2 = 6 = 3!$
the Galois group is all of S_3. ☺

Have a great summer!