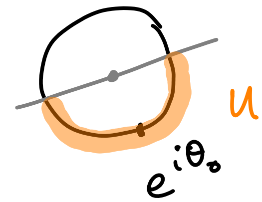
Midterm 1 MAT 5263.001 Sp. 2013

1. Let $e^{i\theta_0} \in S^1$.

Let $U = \{ e^{i\theta} : \theta_0 - \frac{\pi}{2} < \theta < \theta_0 + \frac{\pi}{2} \}$



then U is an open nbd of z in S^1.

The preimage of U under the exponential map is $\bigcup_{k \in \mathbb{Z}} I_k$, where $I_k = \left(\theta_0 + (2k - \frac{1}{2})\pi, \ \theta_0 + (2k + \frac{1}{2})\pi\right)$ are open intervals in $\mathbb{R}$.

The exponential map restricted to each I_k gives a homeo $I_k \to U$

We need to show I_k's are disjoint.

Suppose $\theta \in I_j$ and let $k \geq j+1$

Then $\theta < \theta_0 + (2j + \frac{1}{2})\pi = \theta_0 + (2j + 2 - 2 + \frac{1}{2})\pi$

$= \theta_0 + (2(j+1) - \frac{3}{2})\pi < \theta_0 + (2k - \frac{1}{2})\pi$, so

$\theta \notin I_k$. For $k < j$, reverse the roles of j and k.

$\therefore$ We have a covering space.

$Id_{\mathbb{R}}$ is homotopic to the zero map via $H(x,t) = xt$, so $\mathbb{R}$ is contractible, so simply connected.

$\therefore$ The cover is universal.

2. If X & Y are homotopically equivalent, $\exists$ maps $f: X \to Y$, $g: Y \to X$ s.t.

$$fg \simeq Id_Y \quad , \quad gf \simeq Id_X$$

Apply the π_1 functor:

$$f_* g_* = (fg)_* = (Id_Y)_* = Id_{\pi_1(Y)}$$

similarly $g_* f_* = Id_{\pi_1(X)}$

$\therefore$ f_* and g_* are inverses and we have $\pi_1(X) \cong \pi_1(Y)$

3. $\omega = y^2 z^4 \, dx + 2xyz^4 \, dy + 4xy^2 z^3 \, dz$

$$d\omega = \left(\cancel{2yz^4 \, dy} + \cancel{4y^2 z^3 \, dz}\right) dx + \left(\cancel{2yz^4 \, dx} + \cancel{8xyz^3 \, dz}\right) dy$$
$$+ \left(\cancel{4y^2 z^3 \, dx} + \cancel{8xyz^3 \, dy}\right) dz = 0$$

$$d(xy^2 z^4) = \omega$$

4. Let $f(z) = u(x,y) + i v(x,y)$

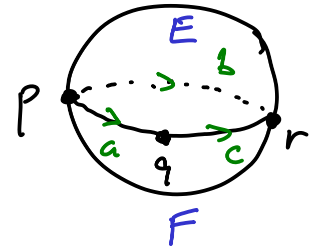
$$d(f(z)dz) = d\left([u(x,y) + i v(x,y)][dx + i dy]\right)$$
$$= (u_x dx + u_y dy + i v_x dx + i v_y dy)(dx + i dy)$$
$$= (i u_x - u_y - v_x - i v_y) \, dx \, dy$$

$$\mathrm{Re}[d(f dz)] = (-u_y - v_x) \, dx \, dy$$
$$\mathrm{Im}\{d(f dz)\} = (u_x - v_y) \, dx \, dy$$

$\therefore$ $d(f dz) = 0 \iff u_x = v_y$ and $u_y = -v_x$

(Cauchy-Riemann eqs.)

5.



$$0 \to C_2 \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_0} C_0 \to 0$$

$\mathbb{Z}^2$ (gen: E, F) $\quad \mathbb{Z}^3$ (gen: a, b, c) $\quad \mathbb{Z}^3$ (gen by p, q, r)

$\partial_1 E = a - b + c$

$\partial_2 F = a - b + c$

$\partial_0 a = q - p$

$\partial_0 b = r - p$

$\partial_0 c = r - q$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 1 \end{bmatrix}$$

rank 1

so image $\partial_1 \cong \mathbb{Z}$

and $\ker \partial_1 \cong \mathbb{Z}$

$\therefore H_2 \cong \mathbb{Z}$

$$\begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

Gaussian elimination $\rightarrow$ $\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ $\rightarrow$ $\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

rank = 2

$\therefore \ker \partial_0 \cong \mathbb{Z}$

$$\therefore H_1 = \frac{\ker \partial_0}{\text{Image } \partial_1} \cong \frac{\mathbb{Z}}{\mathbb{Z}} = 0$$

$$H_0 = \frac{\mathbb{Z}^3}{\text{Image } \partial_0} \cong \frac{\mathbb{Z}^3}{\mathbb{Z}^2} \cong \mathbb{Z}$$

For $n > 2$ $H_n = 0$