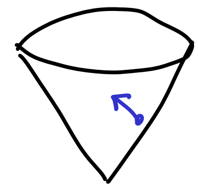
Algebraic Topology MAT 5263 Midterm 2 Sp. 2013

① Parametrize the cone:

$$\Phi(r,\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ r \end{bmatrix} \qquad \begin{matrix} 0 \le r \le 2 \\ -\pi < \theta \le \pi \end{matrix}$$



$$dx = \cos\theta\, dr - r\sin\theta\, d\theta$$
$$dy = \sin\theta\, dr + r\cos\theta\, d\theta$$
$$dz = dr$$

$$d\vec{S} = \begin{bmatrix} dy\,dz \\ dz\,dx \\ dx\,dy \end{bmatrix} = \begin{bmatrix} -r\cos\theta \\ -r\sin\theta \\ r \end{bmatrix} dr\,d\theta = r\begin{bmatrix} -\cos\theta \\ -\sin\theta \\ 1 \end{bmatrix} dr\,d\theta$$

$$\omega = -y\,dy\,dz + x\,dz\,dx + z\,dx\,dy$$

$$\omega^* = \cancel{-r\sin\theta \cdot r(-\cos\theta)} + \cancel{r\cos\theta \cdot r(-\sin\theta)} + r\cdot r\,dr\,d\theta$$
$$= r^2\,dr\,d\theta$$

$$\int \omega = \int_{-\pi}^{\pi}\int_0^2 r^2\,dr\,d\theta = \int_{-\pi}^{\pi} \frac{r^3}{3}\Big|_0^2\,d\theta = \frac{8}{3}\cdot 2\pi = \boxed{\frac{16\pi}{3}}$$

Alt.: $\displaystyle\int_{\text{cone}} \omega + \int_{\text{disc}} \omega = \int_{\partial\,\text{solid cone}} \omega \underset{\text{FTC}}{=} \int_{\text{solid cone}} d\omega = \int dx\,dy\,dz = \text{vol} = \frac{1}{3}2\pi 2^2 = \frac{8\pi}{3}$

Since on the top disc $z=2$, $dz = 0$ $\displaystyle\int_{\text{disc}} \omega = 2\int_{\text{disc}} dx\,dy = 2\pi 2^2 = 8\pi$

$\therefore \displaystyle\int_{\text{cone}} \omega = \frac{8\pi}{3} - 8\pi = \boxed{-\frac{16\pi}{3}}$ (− because of opposite orientation)

(2)

$$\begin{tikzcd} A \arrow[r, "\phi"] & B \arrow[r, "\pi"] \arrow[dr, "f"'] & \dfrac{B}{\phi(A)} = \operatorname{coker}\phi \arrow[d, dashed, "\exists!\,\psi"] \\ & & C \end{tikzcd}$$

$\exists!\,\psi$ s.t. $f = \psi\pi$

Since $\ker \pi = \phi(A)$, $\pi\phi = 0$

Suppose $f: B \to C$ with $f\phi = 0$

Let $x + \phi(A) \in \operatorname{coker}\phi$. $(x \in B)$

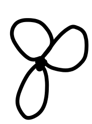
Define $\psi(x) = f(x)$. If $x - x' \in \phi(A)$,

since $f\phi = 0$, $f(x - x') = 0$, so $f(x) = f(x')$

so ψ is well-defined.

Since $f = \psi\pi$, $f(x) = \psi(\pi(x)) = \psi(x + \phi(A))$

so the definition of ψ is unique.

(3) The Mercedes-Benz logo is homotopically equiv. to , so to  $= S^1 \vee S^1 \vee S^1$.

By van Kampen's theorem $\pi_1 = \pi_1(S^1) * \pi_1(S^1) * \pi_1(S^1)$

$= \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$ — the free group on 3 elements.

<u>Homology:</u> $H_1 \cong Ab(\pi_1) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$

Since  is path connected, $H_0 \cong \mathbb{Z}$

Since  is 1-dim, by vanishing,

$H_n = 0$ for $n > 1$.

(4) Suppose $j : A \to X$ is an inclusion and $f : X \to A$ is a retraction. Then $f \circ j = Id_A$

Since homology is a functor we get

$$f_* \circ j_* = (f \circ j)_* = (Id_A)_* = Id_{H_*(A)}$$

Since $Id_{H_*(A)}$ is injective, so is j_* ☺

Example 1 Let X be the discrete space $\{0, 1\}$ and $A = \{0\}$.

The constant map $f : X \to A$, $f(x) = 0$ is a retraction.

Meanwhile $H_0(A) = \mathbb{Z}$ and $H_0(X) = \mathbb{Z} \oplus \mathbb{Z}$, so j_* is not surjective.

Example 2 Let $X = I$ and $A = \{0, 1\}$

A continuous image of a connected space is connected, so any map $X \to A$ cannot be surjective, so A is not a retract of X.

We have $H_0(A) = \mathbb{Z} \oplus \mathbb{Z}$, $H_0(X) = \mathbb{Z}$

so j_* is not injective.

(5) Long exact sequence of homology:

$$\cdots \to H_n(A) \xrightarrow{j_*} H_n(X) \xrightarrow{\pi} H_n(X, A) \xrightarrow{\bar{\partial}} H_{n-1}(A) \xrightarrow{j_*} H_{n-1}(X) \to \cdots$$

If $H_n(X, A) = 0$, $\ker \pi = H_n(X)$, so j_* are onto

Also $\operatorname{Im} \bar{\partial} = 0$, so j_* are 1-1

Conversely if j_* are onto, $\ker \pi = H_n(X)$,

so $\pi = 0$, so $\ker \bar{\partial} = 0$, but if

j_* are 1-1, $\bar{\partial} = 0$, so $H_n(X, A) = 0$ ☺