1. Let b_1, b_2 be a basis for F and let $f \in \mathrm{Alt}_2 F$. Given $x_1, x_2 \in F$, there exist unique $\kappa_{11}, \kappa_{12}, \kappa_{21}, \kappa_{22} \in K$ such that $x_1 = \kappa_{11} b_1 + \kappa_{12} b_2$ and $x_2 = \kappa_{21} b_1 + \kappa_{22} b_2$.

   Since f is bilinear, $f(x_1, x_2) = f(\kappa_{11} b_1 + \kappa_{12} b_2, \kappa_{21} b_1 + \kappa_{22} b_2) = \kappa_{11}\kappa_{21} f(b_1, b_1) + \kappa_{11}\kappa_{22} f(b_1, b_2) + \kappa_{12}\kappa_{21} f(b_2, b_1) + \kappa_{12}\kappa_{22} f(b_2, b_2)$. Since f is alternating, $f(b_1, b_1) = f(b_2, b_2) = 0$ and $f(b_2, b_1) = -f(b_1, b_2)$, so $f(x_1, x_2) = (\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21}) f(b_1, b_2)$.

   Thus, any $f \in \mathrm{Alt}_2 F$ is a scalar multiple of the 2×2 determinant map and the scalar in question is unique. Therefore, the determinant map forms a basis for $\mathrm{Alt}_2 F$.

2. Choose a basis $(b_i, i = 1..n)$ for F. We may represent L by a matrix $A = (a_{ij})$, whose columns are the coefficients of the images of the basis elements under L expanded in the same basis, i.e. $L(b_i) = \sum_i a_{ij} b_j$. The identity map is represented by the identity matrix, also denoted by I.

   The map $\lambda I - L$ fails to be bijective $\Leftrightarrow$ the matrix $\lambda I - A$ is not invertible $\Leftrightarrow$ $\det(\lambda I - A)$ is not a unit in K. Since K is a field, this is equivalent to $\det(\lambda I - A) = 0$, which is a polynomial equation in λ

$$\sum_{\sigma \in \Sigma_n} (-1)^{\operatorname{sgn} \sigma} \prod_{i=1}^{n} [\lambda \delta_{\sigma(i)i} - a_{\sigma(i)i}] = 0$$

   The highest power of λ occurs when σ is the identity permutation, i.e. when taking the product of the diagonal entries $\prod_i (\lambda - a_{ii}) = \lambda^n + \ldots$. Thus, the leading coefficient of the polynomial is 1, so the equation is nontrivial and therefore has finitely many solutions.

   Note: $\det(\lambda I - A) = \lambda^n - \operatorname{tr} A\, \lambda^{n-1} + \ldots + (-1)^n \det A$ is called the *characteristic polynomial* of A.

3. (a) Let $a, b \in A$ and $\alpha, \beta \in K$. Then $f(\alpha a + \beta b, (\kappa, \lambda)) = (\kappa(\alpha a + \beta b), \lambda(\alpha a + \beta b)) = (\kappa\alpha a + \kappa\beta b, \lambda\alpha a + \lambda\beta b) = \alpha(\kappa a, \lambda a) + \beta(\kappa b, \lambda b) = \alpha f(a, (\kappa, \lambda)) + \beta f(b, (\kappa, \lambda))$.

   On the other hand, $f(a, \alpha(\kappa, \lambda) + \beta(\mu, \nu)) = f(a, (\alpha\kappa + \beta\mu, \alpha\lambda + \beta\nu)) = ((\alpha\kappa + \beta\mu)a, (\alpha\lambda + \beta\nu)a) = (\alpha\kappa a + \beta\mu a, \alpha\lambda a + \beta\nu a) = \alpha(\kappa a, \lambda a) + \beta(\mu a, \nu a) = \alpha f(a, (\kappa, \lambda)) + \beta f(a, (\mu, \nu))$.

   (b) Suppose $g\colon A \times K^2 \to C$ is bilinear. If we expect g' to be linear and $g = g' \circ f$, we *must* have $g'(a, b) = g'((a, 0) + (0, b)) = g'(a, 0) + g'(0, b) = g'(f[a, (1, 0)]) + g'(f[b, (0, 1)]) = g(a, (1, 0)) + g(b, (0, 1))$, so define $g'(a, b) = g(a, (1, 0)) + g(b, (0, 1))$.

   Linearity: $g'(\mu(a, b) + \nu(c, d)) = g'(\mu a + \nu c, \mu b + \nu d) = g(\mu a + \nu c, (1, 0)) + g(\mu b + \nu d, (0, 1)) = \mu g(a, (1, 0)) + \nu g(c, (1, 0) + \mu g(b, (0, 1)) + \nu g(d, (0, 1)) = \mu(g[a, (1, 0)] + g[b, (0, 1)]) + \nu(g[c, (1, 0)] + g[d, (0, 1)]) = \mu g'(a, b) + \nu g'(c, d)$.

   Note: The universality of f, in particular, implies $A \otimes_K K^2 \cong A^2$, which is a special case of ((33), p. 322).

4. (a) $\mathbf{Z}^2 \otimes \mathbf{Z}^3 \cong \mathbf{Z}^6$
   (b) $\mathbf{Z}^2 \otimes \mathbf{Z}_3 \cong (\mathbf{Z}_3)^2$
   (c) $\mathbf{Z}_2 \otimes \mathbf{Z}_3 \cong 0$
   (d) $\mathbf{Z}^2 \otimes \mathbf{Q} \cong \mathbf{Q}^2$
   (e) $\mathbf{Z}_2 \otimes \mathbf{Q} \cong 0$

   Notes: Parts (a), (b), and (d) are special cases of $A \otimes_K K^n \cong A^n$ ((33), p. 322).

   (c) Since $-1 \equiv 2 \bmod 3$, any pure tensor $a \otimes b = a \otimes (-2b) = (-2a) \otimes b = 0 \otimes b = 0$, but the tensor product is generated by pure tensors. By the way, this is a special case of $\mathbf{Z}_m \otimes \mathbf{Z}_n \cong \mathbf{Z}_{\gcd(m,n)}$ (IX.8.1a).

   (e) $a \otimes b = a \otimes 2b/2 = 2a \otimes b/2 = 0 \otimes b/2 = 0$.